AH

National Qualifications
2021 ASSESSMENT RESOURCE

X847/77/12

Mathematics Paper 2

Duration — 2 hours

Total marks — 60

SECTION 1 — 45 marks

Attempt ALL questions.

SECTION 2 — 15 marks

Attempt EITHER Part A OR Part B.

You may use a calculator.

To earn full marks you must show your working in your answers.

State the units for your answer where appropriate.

You will not earn marks for answers obtained by readings from scale drawings.

Write your answers clearly in the spaces provided in the answer booklet. The size of the space provided for an answer is not an indication of how much to write. You do not need to use all the space.

Additional space for answers is provided at the end of the answer booklet. If you use this space you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

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FORMULAE LIST

Standard derivatives

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals

$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)+c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\frac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $$S_n = \frac{1}{2}n\left[2a+(n-1)d\right]$$

(Geometric series) $$S_n = \frac{a\left(1-r^n\right)}{1-r},\ r \neq 1$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2},\quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6},\quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\frac{f^{iv}(0)x^4}{4!}+\ldots$$

De Moivre's theorem

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ, about the origin, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

[Turn over

MARKS

SECTION 1 — 45 marks

Attempt ALL questions

1. Given $f(x) = 3\sec 2x$, find the exact value of $f'\left(\frac{\pi}{8}\right)$. **2**

2. (a) Use the Euclidean algorithm to find integers a and b such that $105a + 72b = 3$. **3**

   (b) Hence find integers x and y such that $105x + 72y = 360$. **1**

3. Use integration by parts to find $\int (2x+3)\cos 4x\, dx$. **3**

4. A curve is defined parametrically by

$$x = \sin^{-1} 2t \text{ and } y = \tan^{-1} t.$$

   (a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$. **3**

   (b) When $t = 0$ find the equation of the tangent to the curve. **2**

5. A non-singular matrix A satisfies the equation

$$A^2 = 2A + 5I,$$

where I is the identity matrix.

   (a) Express A^4 in the form $pA + qI$, where $p, q \in \mathbb{Z}$. **2**

   (b) Express A^{-1} in the form $rA + sI$, where $r, s \in \mathbb{Q}$. **2**

MARKS

6. Solve the differential equation

$$\frac{dy}{dx} + 2xy = 14xe^{-x^2}$$

given that when $x = 0,\ y = 3$.

Express y in terms of x. **4**

7. A complex number is defined by $z = a + 2i$ where a is a positive real number.

(a) State and simplify the binomial expansion of z^3. **3**

(b) Given that $z^3 + 3z = b + 148i$, where b is a real number, find the values of a and b. **3**

8. A curve is defined by $x^2y^3 + e^{2y} = 5$.

(a) Find $\frac{dy}{dx}$ in terms of x and y. **4**

(b) Show that there is only one stationary point on the curve. **3**

9. (a) Express $\frac{1}{x(5-x)}$ in partial fractions. **2**

(b) A small island is being populated by seals. The size of the seal population can be modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{100}P(5-P),\ 0 < P < 5$$

where P (in hundreds) is the number of seals on the island t years after the seals arrive.

Given that there are 250 seals after 10 years, find an expression for P in terms of t. **8**

[END OF SECTION 1]

[Turn over

MARKS

SECTION 2 — 15 marks

Attempt EITHER Part A OR Part B

Part A

10. Prove by induction that $\sum_{r=2}^{n} \frac{1}{r(r-1)} = \frac{n-1}{n}$ for all positive integers $n \geq 2$. **5**

11. Three consecutive terms of an arithmetic sequence are given by

$$x-1, \quad x-7, \quad 2x-9.$$

(a) (i) Find the common difference. **1**

(ii) Hence find the value of x. **1**

(b) Given that $x-1$ is the 21^{st} term, find

(i) the value of the first term **1**

(ii) a simplified expression for the n^{th} term of the sequence. **1**

Three consecutive terms of a geometric sequence are given by

$$y-1, \quad y-7, \quad 2y-9.$$

(c) Find the two possible values of y and the corresponding common ratios. **3**

One of the values of y gives an associated geometric series which has a sum to infinity.

(d) (i) Identify the value of y and justify your answer. **1**

(ii) Determine whether $\frac{64}{3}$ is a possible value for this sum to infinity. Give a reason for your answer. **2**

MARKS

Part B

12. The points A(4, 0, 8), B(6, –5, 4) and C(3, 4, 11) all lie on the plane π_1.

(a) Find the Cartesian equation of π_1. **4**

The plane π_2 is parallel to π_1 and passes through the origin.

(b) State the equation of π_2. **1**

A sphere touches π_1, where A is the point of contact. The sphere also has a single point of contact, Q, with π_2.

(c) (i) Find parametric equations for the line AQ. **1**

(ii) Hence find the coordinates for Q. **2**

[Turn over

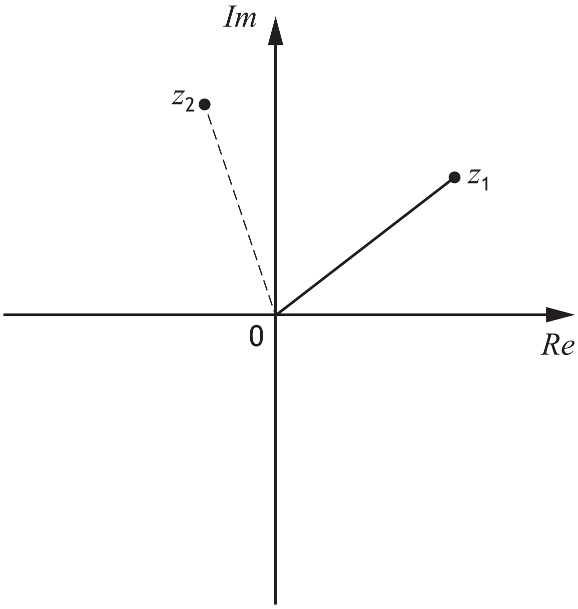
MARKS

13. (a) Express -1 in the form $\cos\theta + i\sin\theta$. **1**

The complex number z_1 is defined by $z_1 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$.

(b) Use de Moivre's theorem to show that z_1 is a root of the equation $z^5 + 1 = 0$. **1**

The complex number z_2 is also a root of the equation $z^5 + 1 = 0$. Roots z_1 and z_2 have been plotted on an Argand diagram, as shown.



(c) Express z_2 in the form $\cos\theta + i\sin\theta$. **1**

The remaining roots of the equation $z^5 + 1 = 0$ are z_3, z_4 and z_5.

(d) Express z_3, z_4 and z_5 in the form $\cos\theta + i\sin\theta$, where $-\pi < \theta \leq \pi$. **2**

(e) Given $z_1 + z_2 + z_3 + z_4 + z_5 = 0$, show algebraically that

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}.$$ **2**

[END OF SECTION 2]

[END OF QUESTION PAPER]