ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Applications of Calculus

1. $a(t) = 20 + 6t + 4t^2$

$$v(t) = \int a(t)dt = \int (20 + 6t + 4t^2)dt$$
$$= 20t + \frac{6t^2}{2} + \frac{4t^3}{3} + C$$
$$= 20t + 3t^2 + \frac{4t^3}{3} + C$$

The car started from rest at time t = 0, so when $t = 0$, $v = 0$.

Hence $0 = 20(0) + 3(0)^2 + \frac{4(0)^3}{3} + C \quad \Rightarrow \quad C = 0$

The velocity of the car after t seconds is given by $v(t) = 20t + 3t^2 + \frac{4t^3}{3}$.

$$s(t) = \int v(t)dt = \int\left(20t + 3t^2 + \frac{4}{3}t^3\right)dt$$
$$= \frac{20t^2}{2} + \frac{3t^3}{3} + \frac{4}{3}\left(\frac{t^4}{4}\right) + D$$
$$= 10t^2 + t^3 + \frac{t^4}{3} + D$$

Let $s(t)$ be the displacement of the car from the point O on the road after t seconds.

The car started from point O, so when $t = 0$, $s = 0$.

Hence $0 = 10(0)^2 + 0^3 + \frac{0^4}{3} + D \quad \Rightarrow \quad D = 0$

The displacement of the car from O after t seconds is given by $s(t) = 10t^2 + t^3 + \frac{t^4}{3}$.

2.(a) $a(t) = 8 + 10t - \frac{3}{4}t^2$

$$v(t) = \int a(t)dt = \int \left(8 + 10t - \frac{3}{4}t^2 \right) dt$$

$$= 8t + \frac{10t^2}{2} - \frac{3}{4}\left(\frac{t^3}{3} \right) + C$$

$$= 8t + 5t^2 - \frac{t^3}{4} + C$$

The missile was fired from rest, so when $t = 0$, $v = 0$.

Hence $0 = 8(0) + 5(0)^2 - \frac{0^3}{4} + C \quad \Rightarrow \quad C = 0$

The velocity of the missile t seconds after firing is given by $v(t) = 8t + 5t^2 - \frac{t^3}{4}$.

(b) $$s(t) = \int v(t)dt = \int \left(8t + 5t^2 - \frac{1}{4}t^3 \right) dt$$

$$= \frac{8t^2}{2} + \frac{5t^3}{3} - \frac{1}{4}\left(\frac{t^4}{4} \right) + D$$

$$= 4t^2 + \frac{5t^3}{3} - \frac{t^4}{16} + D$$

Let $s(t)$ be the displacement of the missile from its firing point after t seconds.

When $t = 0$, $s = 0 \quad \Rightarrow \quad 0 = 4(0)^2 + \frac{5(0)^3}{3} - \frac{0^4}{16} + D \quad \Rightarrow \quad D = 0$

Hence $s = 4t^2 + \frac{5t^3}{3} - \frac{t^4}{16}$.

When $t = 10$: $s = 4(10)^2 + \frac{5(10)^3}{3} - \frac{10^4}{16} = \frac{4325}{3}$

The displacement of the missile from its firing point when its fuel was exhausted was $\frac{4325}{3}$ metres.

3. $a(t) = \dfrac{1}{1+t^2}$

$$v(t) = \int a(t)dt = \int \frac{1}{1+t^2}dt = \tan^{-1} t + C$$

$v = 0$ when $t = 1 \quad \Rightarrow \quad 0 = \tan^{-1} 1 + C \quad \Rightarrow \quad 0 = \dfrac{\pi}{4} + C \quad \Rightarrow \quad C = -\dfrac{\pi}{4}$

Hence $\quad v(t) = \tan^{-1} t - \dfrac{\pi}{4}$.

When $t = \sqrt{3}$: $\quad v = \tan^{-1} \sqrt{3} - \dfrac{\pi}{4} = \dfrac{\pi}{3} - \dfrac{\pi}{4} = \dfrac{4\pi}{12} - \dfrac{3\pi}{12} = \dfrac{\pi}{12}$

The velocity of the particle when $t = \sqrt{3}$ is $\dfrac{\pi}{12}$ ms^{-1}.

4.(a) $v = t^3 - 12t^2 + 32t$

$$a = \frac{dv}{dt} = 3t^2 - 24t + 32$$

When $t = 0$: $a = 3(0)^2 - 24(0) + 32 = 32$

The acceleration when $t = 0$ is 32.

(b) $s = \int v dt = \int (t^3 - 12t^2 + 32t) dt$

$$= \frac{t^4}{4} - \frac{12t^3}{3} + \frac{32t^2}{2} + C$$

$$= \frac{t^4}{4} - 4t^3 + 16t^2 + C$$

Let *s* be the displacement of the body from the origin *O* at time *t*.

When $t = 0$, $s = 0$ $\Rightarrow$ $0 = \frac{0^4}{4} - 4(0)^3 + 16(0)^2 + C$ $\Rightarrow$ $C = 0$

The displacement of the body at time *t* is given by $s = \frac{t^4}{4} - 4t^3 + 16t^2$.

The body returns to *O* when $s = 0$ $\Rightarrow$ $0 = \frac{t^4}{4} - 4t^3 + 16t^2$ $[\times 4]$

$$\Rightarrow t^4 - 16t^3 + 64t^2 = 0$$

$$\Rightarrow t^2(t^2 - 16t + 64) = 0$$

$$\Rightarrow t^2(t-8)(t-8) = 0$$

$$\Rightarrow t = 0,\ t = 8$$

Hence the body returns to *O* at time $T = 8$.

Note

No units are given in this question.

5.(a) $v = e^{3t} + 2e^{t}$

$$a = \frac{dv}{dt} = 3e^{3t} + 2e^{t}$$

The acceleration of P at time t is given by $a = 3e^{3t} + 2e^{t}$.

(b) $s = \int v dt = \int (e^{3t} + 2e^{t})dt = \frac{1}{3}e^{3t} + 2e^{t} + C$

Let s be the displacement of P from its position at time $t = 0$.

When $t = 0$, $s = 0$ $\Rightarrow$ $0 = \frac{1}{3}e^{3(0)} + 2e^{0} + C$

$$\Rightarrow \quad 0 = \frac{1}{3}e^{0} + 2e^{0} + C$$

$$\Rightarrow \quad 0 = \frac{1}{3}(1) + 2(1) + C$$

$$\Rightarrow \quad 0 = \frac{7}{3} + C$$

$$\Rightarrow \quad C = -\frac{7}{3}$$

Hence $s = \frac{1}{3}e^{3t} + 2e^{t} - \frac{7}{3}$.

When $t = \ln 3$: $s = \frac{1}{3}e^{3\ln 3} + 2e^{\ln 3} - \frac{7}{3}$

$$= \frac{1}{3}e^{\ln 27} + 2e^{\ln 3} - \frac{7}{3} \qquad [\text{since } 3\ln 3 = \ln 3^{3} = \ln 27]$$

$$= \frac{1}{3}(27) + 2(3) - \frac{7}{3} \qquad [\text{since } e^{\ln 27} = 27 \text{ and } e^{\ln 3} = 3]$$

$$= \frac{38}{3}$$

The distance covered by P between $t = 0$ and $t = \ln 3$ is $\frac{38}{3}$.

Note

No units are given in this question.

6.(a) $a(t) = -\frac{3}{2}\sqrt{4+t}$

$$v(t) = \int a(t)dt = \int -\frac{3}{2}\sqrt{4+t}dt = \int -\frac{3}{2}(4+t)^{\frac{1}{2}}dt$$

$$= -\frac{3}{2}\left(\frac{(4+t)^{\frac{3}{2}}}{\frac{3}{2}\times 1}\right) + C$$

$$= -\frac{3}{2}\left(\frac{2(4+t)^{\frac{3}{2}}}{3}\right) + C$$

$$= -(4+t)^{\frac{3}{2}} + C$$

$$= C - \sqrt{(4+t)^3}$$

When $t = 0$, $v = 19$ $\Rightarrow$ $19 = C - \sqrt{4^3}$ $\Rightarrow$ $19 = C - 8$ $\Rightarrow$ $C = 27$

Hence $v = 27 - \sqrt{(4+t)^3}$.

The car comes to a complete stop after T seconds, so $v = 0$ when $t = T$.

Hence $0 = 27 - \sqrt{(4+T)^3}$ $\Rightarrow$ $\sqrt{(4+T)^3} = 27$

$$\Rightarrow (4+T)^3 = 27^2$$

$$\Rightarrow (4+T)^3 = 729$$

$$\Rightarrow 4+T = \sqrt[3]{729}$$

$$\Rightarrow 4+T = 9$$

$$\Rightarrow T = 5$$

(b) $s(t) = \int v(t)dt = \int\left(27 - \sqrt{(4+t)^3}\right)dt = \int\left(27 - (4+t)^{\frac{3}{2}}\right)dt$

$$= 27t - \frac{(4+t)^{\frac{5}{2}}}{\frac{5}{2}\times 1} + D$$

$$= 27t - \frac{2(4+t)^{\frac{5}{2}}}{5} + D$$

$$= 27t - \frac{2\sqrt{(4+t)^5}}{5} + D$$

Let $s(t)$ be the displacement of the car from its position at time $t = 0$.

When $t = 0$, $s = 0$ $\Rightarrow$ $0 = 27(0) - \frac{2\sqrt{4^5}}{5} + D$

$$\Rightarrow \quad 0 = -\frac{2(32)}{5} + D$$

$$\Rightarrow \quad D = \frac{64}{5}$$

Hence $s(t) = 27t - \frac{2\sqrt{(4+t)^5}}{5} + \frac{64}{5}$.

When $t = 5$: $s = 27(5) - \frac{2\sqrt{9^5}}{5} + \frac{64}{5} = 135 - \frac{2(243)}{5} + \frac{64}{5} = \frac{253}{5}$

The distance travelled by the car during the 5 seconds that it takes to come to a stop is $\frac{253}{5}$ metres (or 50·6 metres).

7. The rotation is about the *x*-axis, so the volume of the solid formed is given by $V = \pi\int_0^1 y^2 dx$, where $y = e^{-2x}$.

$$\int_0^1 y^2 dx = \int_0^1 (e^{-2x})^2 dx = \int_0^1 e^{-4x} dx = \left[-\frac{1}{4}e^{-4x}\right]_0^1 = \left[-\frac{1}{4}e^{-4(1)}\right] - \left[-\frac{1}{4}e^{-4(0)}\right]$$

$$= \left[-\frac{1}{4}e^{-4}\right] - \left[-\frac{1}{4}e^{0}\right]$$

$$= \left[-\frac{1}{4}e^{-4}\right] - \left[-\frac{1}{4}(1)\right]$$

$$= -\frac{1}{4}e^{-4} + \frac{1}{4}$$

$$= \frac{1}{4}(1 - e^{-4})$$

Hence $V = \pi \times \frac{1}{4}(1 - e^{-4}) = \frac{\pi}{4}(1 - e^{-4})$ units^3

8. The rotation is about the *x*-axis, so the volume of the solid of revolution formed is given by $V = \pi \int_{-2}^{3} y^2 dx$.

To find $\int_{-2}^{3} y^2 dx$, we need a formula for y^2 in terms of *x*.

$x^2 + y^2 = 9 \quad \Rightarrow \quad y^2 = 9 - x^2$

Hence $\int_{-2}^{3} y^2 dx = \int_{-2}^{3} (9 - x^2) dx = \left[9x - \frac{x^3}{3} \right]_{-2}^{3}$

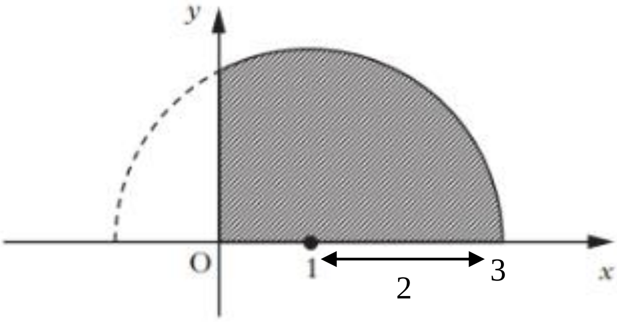
$$= \left[9(3) - \frac{3^3}{3} \right] - \left[9(-2) - \frac{(-2)^3}{3} \right]$$

$$= \left[27 - \frac{27}{3} \right] - \left[-18 - \frac{(-8)}{3} \right]$$

$$= \frac{100}{3}$$

Hence $V = \pi \times \frac{100}{3} = \frac{100\pi}{3}$ units^3

9. The semi-circle crosses the *x*-axis at the point (3, 0) since the semi-circle has centre (1, 0) and radius 2 (see diagram below).



The rotation is about the *x*-axis, so the volume of the solid of revolution formed is given by $V = \pi \int_0^3 y^2 dx$.

To find $\int_0^3 y^2 dx$, we need a formula for y^2 in terms of *x*.

The equation of the circle with centre (1, 0) and radius 2 is

$(x-1)^2 + (y-0)^2 = 2^2 \quad \Rightarrow \quad (x-1)^2 + y^2 = 4 \quad \Rightarrow \quad y^2 = 4-(x-1)^2$

Hence $\int_0^3 y^2 dx = \int_0^3 \left(4-(x-1)^2\right) dx = \left[4x - \frac{(x-1)^3}{3\times 1}\right]_0^3$

$$= \left[4x - \frac{(x-1)^3}{3}\right]_0^3$$

$$= \left[4(3) - \frac{(3-1)^3}{3}\right] - \left[4(0) - \frac{(0-1)^3}{3}\right]$$

$$= \left[12 - \frac{8}{3}\right] - \left[0 - \frac{(-1)}{3}\right]$$

$$= 9$$

Hence $V = 9\pi$ units^3.

Notes

(1) The equation of the circle with centre (a, b) and radius r is $(x-a)^2 + (y-b)^2 = r^2$.

(2) $\int_0^3 \left(4-(x-1)^2\right)dx$ can also be found by expanding the brackets before integrating.

$$\int_0^3 \left(4-(x-1)^2\right)dx = \int_0^3 (4-(x^2-2x+1))dx$$

$$= \int_0^3 (4-x^2+2x-1)dx$$

$$= \int_0^3 (3+2x-x^2)dx$$

$$= \left[3x + \frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \left[3x + x^2 - \frac{x^3}{3}\right]_0^3$$

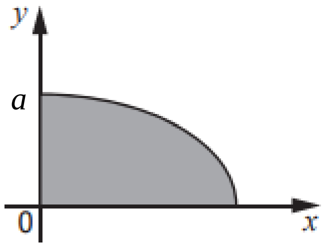
$$= \left[3(3) + 3^2 - \frac{3^3}{3}\right] - [0]$$

$$= 9$$

10. The rotation is about the *y*-axis, so the volume of the solid generated is given by

$V = \pi \int_0^a x^2 dy$, where the upper limit *a* is a value on the *y*-axis.

To find the value of *a*, we need to find where the curve crosses the *y*-axis (see diagram below).



The curve crosses the *y*-axis when $x = 0$:

$$4x^2 + 9y^2 = 36$$
$$\Rightarrow \quad 4(0)^2 + 9y^2 = 36$$
$$\Rightarrow \quad 9y^2 = 36$$
$$\Rightarrow \quad y^2 = 4$$
$$\Rightarrow \quad y = \pm 2$$

Hence $a = 2$ since $a > 0$ and $V = \pi \int_0^2 x^2 dy$.

To find $\int_0^2 x^2 dy$, we need to express x^2 in terms of *y* by rearranging the equation of the curve.

$$4x^2 + 9y^2 = 36 \quad \Rightarrow \quad 4x^2 = 36 - 9y^2 \quad \Rightarrow \quad x^2 = 9 - \frac{9}{4}y^2$$

Hence $\int_0^2 x^2 dy = \int_0^2 \left(9 - \frac{9}{4}y^2\right) dy = \left[9y - \frac{9}{4}\left(\frac{y^3}{3}\right)\right]_0^2 = \left[9y - \frac{3y^3}{4}\right]_0^2$

$$= \left[9(2) - \frac{3(2)^3}{4}\right] - [0]$$
$$= 18 - 6$$
$$= 12$$

Hence $V = 12\pi$ units^3.

11.(a) First factorise the denominator: $x^2 + x = x(x+1))$

The denominator contains distinct linear factors.

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+Bx}{x(x+1)}$$

$1 = A(x+1) + Bx$

Let $x = -1 \quad \Rightarrow \quad 1 = A(0) + B(-1) \quad \Rightarrow \quad 1 = -B \quad \Rightarrow \quad B = -1$

Let $x = 0 \quad \Rightarrow \quad 1 = A(1) + B(0) \quad \Rightarrow \quad 1 = A \quad \Rightarrow \quad A = 1$

Hence $\dfrac{1}{x^2+x} = \dfrac{1}{x} + \dfrac{-1}{x+1}$ *or* $\dfrac{1}{x^2+x} = \dfrac{1}{x} - \dfrac{1}{x+1}$.

(b) The rotation is about the x-axis, so the volume of the solid of revolution formed is given by $V = \pi\int_1^3 y^2\,dx$, where $y = \dfrac{1}{\sqrt{x^2+x}}$.

$$\int_1^3 y^2\,dx = \int_1^3 \left(\frac{1}{\sqrt{x^2+x}}\right)^2 dx = \int_1^3 \frac{1}{x^2+x}\,dx = \int_1^3 \left(\frac{1}{x} - \frac{1}{x+1}\right)dx \quad \text{[using the partial fractions in (a)]}$$
$$= \Big[\ln|x| - \ln|x+1|\Big]_1^3$$
$$= \Big[\ln|3| - \ln|4|\Big] - \Big[\ln|1| - \ln|2|\Big]$$
$$= \Big[\ln 3 - \ln 4\Big] - [\ln 1 - \ln 2]$$
$$= \left[\ln\left(\frac{3}{4}\right)\right] - [0 - \ln 2]$$
$$= \ln\left(\frac{3}{4}\right) + \ln 2$$
$$= \ln\left(\frac{3}{4}\times 2\right)$$
$$= \ln\left(\frac{3}{2}\right)$$

Hence $V = \pi \ln\left(\dfrac{3}{2}\right)$ units^3.

<u>Note</u>

An exact value for the volume is not required and an approximate value can be given to any reasonable degree of accuracy, eg 1·2738 units^3 (to 4 dp).

12.(a) Let $I = \int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$.

Using the substitution $u = 1 + x^2$, rewrite the entire integral in terms of *u* (including the limits).

$$u = 1 + x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx$$

$$(1+x^2)^4 = u^4 \quad \text{and} \quad u = 1 + x^2 \quad \Rightarrow \quad x^2 = u - 1$$

When $x = 0$: $u = 1 + 0^2 = 1$ When $x = 1$: $u = 1 + 1^2 = 2$

$$\begin{aligned}
I = \int_0^1 \frac{x^3}{(1+x^2)^4}\,dx = \int_0^1 \frac{x^2}{(1+x^2)^4}(x\,dx) &= \int_1^2 \frac{u-1}{u^4}\left(\frac{1}{2}du\right) \\
&= \int_1^2 \frac{1}{2}u^{-4}(u-1)\,du \\
&= \int_1^2 \left(\frac{1}{2}u^{-3} - \frac{1}{2}u^{-4}\right)du \\
&= \left[\frac{1}{2}\left(\frac{u^{-2}}{-2}\right) - \frac{1}{2}\left(\frac{u^{-3}}{-3}\right)\right]_1^2 \\
&= \left[\frac{u^{-2}}{-4} + \frac{u^{-3}}{6}\right]_1^2 \\
&= \left[-\frac{1}{4u^2} + \frac{1}{6u^3}\right]_1^2 \\
&= \left[-\frac{1}{4(2)^2} + \frac{1}{6(2)^3}\right] - \left[-\frac{1}{4(1)^2} + \frac{1}{6(1)^3}\right] \\
&= \left[-\frac{1}{16} + \frac{1}{48}\right] - \left[-\frac{1}{4} + \frac{1}{6}\right] \\
&= \frac{1}{24}
\end{aligned}$$

(b) The rotation is about the *x*-axis, so the volume of the solid formed is given by $V = \pi\int_0^1 y^2\,dx$.

$$V = \pi\int_0^1 y^2\,dx = \pi\int_0^1 \left(\frac{x^{\frac{3}{2}}}{(1+x^2)^2}\right)^2 dx = \pi\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx = \pi\left(\frac{1}{24}\right) = \frac{\pi}{24} \text{ units}^3$$

13.(a) The rotation is about the x-axis, so the volume of the solid formed is given by $V = \pi\int_{15}^{30} y^2 dx$,

where $y = e^{\frac{x}{12}}$.

Note that $y^2 = \left(e^{\frac{x}{12}}\right)^2 = e^{\frac{x}{12}\times 2} = e^{\frac{x}{6}}$.

$$\int_{15}^{30} y^2 dx = \int_{15}^{30} e^{\frac{x}{6}} dx = \int_{15}^{30} e^{\frac{1}{6}x} dx = \left[6e^{\frac{1}{6}x}\right]_{15}^{30} \qquad \text{using } \int e^{ax} dx = \frac{1}{a}e^{ax} + C$$

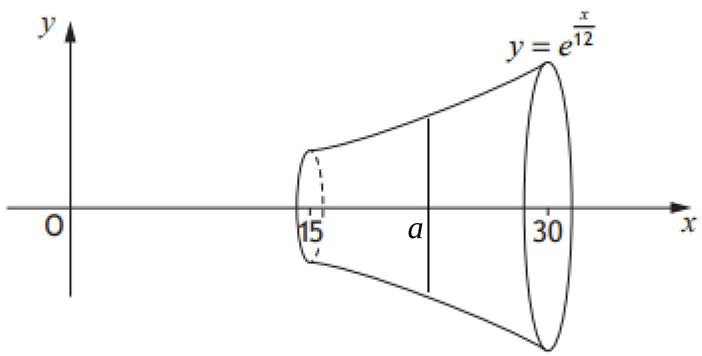
$$= \left[6e^{\frac{1}{6}(30)}\right] - \left[6e^{\frac{1}{6}(15)}\right]$$

$$= 6e^5 - 6e^{2\cdot 5}$$

$$= 6(e^5 - e^{2\cdot 5})$$

Hence $V = \pi \times 6(e^5 - e^{2\cdot 5}) = 6\pi(e^5 - e^{2\cdot 5}) = 2567 \cdot 887... = 2570$ cm^3 (to 3 sf)

(b) Let the line with equation $x = a$, where $15 < a < 30$, indicate when the bowl is half full as shown in the diagram below.



Then the volume of the part of the bowl generated by rotating the curve between $x = 15$ and $x = a$ through 2π radians about the x-axis will be $\dfrac{2570}{2} = 1285$ cm^3.

This gives the equation $\pi\int_{15}^{a} y^2 dx = 1285$.

$$\int_{15}^{a} y^2 dx = \left[6e^{\frac{1}{6}x}\right]_{15}^{a} = \left[6e^{\frac{1}{6}a}\right] - \left[6e^{\frac{1}{6}(15)}\right] = 6e^{\frac{a}{6}} - 6e^{2\cdot 5} = 6\left(e^{\frac{a}{6}} - e^{2\cdot 5}\right)$$

Hence $\pi \times 6\left(e^{\frac{a}{6}} - e^{2\cdot5}\right) = 1285 \;\Rightarrow\; 6\pi\left(e^{\frac{a}{6}} - e^{2\cdot5}\right) = 1285$

$$\Rightarrow\; e^{\frac{a}{6}} - e^{2\cdot5} = \frac{1285}{6\pi}$$

$$\Rightarrow\; e^{\frac{a}{6}} = \frac{1285}{6\pi} + e^{2\cdot5}$$

$$\Rightarrow\; \frac{a}{6} = \ln\left(\frac{1285}{6\pi} + e^{2\cdot5}\right)$$

$$\Rightarrow\; a = 6\ln\left(\frac{1285}{6\pi} + e^{2\cdot5}\right) = 26\cdot3 \text{ (to 1 dp)}$$

26·3 − 15 = 11·3, so the line should be marked 11·3 cm above the base of the bowl.

Note

Alternatively, the volume of the part of the bowl generated by rotating the curve between $x = a$ and $x = 30$ through 2π radians about the *x*-axis will also be $\frac{2570}{2} = 1285$ cm^3.

This gives the equation $\pi\int_a^{30} y^2\,dx = 1285$.

$$\int_a^{30} y^2\,dx = \left[6e^{\frac{1}{6}x}\right]_a^{30} = \left[6e^{\frac{1}{6}(30)}\right] - \left[6e^{\frac{1}{6}a}\right] = 6e^5 - 6e^{\frac{a}{6}} = 6\left(e^5 - e^{\frac{a}{6}}\right)$$

Hence $\pi \times 6\left(e^5 - e^{\frac{a}{6}}\right) = 1285 \;\Rightarrow\; 6\pi\left(e^5 - e^{\frac{a}{6}}\right) = 1285$

$$\Rightarrow\; e^5 - e^{\frac{a}{6}} = \frac{1285}{6\pi}$$

$$\Rightarrow\; e^{\frac{a}{6}} = e^5 - \frac{1285}{6\pi}$$

$$\Rightarrow\; \frac{a}{6} = \ln\left(e^5 - \frac{1285}{6\pi}\right)$$

$$\Rightarrow\; a = 6\ln\left(e^5 - \frac{1285}{6\pi}\right) = 26\cdot3 \text{ (to 1 dp)}$$

26·3 − 15 = 11·3, so the line should be marked 11·3 cm above the base of the bowl.