ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Basic Differentiation

1. $y = \sin(e^{5x})$

To differentiate *y*, use the **chain rule** as *y* is a function of a function.

$$\frac{dy}{dx} = \cos(e^{5x}) \times 5e^{5x}$$
$$= 5e^{5x}\cos(e^{5x})$$

2. $f(x) = \dfrac{x-1}{1+x^2}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$f'(x) = \frac{(1+x^2)1-(x-1)2x}{(1+x^2)^2}$$
$$= \frac{1+x^2-2x(x-1)}{(1+x^2)^2}$$
$$= \frac{1+x^2-2x^2+2x}{(1+x^2)^2}$$
$$= \frac{1+2x-x^2}{(1+x^2)^2}$$

3. $f(x) = \dfrac{x^2-1}{x^2+1}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$f'(x) = \frac{(x^2+1)2x-(x^2-1)2x}{(x^2+1)^2}$$
$$= \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$$
$$= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$
$$= \frac{4x}{(x^2+1)^2}$$

4. $f(x) = \dfrac{1 - x^2}{1 + 4x^2}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$f'(x) = \frac{(1+4x^2)(-2x) - (1-x^2)8x}{(1+4x^2)^2}$$
$$= \frac{-2x(1+4x^2) - 8x(1-x^2)}{(1+4x^2)^2}$$
$$= \frac{-2x - 8x^3 - 8x + 8x^3}{(1+4x^2)^2}$$
$$= \frac{-10x}{(1+4x^2)^2}$$

5. $y = \dfrac{x}{x^2 + 4}$

To differentiate *y*, use the **quotient rule** as *y* is the quotient of two functions.

$$\frac{dy}{dx} = \frac{(x^2+4)1 - x(2x)}{(x^2+4)^2} = \frac{x^2 + 4 - 2x^2}{(x^2+4)^2} = \frac{4 - x^2}{(x^2+4)^2}$$

When $x = 2$: $m = \dfrac{dy}{dx} = \dfrac{4 - 2^2}{(2^2+4)^2} = 0$

The gradient of the curve when $x = 2$ is 0.

6. $y = \dfrac{e^{5x}}{7x+1}$

To differentiate *y*, use the **quotient rule** as *y* is the quotient of two functions.

$$\frac{dy}{dx} = \frac{(7x+1)5e^{5x} - e^{5x}(7)}{(7x+1)^2}$$
$$= \frac{5e^{5x}(7x+1) - 7e^{5x}}{(7x+1)^2}$$
$$= \frac{35xe^{5x} + 5e^{5x} - 7e^{5x}}{(7x+1)^2}$$
$$= \frac{35xe^{5x} - 2e^{5x}}{(7x+1)^2}$$
$$= \frac{e^{5x}(35x-2)}{(7x+1)^2}$$

7. $f(x) = \dfrac{e^{x^2-1}}{x^2-1}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions and use the **chain rule** to differentiate e^{x^2-1}.

$$f'(x) = \frac{(x^2-1)2xe^{x^2-1} - e^{x^2-1}(2x)}{(x^2-1)^2}$$
$$= \frac{2xe^{x^2-1}(x^2-1) - 2xe^{x^2-1}}{(x^2-1)^2}$$
$$= \frac{2x^3e^{x^2-1} - 2xe^{x^2-1} - 2xe^{x^2-1}}{(x^2-1)^2}$$
$$= \frac{2x^3e^{x^2-1} - 4xe^{x^2-1}}{(x^2-1)^2}$$
$$= \frac{2xe^{x^2-1}(x^2-2)}{(x^2-1)^2}$$

8. $y = 2x\sqrt{x-1}$

To differentiate *y*, use the **product rule** as *y* is the product of two functions and use the **chain rule** to differentiate $\sqrt{x-1}$.

$$\frac{d}{dx}\sqrt{x-1} = \frac{d}{dx}(x-1)^{\frac{1}{2}} = \frac{1}{2}(x-1)^{-\frac{1}{2}} \times 1 = \frac{1}{2\sqrt{x-1}}$$

Hence $\frac{dy}{dx} = 2x\left(\frac{1}{2\sqrt{x-1}}\right) + \sqrt{x-1}(2)$

$$= \frac{x}{\sqrt{x-1}} + 2\sqrt{x-1}$$

When $x = 10$: $m = \frac{dy}{dx} = \frac{10}{\sqrt{9}} + 2\sqrt{9} = \frac{10}{3} + 6 = \frac{28}{3}$

The gradient of the tangent to the curve when $x = 10$ is $\frac{28}{3}$.

9. $f(x) = x(1+x)^{10}$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions and use the **chain rule** to differentiate $(1+x)^{10}$.

$$\frac{d}{dx}(1+x)^{10} = 10(1+x)^9 \times 1 = 10(1+x)^9$$

Hence $f'(x) = x\left(10(1+x)^9\right) + (1+x)^{10}(1)$

$$= 10x(1+x)^9 + (1+x)^{10}$$

$$= (1+x)^9\left(10x + (1+x)\right)$$ [removing a common factor of $(1+x)^9$]

$$= (1+x)^9(11x+1)$$

10. $f(x) = (x+1)(x-2)^3$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions and use the **chain rule** to differentiate $(x-2)^3$.

$$\frac{d}{dx}(x-2)^3 = 3(x-2)^2 \times 1 = 3(x-2)^2$$

Hence
$$\begin{aligned} f'(x) &= (x+1)\left(3(x-2)^2\right) + (x-2)^3(1) \\ &= 3(x+1)(x-2)^2 + (x-2)^3 \\ &= (x-2)^2\left(3(x+1) + (x-2)\right) \quad \text{[removing a common factor of } (x-2)^2\text{]} \\ &= (x-2)^2(3x+3+x-2) \\ &= (x-2)^2(4x+1) \end{aligned}$$

Then $f'(x) = 0 \quad \Rightarrow \quad (x-2)^2(4x+1) = 0 \quad \Rightarrow \quad x = -\frac{1}{4},\ x = 2$

<u>Note</u>

In order to solve the equation $f'(x) = 0$, the expression for $f'(x)$ must be expressed in factorised form.

11. $f(x) = \frac{2x}{x-1}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$f'(x) = \frac{(x-1)2 - 2x(1)}{(x-1)^2} = \frac{2(x-1)-2x}{(x-1)^2} = \frac{2x-2-2x}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

If $x \neq 1$, $(x-1)^2 > 0$ and $f'(x) < 0$.
Hence $f'(x) < 0$ for all x ($x \neq 1$) and the derivative is always negative.

12.(a) $f(x) = e^{x^2+3}$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = e^{x^2+3} \times 2x = 2xe^{x^2+3}$$

(b) **<u>Method 1</u>**

$$g(x) = \ln\sqrt{x^2+3}$$

To differentiate $g(x)$, use the **chain rule** as $g(x)$ is a function of a function.

$$g'(x) = \frac{1}{\sqrt{x^2+3}} \times \frac{d}{dx}\sqrt{x^2+3}$$

To differentiate $\sqrt{x^2+3}$, use the **chain rule** again:

$$\frac{d}{dx}\sqrt{x^2+3} = \frac{d}{dx}(x^2+3)^{\frac{1}{2}} = \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \times 2x = \frac{x}{\sqrt{x^2+3}}$$

Hence $g'(x) = \dfrac{1}{\sqrt{x^2+3}} \times \dfrac{x}{\sqrt{x^2+3}} = \dfrac{x}{x^2+3}$

<u>Method 2</u>

Use the laws of logarithms to simplify $g(x)$ before differentiating.

$$g(x) = \ln\sqrt{x^2+3} = \ln(x^2+3)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+3)$$

To differentiate $\ln(x^2+3)$, use the **chain rule** as $\ln(x^2+3)$ is a function of a function.

$$g'(x) = \frac{1}{2}\left(\frac{1}{(x^2+3)} \times 2x\right) = \frac{x}{x^2+3}$$

13. **<u>Method 1</u>**

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = \frac{1}{\left(\frac{1+x}{1-x}\right)} \times \frac{d}{dx}\left(\frac{1+x}{1-x}\right) = \left(\frac{1-x}{1+x}\right) \times \frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

To differentiate $\frac{1+x}{1-x}$, use the **quotient rule**:

$$\frac{d}{dx}\left(\frac{1+x}{1-x}\right) = \frac{(1-x)1-(1+x)(-1)}{(1-x)^2} = \frac{1(1-x)+1(1+x)}{(1-x)^2} = \frac{1-x+1+x}{(1-x)^2} = \frac{2}{(1-x)^2}$$

Hence $\quad f'(x) = \left(\frac{1-x}{1+x}\right) \times \frac{2}{(1-x)^2} = \frac{2}{(1+x)(1-x)} = \frac{2}{1-x^2}$

<u>Method 2</u>

Use the laws of logarithms to simplify $f(x)$ before differentiating.

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

To differentiate $\ln(1+x)$ and $\ln(1-x)$, use the **chain rule** as each of these terms is a function of a function.

$$\begin{aligned} f'(x) = \frac{1}{(1+x)} \times 1 - \frac{1}{(1-x)} \times (-1) &= \frac{1}{1+x} + \frac{1}{1-x} \\ &= \frac{1(1-x)+1(1+x)}{(1+x)(1-x)} \\ &= \frac{1-x+1+x}{(1+x)(1-x)} \\ &= \frac{2}{(1+x)(1-x)} \\ &= \frac{2}{1-x^2} \end{aligned}$$

14. $f(x) = \exp(\sin 2x) = e^{\sin 2x}$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = e^{\sin 2x} \times 2\cos 2x = 2\cos 2x \exp(\sin 2x)$$

15. $f(x) = x \ln x$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions.

$$f'(x) = x\left(\frac{1}{x}\right) + \ln x(1) = 1 + \ln x$$

16. $y = x \ln x$

To differentiate *y*, use the **product rule** as *y* is the product of two functions.

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x(1) = 1 + \ln x$$

When $x = e$: $y = e \ln e = e(1) = e \quad \Rightarrow \quad$ point = (e, e)

$$m = \frac{dy}{dx} = 1 + \ln e = 1 + 1 = 2$$

Equation of tangent: $y - b = m(x - a) \quad \Rightarrow \quad y - e = 2(x - e)$

$$\Rightarrow \quad y - e = 2x - 2e$$

$$\Rightarrow \quad y = 2x - e$$

17. Let $f(x) = \dfrac{\sin x}{x}$.

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$f'(x) = \frac{x\cos x - \sin x(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

18.(a) $f(x) = \ln(1 + x^2)$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = \frac{1}{(1+x^2)} \times 2x = \frac{2x}{1+x^2}$$

(b) $g(x) = \dfrac{2 + \sin x}{2 + \cos x}$

To differentiate $g(x)$, use the **quotient rule** as $g(x)$ is the quotient of two functions.

$$\begin{aligned} g'(x) &= \frac{(2+\cos x)\cos x - (2+\sin x)(-\sin x)}{(2+\cos x)^2} \\ &= \frac{\cos x(2+\cos x) + \sin x(2+\sin x)}{(2+\cos x)^2} \\ &= \frac{2\cos x + \cos^2 x + 2\sin x + \sin^2 x}{(2+\cos x)^2} \\ &= \frac{2\cos x + 2\sin x + 1}{(2+\cos x)^2} \quad [\text{since } \sin^2 x + \cos^2 x = 1] \end{aligned}$$

19.(a) $f(x) = \dfrac{1+\sin x}{1+2\sin x}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$f'(x) = \frac{(1+2\sin x)\cos x - (1+\sin x)2\cos x}{(1+2\sin x)^2}$$

$$= \frac{\cos x(1+2\sin x) - 2\cos x(1+\sin x)}{(1+2\sin x)^2}$$

$$= \frac{\cos x + 2\sin x\cos x - 2\cos x - 2\sin x\cos x}{(1+2\sin x)^2}$$

$$= \frac{-\cos x}{(1+2\sin x)^2}$$

(b) $g(x) = \ln(1+e^{2x})$

To differentiate $g(x)$, use the **chain rule** as $g(x)$ is a function of a function.

$$g'(x) = \frac{1}{(1+e^{2x})} \times 2e^{2x} = \frac{2e^{2x}}{1+e^{2x}}$$

20.(a) $y = x^3 e^{-x^2}$

To differentiate *y*, use the **product rule** as *y* is the product of two functions and use the **chain rule** to differentiate e^{-x^2}.

$$\begin{aligned}\frac{dy}{dx} &= x^3(-2xe^{-x^2}) + e^{-x^2}(3x^2)\\ &= -2x^4e^{-x^2} + 3x^2e^{-x^2}\\ &= 3x^2e^{-x^2} - 2x^4e^{-x^2}\\ &= x^2e^{-x^2}(3-2x^2) \qquad \text{[removing a common factor of } x^2e^{-x^2}\text{]}\end{aligned}$$

(b) $f(x) = \dfrac{x^2}{\cos x}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$\begin{aligned}f'(x) &= \frac{\cos x(2x) - x^2(-\sin x)}{\cos^2 x}\\ &= \frac{2x\cos x + x^2\sin x}{\cos^2 x}\\ &= \frac{x(2\cos x + x\sin x)}{\cos^2 x} \qquad \text{[removing a common factor of } x \text{ in the numerator]}\end{aligned}$$

21. Let $f(x) = \dfrac{1+\ln x}{3x}$.

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$\begin{aligned}f'(x) &= \frac{3x\left(\frac{1}{x}\right) - (1+\ln x)3}{(3x)^2}\\ &= \frac{3-3(1+\ln x)}{9x^2}\\ &= \frac{3-3-3\ln x}{9x^2}\\ &= \frac{-3\ln x}{9x^2}\\ &= \frac{-\ln x}{3x^2}\end{aligned}$$

22. $y = e^{2x}\cos x$

To differentiate *y*, use the **product rule** as *y* is the product of two functions.

$$\begin{aligned}\frac{dy}{dx} &= e^{2x}(-\sin x) + \cos x(2e^{2x}) \\ &= -e^{2x}\sin x + 2e^{2x}\cos x \\ &= 2e^{2x}\cos x - e^{2x}\sin x \\ &= e^{2x}(2\cos x - \sin x) \qquad \text{[removing a common factor of } e^{2x}\text{]}\end{aligned}$$

23.(a) $f(x) = x^3 \tan 2x$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions.

$$\begin{aligned}f'(x) &= x^3(2\sec^2 2x) + \tan 2x(3x^2) \\ &= 2x^3\sec^2 2x + 3x^2\tan 2x \\ &= x^2(2x\sec^2 2x + 3\tan 2x) \qquad \text{[removing a common factor of } x^2\text{]}\end{aligned}$$

(b) $y = \dfrac{1+x^2}{1+x}$

To differentiate *y*, use the **quotient rule** as *y* is the quotient of two functions.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+x)2x - (1+x^2)1}{(1+x)^2} \\ &= \frac{2x(1+x) - (1+x^2)}{(1+x)^2} \\ &= \frac{2x + 2x^2 - 1 - x^2}{(1+x)^2} \\ &= \frac{x^2 + 2x - 1}{(1+x)^2} \qquad \text{[note that } x^2 + 2x - 1 \text{ does not factorise]}\end{aligned}$$

24. $y = e^{5x}\tan 2x$

To differentiate *y*, use the **product rule** as *y* is the product of two functions.

$$\begin{aligned}\frac{dy}{dx} &= e^{5x}(2\sec^2 2x) + \tan 2x(5e^{5x}) \\ &= 2e^{5x}\sec^2 2x + 5e^{5x}\tan 2x \\ &= e^{5x}(2\sec^2 2x + 5\tan 2x) \qquad \text{[removing a common factor of } e^{5x}\text{]}\end{aligned}$$

25.(a) $f(x) = e^x \sin(x^2)$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions and use the **chain rule** to differentiate $\sin(x^2)$.

$$\begin{aligned} f'(x) &= e^x(2x\cos(x^2)) + \sin(x^2)e^x \\ &= 2xe^x\cos(x^2) + e^x\sin(x^2) \\ &= e^x(2x\cos(x^2) + \sin(x^2)) \end{aligned}$$

[removing a common factor of e^x]

(b) $g(x) = \dfrac{x^3}{(1+\tan x)}$

To differentiate $g(x)$, use the **quotient rule** as $g(x)$ is the quotient of two functions.

$$\begin{aligned} g'(x) &= \frac{(1+\tan x)3x^2 - x^3\sec^2 x}{(1+\tan x)^2} \\ &= \frac{3x^2(1+\tan x) - x^3\sec^2 x}{(1+\tan x)^2} \\ &= \frac{3x^2 + 3x^2\tan x - x^3\sec^2 x}{(1+\tan x)^2} \\ &= \frac{x^2(3 + 3\tan x - x\sec^2 x)}{(1+\tan x)^2} \end{aligned}$$

[removing a common factor of x^2 in the numerator]

26.(a) $f(x) = e^{2x} \tan x$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions.

$$\begin{aligned} f'(x) &= e^{2x} \sec^2 x + \tan x(2e^{2x}) \\ &= e^{2x} \sec^2 x + 2e^{2x} \tan x \\ &= e^{2x}(\sec^2 x + 2\tan x) \end{aligned}$$

(b) $g(x) = \dfrac{\cos 2x}{x^3}$

To differentiate $g(x)$, use the **quotient rule** as $g(x)$ is the quotient of two functions.

$$\begin{aligned} g'(x) &= \frac{x^3(-2\sin 2x) - \cos 2x(3x^2)}{(x^3)^2} \\ &= \frac{-2x^3 \sin 2x - 3x^2 \cos 2x}{x^6} \qquad \text{[divide all terms in the fraction by } x^2 \text{ to simplify]} \\ &= \frac{-2x \sin 2x - 3\cos 2x}{x^4} \end{aligned}$$

27. $f(x) = e^{\cos x} \sin^2 x$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions and use the **chain rule** to differentiate both $e^{\cos x}$ and $\sin^2 x$.

$$\frac{d}{dx} e^{\cos x} = e^{\cos x} \times (-\sin x) = -\sin x e^{\cos x}$$

$$\frac{d}{dx} \sin^2 x = \frac{d}{dx} (\sin x)^2 = 2\sin x \cos x$$

Hence
$$\begin{aligned} f'(x) &= e^{\cos x}(2\sin x \cos x) + \sin^2 x(-\sin x e^{\cos x}) \\ &= 2e^{\cos x} \sin x \cos x - e^{\cos x} \sin^3 x \\ &= e^{\cos x} \sin x(2\cos x - \sin^2 x) \end{aligned}$$

28. $f(x) = \sin x \cos^3 x$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions and use the **chain rule** to differentiate $\cos^3 x$.

$$\frac{d}{dx}\cos^3 x = \frac{d}{dx}(\cos x)^3 = 3(\cos x)^2(-\sin x) = -3\sin x \cos^2 x$$

Hence $f'(x) = \sin x(-3\sin x \cos^2 x) + \cos^3 x \cos x$

$$= -3\sin^2 x \cos^2 x + \cos^4 x$$
$$= \cos^4 x - 3\sin^2 x \cos^2 x$$
$$= \cos^2 x(\cos^2 x - 3\sin^2 x)$$ [removing a common factor of $\cos^2 x$]

29.(a) $y = \dfrac{5x+1}{x^2+2}$

To differentiate y, use the **quotient rule** as y is the quotient of two functions.

$$\frac{dy}{dx} = \frac{(x^2+2)5 - (5x+1)2x}{(x^2+2)^2}$$
$$= \frac{5(x^2+2) - 2x(5x+1)}{(x^2+2)^2}$$
$$= \frac{5x^2 + 10 - 10x^2 - 2x}{(x^2+2)^2}$$
$$= \frac{10 - 2x - 5x^2}{(x^2+2)^2}$$ [note that $10 - 2x - 5x^2$ does not factorise]

(b) $f(x) = e^{2x}\sin^2 3x$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions and use the **chain rule** to differentiate $\sin^2 3x$.

$$\frac{d}{dx}\sin^2 3x = \frac{d}{dx}(\sin 3x)^2 = 2\sin 3x(3\cos 3x) = 6\sin 3x \cos 3x$$

Hence $f'(x) = e^{2x}(6\sin 3x \cos 3x) + \sin^2 3x(2e^{2x})$

$$= 6e^{2x}\sin 3x \cos 3x + 2e^{2x}\sin^2 3x$$
$$= 2e^{2x}\sin 3x(3\cos 3x + \sin 3x)$$ [removing a common factor of $2e^{2x}\sin 3x$]

30.(a) $f(x) = \dfrac{3x+1}{x^2+1}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$f'(x) = \frac{(x^2+1)3-(3x+1)2x}{(x^2+1)^2}$$
$$= \frac{3(x^2+1)-2x(3x+1)}{(x^2+1)^2}$$
$$= \frac{3x^2+3-6x^2-2x}{(x^2+1)^2}$$
$$= \frac{3-2x-3x^2}{(x^2+1)^2} \qquad \text{[note that } 3-2x-3x^2 \text{ does not factorise]}$$

(b) $g(x) = \cos^2 x \exp(\tan x) = \cos^2 x e^{\tan x}$

To differentiate $g(x)$, use the **product rule** as $g(x)$ is the product of two functions and use the **chain rule** to differentiate both $\cos^2 x$ and $e^{\tan x}$.

$$\frac{d}{dx}\cos^2 x = \frac{d}{dx}(\cos x)^2 = 2\cos x(-\sin x) = -2\sin x\cos x$$

$$\frac{d}{dx}e^{\tan x} = e^{\tan x} \times \sec^2 x = \sec^2 x e^{\tan x}$$

Hence $g'(x) = \cos^2 x(\sec^2 x e^{\tan x}) + e^{\tan x}(-2\sin x\cos x)$ ...(*)

$= e^{\tan x} - 2e^{\tan x}\sin x\cos x$

$= e^{\tan x}(1-2\sin x\cos x)$ ...(**)

(*) Note that $\cos^2 x\sec^2 x = \cos^2 x\left(\dfrac{1}{\cos^2 x}\right) = 1$

(**) Note that $g'(x)$ can be written as $g'(x) = e^{\tan x}(1-\sin 2x)$ since $\sin 2x = 2\sin x\cos x$.

31.(a) $f(x) = \dfrac{\ln x}{2x^2}$

To differentiate $f(x)$, use the **quotient rule** as *y* is the quotient of two functions.

$$f'(x) = \frac{2x^2\left(\frac{1}{x}\right) - \ln x(4x)}{(2x^2)^2} = \frac{2x - 4x\ln x}{4x^4}$$ [divide all terms in the fraction by $2x$ to simplify]

$$= \frac{1 - 2\ln x}{2x^3}$$

(b) $y = \operatorname{cosec}^2 3x = (\operatorname{cosec} 3x)^2$

To differentiate *y*, use the **chain rule** as *y* is a function of a function.

$$\frac{dy}{dx} = 2\operatorname{cosec} 3x \times \frac{d}{dx}\operatorname{cosec} 3x$$

$$= 2\operatorname{cosec} 3x\ (-3\operatorname{cosec} 3x\ \cot 3x)$$ [using the chain rule to differentiate cosec $3x$]

$$= -6\operatorname{cosec}^2 3x\ \cot 3x$$

Hence $\dfrac{dy}{dx} + 6y\cot 3x = -6\operatorname{cosec}^2 3x\ \cot 3x + 6\operatorname{cosec}^2 3x\ \cot 3x = 0$.

32. $y = \dfrac{\ln x}{x^3 + 1}$

To differentiate *y*, use the **quotient rule** as *y* is the quotient of two functions.

$$\frac{dy}{dx} = \frac{(x^3+1)\frac{1}{x} - \ln x(3x^2)}{(x^3+1)^2}$$

$$= \frac{\frac{(x^3+1)}{x} - 3x^2\ln x}{(x^3+1)^2}$$ [multiply all terms in the fraction by *x* to simplify]

$$= \frac{x^3 + 1 - 3x^3\ln x}{x(x^3+1)^2}$$

33.(a) $h(x) = \sin(x^2)\cos(3x)$

To differentiate $h(x)$, use the **product rule** as $h(x)$ is the product of two functions and use the **chain rul**e to differentiate both $\sin(x^2)$ and $\cos(3x)$.

$$\begin{aligned} h'(x) &= \sin(x^2)(-3\sin(3x)) + \cos(3x)(2x\cos(x^2)) \\ &= -3\sin(x^2)\sin(3x) + 2x\cos(x^2)\cos(3x) \\ &= 2x\cos(x^2)\cos(3x) - 3\sin(x^2)\sin(3x) \end{aligned}$$

(b) $y = \dfrac{\ln(x+3)}{(x+3)}$

To differentiate *y*, use the **quotient rule** as *y* is the quotient of two functions and use the chain rule to differentiate $\ln(x+3)$.

$$\frac{d}{dx}\ln(x+3) = \frac{1}{(x+3)} \times 1 = \frac{1}{x+3}$$

Hence $\dfrac{dy}{dx} = \dfrac{(x+3)\dfrac{1}{(x+3)} - \ln(x+3) \times 1}{(x+3)^2} = \dfrac{1 - \ln(x+3)}{(x+3)^2}$

34. $g(x) = e^{\cot 2x}$

To differentiate $g(x)$, use the **chain rule** as $g(x)$ is a function of a function.

$$g'(x) = e^{\cot 2x} \times (-2\,\text{cosec}^2 2x) = -2\,\text{cosec}^2 2x \times e^{\cot 2x}$$

35. $f(x) = e^{\sec^2 x}$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = e^{\sec^2 x} \times \frac{d}{dx}\sec^2 x$$

To differentiate $\sec^2 x$, use the **chain rule**: $\frac{d}{dx}\sec^2 x = \frac{d}{dx}(\sec x)^2 = 2\sec x \times \sec x \tan x$

$$= 2\sec^2 x \tan x$$

Hence $f'(x) = e^{\sec^2 x} \times 2\sec^2 x \tan x = 2\sec^2 x \tan x \times e^{\sec^2 x}$

$$f'\left(\frac{\pi}{4}\right) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} \times e^{\sec^2 \frac{\pi}{4}}$$

Now $\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{1/\sqrt{2}} = \sqrt{2} \quad \Rightarrow \quad \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$ and $\tan\frac{\pi}{4} = 1$.

Hence $f'\left(\frac{\pi}{4}\right) = 2(2)(1) \times e^2 = 4e^2$

36.(a) $f(x) = \exp\left(\tan\frac{1}{2}x\right) = e^{\tan\frac{1}{2}x}$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = e^{\tan\frac{1}{2}x} \times \frac{1}{2}\sec^2\frac{1}{2}x = \frac{1}{2}\sec^2\frac{1}{2}x \times \exp\left(\tan\frac{1}{2}x\right)$$

(b) $g(x) = (x^3+1)\ln(x^3+1)$

To differentiate $g(x)$, use the **product rule** as $g(x)$ is the product of two functions and use the **chain rule** to differentiate $\ln(x^3+1)$.

$$\frac{d}{dx}\ln(x^3+1) = \frac{1}{(x^3+1)} \times 3x^2 = \frac{3x^2}{x^3+1}$$

Hence $g'(x) = (x^3+1)\dfrac{3x^2}{(x^3+1)} + \ln(x^3+1) \times 3x^2$

$= 3x^2 + 3x^2\ln(x^3+1)$

$= 3x^2(1+\ln(x^3+1))$ [removing a common factor of $3x^2$]

37. $f(x) = \sqrt{x}e^{-x}$

To differentiate $f(x)$, use the **product rule** as $f(x)$ is the product of two functions.

Note that $\dfrac{d}{dx}\sqrt{x} = \dfrac{d}{dx}x^{\frac{1}{2}} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$.

$f'(x) = \sqrt{x}(-e^{-x}) + e^{-x}\left(\dfrac{1}{2\sqrt{x}}\right)$

$= \dfrac{e^{-x}}{2\sqrt{x}} - \sqrt{x}e^{-x}$ [the answer must now be expressed as a single fraction to simplify]

$= \dfrac{e^{-x}}{2\sqrt{x}} - \dfrac{\sqrt{x}e^{-x}}{1}$

$= \dfrac{e^{-x}}{2\sqrt{x}} - \dfrac{\sqrt{x}e^{-x} \times 2\sqrt{x}}{1 \times 2\sqrt{x}}$ [making a common denominator]

$= \dfrac{e^{-x}}{2\sqrt{x}} - \dfrac{2xe^{-x}}{2\sqrt{x}}$

$= \dfrac{e^{-x} - 2xe^{-x}}{2\sqrt{x}} = \dfrac{e^{-x}(1-2x)}{2\sqrt{x}}$ [removing a common factor of e^{-x} in the numerator]

38. $g(x) = \dfrac{\sin x}{1+\cos x}$

To differentiate $g(x)$, use the **quotient rule** as $g(x)$ is the quotient of two functions.

$$g'(x) = \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x(1+\cos x)+\sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x+1}{(1+\cos x)^2} \qquad \text{[since } \sin^2 x+\cos^2 x = 1\text{]}$$

$$= \frac{(1+\cos x)}{(1+\cos x)(1+\cos x)}$$

$$= \frac{1}{1+\cos x} \qquad \text{[cancelling the factor of } (1+\cos x) \text{ to simplify]}$$

39. $y = \ln(1+\sin x)$

To differentiate y, use the **chain rule** as y is a function of a function.

$$\frac{dy}{dx} = \frac{1}{(1+\sin x)} \times \cos x = \frac{\cos x}{1+\sin x}$$

To differentiate $\dfrac{dy}{dx}$ to find the second derivative $\dfrac{d^2y}{dx^2}$, use the **quotient rule** as $\dfrac{dy}{dx}$ is a quotient of two functions.

$$\frac{d^2y}{dx^2} = \frac{(1+\sin x)(-\sin x)-\cos x\cos x}{(1+\sin x)^2}$$

$$= \frac{-\sin x(1+\sin x)-\cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-(\sin x+\sin^2 x+\cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-(\sin x+1)}{(1+\sin x)^2} \qquad \text{[since } \sin^2 x+\cos^2 x = 1\text{]}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)(1+\sin x)} = \frac{-1}{1+\sin x} \qquad \text{[cancelling the factor of } (1+\sin x) \text{ to simplify]}$$

40. $y = \dfrac{\sin x}{2 - \cos x}$

To differentiate *y*, use the **quotient rule** as *y* is the quotient of two functions.

$$\frac{dy}{dx} = \frac{(2-\cos x)\cos x - \sin x \sin x}{(2-\cos x)^2}$$

$$= \frac{\cos x(2-\cos x) - \sin^2 x}{(2-\cos x)^2}$$

$$= \frac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2}$$

$$= \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2-\cos x)^2}$$

$$= \frac{2\cos x - 1}{(2-\cos x)^2} \qquad [\text{since } \sin^2 x + \cos^2 x = 1]$$

Stationary points occur when $\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{2\cos x - 1}{(2-\cos x)^2} = 0$

$$\Rightarrow \quad 2\cos x - 1 = 0$$

$$\Rightarrow \quad 2\cos x = 1$$

$$\Rightarrow \quad \cos x = \frac{1}{2}$$

$$\Rightarrow \quad x = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Note that $x = \dfrac{\pi}{3}$ is the only solution of the equation $\cos x = \dfrac{1}{2}$ in the interval $0 \le x \le \pi$.

When $x = \dfrac{\pi}{3}$: $\quad y = \dfrac{\sin x}{2-\cos x} = \dfrac{\sin \frac{\pi}{3}}{2 - \cos\frac{\pi}{3}} = \dfrac{\frac{\sqrt{3}}{2}}{2 - \frac{1}{2}} = \dfrac{\sqrt{3}/2}{3/2} = \dfrac{\sqrt{3}}{3}$

Hence the coordinates of the stationary point are $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}}{3}\right)$.

41.(a) $f(x) = \dfrac{x}{\ln x}$

To differentiate $f(x)$, use the **quotient rule** as $f(x)$ is the quotient of two functions.

$$f'(x) = \frac{\ln x \times 1 - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

To differentiate $f'(x)$ to find the second derivative $f''(x)$, use the **quotient rule** as $f'(x)$ is the quotient of two functions and use the **chain rule** to differentiate $(\ln x)^2$.

$$\frac{d}{dx}(\ln x)^2 = 2\ln x \times \frac{1}{x} = \frac{2\ln x}{x}$$

Hence $f''(x) = \dfrac{(\ln x)^2 \frac{1}{x} - (\ln x - 1)\frac{2\ln x}{x}}{((\ln x)^2)^2}$

$= \dfrac{\frac{(\ln x)^2}{x} - \frac{2\ln x(\ln x - 1)}{x}}{(\ln x)^4}$ [multiply all terms in the fraction by x to simplify]

$= \dfrac{(\ln x)^2 - 2\ln x(\ln x - 1)}{x(\ln x)^4}$ [now divide all terms in the fraction by $\ln x$ to simplify]

$= \dfrac{\ln x - 2(\ln x - 1)}{x(\ln x)^3}$

$= \dfrac{\ln x - 2\ln x + 2}{x(\ln x)^3}$

$= \dfrac{2 - \ln x}{x(\ln x)^3}$

(b) Stationary points occur when $f'(x) = 0 \Rightarrow \dfrac{\ln x - 1}{(\ln x)^2} = 0$

$\Rightarrow \ln x - 1 = 0$

$\Rightarrow \ln x = 1$

$\Rightarrow x = e$

$f(e) = \dfrac{e}{\ln e} = \dfrac{e}{1} = e \Rightarrow (e, e)$ is a stationary point

To determine the nature of the stationary point, use the second derivative test.

$f''(e) = \dfrac{2 - \ln e}{e(\ln e)^3} = \dfrac{2 - 1}{e(1)^3} = \dfrac{1}{e} > 0 \Rightarrow (e, e)$ is a minimum turning point

(c) Points of inflexion occur when $f''(x) = 0 \quad \Rightarrow \quad \dfrac{2 - \ln x}{x(\ln x)^3} = 0$

$$\Rightarrow \quad 2 - \ln x = 0$$

$$\Rightarrow \quad \ln x = 2$$

$$\Rightarrow \quad x = e^2$$

$$f(e^2) = \frac{e^2}{\ln e^2} = \frac{e^2}{2} \quad \Rightarrow \quad \left(e^2, \frac{e^2}{2}\right) \text{ is a point of inflexion}$$