ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Basic Integration

1. To find $\int e^{-2t}dt$, use the standard integral $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$.

$$\int e^{-2t}dt = -\frac{1}{2}e^{-2t} + C$$

2. Multiply out the brackets first before integrating.

$$\int_0^{\frac{\pi}{4}} (\sec x - x)(\sec x + x)dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - x^2)dx$$

$$= \left[\tan x - \frac{x^3}{3}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\tan\frac{\pi}{4} - \frac{\left(\pi/4\right)^3}{3}\right] - \left[\tan 0 - \frac{0^3}{3}\right]$$

$$= \left[1 - \frac{\pi^3/64}{3}\right] - [0]$$

$$= 1 - \frac{\pi^3}{192}$$

3. To find $\int \frac{5x}{x^2+3}dx$, make the numerator the exact derivative of the denominator by adjusting the constant and then integrate using $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$.

Note that $\frac{d}{dx}(x^2+3) = 2x$, so the numerator must be written as $2x$.

$$\int \frac{5x}{x^2+3}dx = \frac{5}{2}\int \frac{2x}{x^2+3}dx = \frac{5}{2}\ln|x^2+3| + C$$

<u>Note</u> Alternatively, the substitution $u = x^2 + 3$ could be used.

4. To find $\int \frac{12x^3 - 6x}{x^4 - x^2 + 1} dx$, make the numerator the exact derivative of the denominator by adjusting the constant and then integrate using $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$.

Note that $\frac{d}{dx}(x^4 - x^2 + 1) = 4x^3 - 2x$, so the numerator must be written as $(4x^3 - 2x)$.

$$\int \frac{12x^3 - 6x}{x^4 - x^2 + 1} dx = \int \frac{3(4x^3 - 2x)}{x^4 - x^2 + 1} dx = 3\int \frac{4x^3 - 2x}{x^4 - x^2 + 1} dx = 3\ln|x^4 - x^2 + 1| + C$$

<u>Note</u> Alternatively, the substitution $u = x^4 - x^2 + 1$ could be used.

5. To find $\int \frac{\sec^2 3x}{1 + \tan 3x} dx$, make the numerator the exact derivative of the denominator by adjusting the constant and then integrate using $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$.

Note that $\frac{d}{dx}(1 + \tan 3x) = 3\sec^2 3x$, so the numerator must be written as $3\sec^2 3x$.

$$\int \frac{\sec^2 3x}{1 + \tan 3x} dx = \frac{1}{3}\int \frac{3\sec^2 3x}{1 + \tan 3x} dx = \frac{1}{3}\ln|1 + \tan 3x| + C$$

<u>Note</u> Alternatively, the substitution $u = 1 + \tan 3x$ could be used.

6. Let $I = \int_0^{\frac{\pi}{3}} \cos^5 x \sin x dx$.

Using the substitution $u = \cos x$, rewrite the entire integral in terms of u (including the limits).

$u = \cos x \quad \Rightarrow \quad \frac{du}{dx} = -\sin x \quad \Rightarrow \quad du = -\sin x dx \quad \Rightarrow \quad -du = \sin x dx$

$\cos^5 x = u^5$

When $x = 0$: $u = \cos 0 = 1$ When $x = \frac{\pi}{3}$: $u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$I = \int_0^{\frac{\pi}{3}} \cos^5 x \sin x dx = \int_1^{\frac{1}{2}} u^5(-du) = \int_1^{\frac{1}{2}} -u^5 du = \left[-\frac{u^6}{6}\right]_1^{\frac{1}{2}}$$

$$= \left[-\frac{\left(\frac{1}{2}\right)^6}{6}\right] - \left[-\frac{1^6}{6}\right]$$

$$= \left[-\frac{\frac{1}{64}}{6}\right] - \left[-\frac{1}{6}\right]$$

$$= -\frac{1}{384} + \frac{1}{6}$$

$$= \frac{21}{128}$$

7. Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sin^4\theta\cos\theta d\theta$.

Using the substitution $u = \sin\theta$, rewrite the entire integral in terms of u (including the limits).

$u = \sin\theta \quad \Rightarrow \quad \frac{du}{d\theta} = \cos\theta \quad \Rightarrow \quad du = \cos\theta d\theta$

$\sin^4\theta = u^4$

When $\theta = \frac{\pi}{6}$: $\quad u = \sin\frac{\pi}{6} = \frac{1}{2}$ When $\theta = \frac{\pi}{2}$: $\quad u = \sin\frac{\pi}{2} = 1$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sin^4\theta\cos\theta d\theta = \int_{\frac{1}{2}}^{1} 2u^4 du = \left[\frac{2u^5}{5}\right]_{\frac{1}{2}}^{1} = \left[\frac{2(1)^5}{5}\right] - \left[\frac{2\left(\frac{1}{2}\right)^5}{5}\right]$$

$$= \left[\frac{2(1)}{5}\right] - \left[\frac{2\left(\frac{1}{32}\right)}{5}\right]$$

$$= \frac{2}{5} - \frac{1}{80}$$

$$= \frac{31}{80}$$

8. Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)^3}\,d\theta$.

Using the substitution $x = 1+\sin\theta$, rewrite the entire integral in terms of *x* (including the limits).

$$x = 1+\sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = \cos\theta \quad \Rightarrow \quad dx = \cos\theta\, d\theta$$

$$(1+\sin\theta)^3 = x^3$$

When $\theta = 0$: $\quad x = 1+\sin 0 = 1+0 = 1$

When $\theta = \frac{\pi}{2}$: $\quad x = 1+\sin\frac{\pi}{2} = 1+1 = 2$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{(1+\sin\theta)^3}(\cos\theta d\theta) = \int_1^2 \frac{1}{x^3}\,dx = \int_1^2 x^{-3}dx = \left[\frac{x^{-2}}{-2}\right]_1^2 = \left[-\frac{1}{2x^2}\right]_1^2$$

$$= \left[-\frac{1}{2(2)^2}\right] - \left[-\frac{1}{2(1)^2}\right]$$

$$= -\frac{1}{8} + \frac{1}{2}$$

$$= \frac{3}{8}$$

9. Let $I = \int_0^3 x\sqrt{x+1}\,dx$.

Using the substitution $u = x + 1$, rewrite the entire integral in terms of u (including the limits).

$$u = x + 1 \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$$

$$\sqrt{x+1} = \sqrt{u} \qquad \text{and} \qquad u = x + 1 \quad \Rightarrow \quad x = u - 1$$

When $x = 0$: $u = 0 + 1 = 1$ When $x = 3$: $u = 3 + 1 = 4$

$$I = \int_0^3 x\sqrt{x+1}\,dx = \int_1^4 (u-1)\sqrt{u}\,du = \int_1^4 u^{\frac{1}{2}}(u-1)\,du = \int_1^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) du$$

$$= \left[\frac{u^{\frac{5}{2}}}{5/2} - \frac{u^{\frac{3}{2}}}{3/2} \right]_1^4$$

$$= \left[\frac{2\sqrt{u^5}}{5} - \frac{2\sqrt{u^3}}{3} \right]_1^4$$

$$= \left[\frac{2\sqrt{4^5}}{5} - \frac{2\sqrt{4^3}}{3} \right] - \left[\frac{2\sqrt{1^5}}{5} - \frac{2\sqrt{1^3}}{3} \right]$$

$$= \left[\frac{2(32)}{5} - \frac{2(8)}{3} \right] - \left[\frac{2(1)}{5} - \frac{2(1)}{3} \right]$$

$$= \left[\frac{64}{5} - \frac{16}{3} \right] - \left[\frac{2}{5} - \frac{2}{3} \right]$$

$$= \frac{116}{15}$$

10. Let $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$.

Using the substitution $u = 1 + x$, rewrite the entire integral in terms of u (including the limits).

$$u = 1 + x \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$$

$$\sqrt{1+x} = \sqrt{u} \qquad \text{and} \qquad u = 1 + x \quad \Rightarrow \quad x = u - 1$$

When $x = 0$: $u = 1 + 0 = 1$ When $x = 3$: $u = 1 + 3 = 4$

$$\int_0^3 \frac{x}{\sqrt{1+x}}\,dx = \int_1^4 \frac{u-1}{\sqrt{u}}\,du = \int_1^4 u^{-\frac{1}{2}}(u-1)\,du = \int_1^4 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$$

$$= \left[\frac{u^{\frac{3}{2}}}{3/2} - \frac{u^{\frac{1}{2}}}{1/2}\right]_1^4$$

$$= \left[\frac{2\sqrt{u^3}}{3} - 2\sqrt{u}\right]_1^4$$

$$= \left[\frac{2\sqrt{4^3}}{3} - 2\sqrt{4}\right] - \left[\frac{2\sqrt{1^3}}{3} - 2\sqrt{1}\right]$$

$$= \left[\frac{2(8)}{3} - 2(2)\right] - \left[\frac{2(1)}{3} - 2(1)\right]$$

$$= \left[\frac{16}{3} - 4\right] - \left[\frac{2}{3} - 2\right]$$

$$= \frac{8}{3}$$

11. Let $I = \int_0^8 \frac{t+2}{\sqrt{t+1}}\,dt$.

Using the substitution $u = t+1$, rewrite the entire integral in terms of u (including the limits).

$$u = t+1 \quad \Rightarrow \quad \frac{du}{dt} = 1 \quad \Rightarrow \quad du = dt$$

$$\sqrt{t+1} = \sqrt{u} \quad \text{and} \quad u = t+1 \quad \Rightarrow \quad t+2 = u+1$$

When $t = 0$: $u = 0+1 = 1$ When $t = 8$: $u = 8+1 = 9$

$$I = \int_0^8 \frac{t+2}{\sqrt{t+1}}\,dt = \int_1^9 \frac{u+1}{\sqrt{u}}\,du = \int_1^9 u^{-\frac{1}{2}}(u+1)\,du = \int_1^9 \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}} \right) du$$

$$= \left[\frac{u^{\frac{3}{2}}}{3/2} + \frac{u^{\frac{1}{2}}}{1/2} \right]_1^9$$

$$= \left[\frac{2\sqrt{u^3}}{3} + 2\sqrt{u} \right]_1^9$$

$$= \left[\frac{2\sqrt{9^3}}{3} + 2\sqrt{9} \right] - \left[\frac{2\sqrt{1^3}}{3} + 2\sqrt{1} \right]$$

$$= \left[\frac{2(27)}{3} + 2(3) \right] - \left[\frac{2(1)}{3} + 2(1) \right]$$

$$= \left[\frac{54}{3} + 6 \right] - \left[\frac{2}{3} + 2 \right]$$

$$= \frac{64}{3}$$

12. Let $I = \int \frac{2}{x \ln x}\,dx$.

Using the substitution $u = \ln x$, rewrite the entire integral in terms of u.

$$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \quad \Rightarrow \quad du = \frac{1}{x}\,dx$$

$$I = \int \frac{2}{x \ln x}\,dx = \int \frac{2}{\ln x}\left(\frac{1}{x}\,dx \right) = \int \frac{2}{u}\,du = 2\ln|u| + C = 2\ln|\ln x| + C$$

13. Let $I = \int \frac{x^3}{\sqrt{1+x^2}}dx$.

Using the substitution $u = 1+x^2$, rewrite the entire integral in terms of u.

$$u = 1+x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2xdx \quad \Rightarrow \quad \frac{1}{2}du = xdx$$

$$\sqrt{1+x^2} = \sqrt{u} \quad \text{and} \quad u = 1+x^2 \quad \Rightarrow \quad x^2 = u-1$$

$$\begin{aligned}
I = \int \frac{x^3}{\sqrt{1+x^2}}dx = \int \frac{x^2}{\sqrt{1+x^2}}(xdx) = \int \frac{u-1}{\sqrt{u}}\left(\frac{1}{2}du\right) &= \int \frac{1}{2}u^{-\frac{1}{2}}(u-1)du \\
&= \int \left(\frac{1}{2}u^{\frac{1}{2}} - \frac{1}{2}u^{-\frac{1}{2}}\right)du \\
&= \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{3/2}\right) - \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{1/2}\right) + C \\
&= \frac{1}{2}\left(\frac{2\sqrt{u^3}}{3}\right) - \frac{1}{2}(2\sqrt{u}) + C \\
&= \frac{\sqrt{u^3}}{3} - \sqrt{u} + C \\
&= \frac{\sqrt{(1+x^2)^3}}{3} - \sqrt{1+x^2} + C
\end{aligned}$$

14. Let $I = \int_0^{\sqrt{5}} \frac{2x^3}{\sqrt{x^2+4}}\,dx$.

Using the substitution $u = x^2 + 4$, rewrite the entire integral in terms of u (including the limits).

$$u = x^2 + 4 \quad \Rightarrow \quad \frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2x\,dx$$

$$\sqrt{x^2+4} = \sqrt{u} \qquad \text{and} \qquad u = x^2 + 4 \quad \Rightarrow \quad x^2 = u - 4$$

When $x = 0$: $u = 0^2 + 4 = 4$ When $x = \sqrt{5}$: $u = (\sqrt{5})^2 + 4 = 9$

$$\begin{aligned}
I = \int_0^{\sqrt{5}} \frac{2x^3}{\sqrt{x^2+4}}\,dx = \int_0^{\sqrt{5}} \frac{x^2}{\sqrt{x^2+4}}(2x\,dx) &= \int_4^9 \frac{u-4}{\sqrt{u}}\,du \\
&= \int_4^9 u^{-\frac{1}{2}}(u-4)\,du \\
&= \int_4^9 \left(u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right)du \\
&= \left[\frac{u^{\frac{3}{2}}}{3/2} - \frac{4u^{\frac{1}{2}}}{1/2}\right]_4^9 \\
&= \left[\frac{2\sqrt{u^3}}{3} - 8\sqrt{u}\right]_4^9 \\
&= \left[\frac{2\sqrt{9^3}}{3} - 8\sqrt{9}\right] - \left[\frac{2\sqrt{4^3}}{3} - 8\sqrt{4}\right] \\
&= \left[\frac{2(27)}{3} - 8(3)\right] - \left[\frac{2(8)}{3} - 8(2)\right] \\
&= \left[\frac{54}{3} - 24\right] - \left[\frac{16}{3} - 16\right] \\
&= \frac{14}{3}
\end{aligned}$$

15. Let $I = \int_3^4 x^3(x^2-8)^{\frac{1}{3}}\,dx$.

Using the substitution $u = x^2 - 8$, rewrite the entire integral in terms of u (including the limits).

$$u = x^2 - 8 \quad \Rightarrow \quad \frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx$$

$$(x^2-8)^{\frac{1}{3}} = u^{\frac{1}{3}} \quad \text{and} \quad u = x^2 - 8 \quad \Rightarrow \quad x^2 = u + 8$$

When $x = 3$: $u = 3^2 - 8 = 1$ When $x = 4$: $u = 4^2 - 8 = 8$

$$I = \int_3^4 x^3(x^2-8)^{\frac{1}{3}}\,dx = \int_3^4 x^2(x^2-8)^{\frac{1}{3}}\,x\,dx = \int_1^8 (u+8)u^{\frac{1}{3}}\left(\frac{1}{2}du\right)$$

$$= \int_1^8 \frac{1}{2}u^{\frac{1}{3}}(u+8)\,du$$

$$= \int_1^8 \left(\frac{1}{2}u^{\frac{4}{3}} + 4u^{\frac{1}{3}}\right)du$$

$$= \left[\frac{1}{2}\left(\frac{u^{\frac{7}{3}}}{7/3}\right) + 4\left(\frac{u^{\frac{4}{3}}}{4/3}\right)\right]_1^8$$

$$= \left[\frac{1}{2}\left(\frac{3\times\sqrt[3]{u^7}}{7}\right) + 4\left(\frac{3\times\sqrt[3]{u^4}}{4}\right)\right]_1^8$$

$$= \left[\frac{3\times\sqrt[3]{u^7}}{14} + 3\times\sqrt[3]{u^4}\right]_1^8$$

$$= \left[\frac{3\times\sqrt[3]{8^7}}{14} + 3\times\sqrt[3]{8^4}\right] - \left[\frac{3\times\sqrt[3]{1^7}}{14} + 3\times\sqrt[3]{1^4}\right]$$

$$= \left[\frac{3(128)}{14} + 3(16)\right] - \left[\frac{3(1)}{14} + 3(1)\right]$$

$$= \left[\frac{384}{14} + 48\right] - \left[\frac{3}{14} + 3\right]$$

$$= \frac{1011}{14}$$

16. Let $I = \int \frac{1}{(1+\sqrt{x})^3}\, dx$.

Using the substitution $x = (u-1)^2$, rewrite the entire integral in terms of u.

$$x = (u-1)^2 \quad \Rightarrow \quad \frac{dx}{du} = 2(u-1)\times 1 \quad \Rightarrow \quad dx = 2(u-1)du$$

$$x = (u-1)^2 \quad \Rightarrow \quad \sqrt{x} = u-1 \quad \Rightarrow \quad 1+\sqrt{x} = u \quad \Rightarrow \quad (1+\sqrt{x})^3 = u^3$$

$$I = \int \frac{1}{(1+\sqrt{x})^3}\, dx = \int \frac{1}{u^3} \times 2(u-1)du = \int \frac{2(u-1)}{u^3} du = \int 2u^{-3}(u-1)du$$

$$= \int (2u^{-2} - 2u^{-3})du$$

$$= \frac{2u^{-1}}{-1} - \frac{2u^{-2}}{(-2)} + C$$

$$= -2u^{-1} + u^{-2} + C$$

$$= \frac{-2}{u} + \frac{1}{u^2} + C$$

$$= \frac{-2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C$$

17. Let $I = \int_0^2 \frac{x+1}{\sqrt{16-x^2}}\,dx$.

Using the substitution $x = 4\sin\theta$, rewrite the entire integral in terms of θ (including the limits).

$$x = 4\sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = 4\cos\theta \quad \Rightarrow \quad dx = 4\cos\theta d\theta$$

$x+1 = 4\sin\theta + 1$ and

$$\begin{aligned}\sqrt{16-x^2} &= \sqrt{16-(4\sin\theta)^2} \\ &= \sqrt{16-16\sin^2\theta} \\ &= \sqrt{16(1-\sin^2\theta)} \\ &= \sqrt{16\cos^2\theta} \qquad [\text{since } \cos^2\theta = 1-\sin^2\theta] \\ &= 4\cos\theta\end{aligned}$$

When $x = 0$: $\quad 0 = 4\sin\theta \quad \Rightarrow \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = \sin^{-1}0 = 0$

When $x = 2$: $\quad 2 = 4\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$\begin{aligned}I = \int_0^2 \frac{x+1}{\sqrt{16-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{(4\sin\theta+1)}{4\cos\theta} \times 4\cos\theta d\theta &= \int_0^{\frac{\pi}{6}} (4\sin\theta+1)d\theta \\ &= \left[-4\cos\theta + \theta\right]_0^{\frac{\pi}{6}} \\ &= \left[-4\cos\frac{\pi}{6} + \frac{\pi}{6}\right] - \left[-4\cos 0 + 0\right] \\ &= \left[-4\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right] - \left[-4(1)\right] \\ &= -2\sqrt{3} + \frac{\pi}{6} + 4\end{aligned}$$

<u>**Notes**</u>

(1) $\sqrt{16-x^2}$ must be simplified to $4\cos\theta$.

(2) The new limits must be expressed in radians.

18. Let $I = \int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{2x - x^2}}$.

Using the substitution $x = 1 - \sin\theta$, rewrite the entire integral in terms of θ (including the limits).

$$x = 1 - \sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = -\cos\theta \quad \Rightarrow \quad dx = -\cos\theta d\theta$$

$$\begin{aligned}
\sqrt{2x - x^2} &= \sqrt{2(1-\sin\theta) - (1-\sin\theta)^2} \\
&= \sqrt{2 - 2\sin\theta - (1 - 2\sin\theta + \sin^2\theta)} \\
&= \sqrt{2 - 2\sin\theta - 1 + 2\sin\theta - \sin^2\theta} \\
&= \sqrt{1 - \sin^2\theta} \\
&= \sqrt{\cos^2\theta} \qquad [\text{since } \cos^2\theta = 1 - \sin^2\theta] \\
&= \cos\theta
\end{aligned}$$

When $x = \frac{1}{2}$: $\frac{1}{2} = 1 - \sin\theta \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

When $x = 1$: $1 = 1 - \sin\theta \quad \Rightarrow \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = \sin^{-1} 0 = 0$

$$I = \int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{2x - x^2}} = \int_{\frac{\pi}{6}}^{0} \frac{-\cos\theta d\theta}{\cos\theta} = \int_{\frac{\pi}{6}}^{0} -1 d\theta = [-\theta]_{\frac{\pi}{6}}^{0} = [0] - \left[-\frac{\pi}{6}\right] = \frac{\pi}{6}$$

Notes

(1) $\sqrt{2x - x^2}$ must be simplified to $\cos\theta$.

(2) The new limits must be expressed in radians.

19. Let $I = \int_0^2 \sqrt{16 - x^2}\, dx$.

Using the substitution $x = 4\sin\theta$, rewrite the entire integral in terms of θ (including the limits).

$$x = 4\sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = 4\cos\theta \quad \Rightarrow \quad dx = 4\cos\theta\, d\theta$$

$$\sqrt{16 - x^2} = \sqrt{16 - (4\sin\theta)^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16(1 - \sin^2\theta)} = \sqrt{16\cos^2\theta} = 4\cos\theta$$

↑

since $\cos^2\theta = 1 - \sin^2\theta$

When $x = 0$: $\quad 0 = 4\sin\theta \quad \Rightarrow \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = \sin^{-1} 0 = 0$

When $x = 2$: $\quad 2 = 4\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$I = \int_0^2 \sqrt{16 - x^2}\, dx = \int_0^{\frac{\pi}{6}} 4\cos\theta \times 4\cos\theta\, d\theta = \int_0^{\frac{\pi}{6}} 16\cos^2\theta\, d\theta$$

To integrate $16\cos^2\theta$, use the trig identity given, ie $\cos 2A = 2\cos^2 A - 1$.

$$\cos 2\theta = 2\cos^2\theta - 1 \quad \Rightarrow \quad 2\cos^2\theta = \cos 2\theta + 1 \quad \Rightarrow \quad 16\cos^2\theta = 8\cos 2\theta + 8$$

Hence $\quad I = \int_0^{\frac{\pi}{6}} 4\cos^2\theta\, d\theta = \int_0^{\frac{\pi}{6}} (8\cos 2\theta + 8)\, d\theta = \left[8\left(\frac{1}{2}\sin 2\theta\right) + 8\theta\right]_0^{\frac{\pi}{6}}$

$$= \left[4\sin 2\theta + 8\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left[4\sin 2\left(\frac{\pi}{6}\right) + 8\left(\frac{\pi}{6}\right)\right] - \left[4\sin(2(0)) + 8(0)\right]$$

$$= \left[4\sin\frac{\pi}{3} + \frac{4\pi}{3}\right] - \left[4\sin 0 + 0\right]$$

$$= \left[4\left(\frac{\sqrt{3}}{2}\right) + \frac{4\pi}{3}\right] - \left[4(0)\right]$$

$$= 2\sqrt{3} + \frac{4\pi}{3}$$

Notes

(1) $\sqrt{16 - x^2}$ must be simplified to $4\cos\theta$.

(2) The new limits must be expressed in radians.

20. Let $I = \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$.

Using the substitution $x = 2\sin\theta$, rewrite the entire integral in terms of θ (including the limits).

$$x = 2\sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = 2\cos\theta \quad \Rightarrow \quad dx = 2\cos\theta d\theta$$

$$x^2 = (2\sin\theta)^2 = 4\sin^2\theta \qquad \text{and} \qquad \sqrt{4-x^2} = \sqrt{4-4\sin^2\theta}$$

$$= \sqrt{4(1-\sin^2\theta)}$$

$$= \sqrt{4\cos^2\theta} \qquad [\text{since } \cos^2\theta = 1-\sin^2\theta]$$

$$= 2\cos\theta$$

When $x = 0$: $\quad 0 = 2\sin\theta \quad \Rightarrow \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = \sin^{-1} 0 = 0$

When $x = \sqrt{2}$: $\quad \sqrt{2} = 2\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$$I = \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{4\sin^2\theta}{2\cos\theta} \times 2\cos\theta d\theta = \int_0^{\frac{\pi}{4}} 4\sin^2\theta d\theta$$

To integrate $4\sin^2\theta$, use the trig identity given, ie $\cos 2A = 1 - 2\sin^2 A$.

$$\cos 2\theta = 1 - 2\sin^2\theta \quad \Rightarrow \quad 2\sin^2\theta = 1 - \cos 2\theta \quad \Rightarrow \quad 4\sin^2\theta = 2 - 2\cos 2\theta$$

Hence $\quad I = \int_0^{\frac{\pi}{4}} 4\sin^2\theta d\theta = \int_0^{\frac{\pi}{4}} (2 - 2\cos 2\theta) d\theta = \left[2\theta - 2\left(\frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{\pi}{4}}$

$$= [2\theta - \sin 2\theta]_0^{\frac{\pi}{4}}$$

$$= \left[2\left(\frac{\pi}{4}\right) - \sin 2\left(\frac{\pi}{4}\right)\right] - [2(0) - \sin 2(0)]$$

$$= \left[\frac{\pi}{2} - \sin\frac{\pi}{2}\right] - [0 - \sin 0]$$

$$= \left[\frac{\pi}{2} - 1\right] - [0]$$

$$= \frac{\pi}{2} - 1$$

Notes

(1) $\sqrt{4-x^2}$ must be simplified to $2\cos\theta$.

(2) The new limits must be expressed in radians.

21.(a) **Method 1**

since $\sin^2 x + \cos^2 x = 1$ (↓)

$$\text{LHS} = 1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x = \text{RHS}$$

(↑) since $\tan x = \dfrac{\sin x}{\cos x}$

Method 2

Start with the trig identity $\sin^2 x + \cos^2 x = 1$ and divide all terms in this identity by $\cos^2 x$.

$$\sin^2 x + \cos^2 x = 1 \quad \Rightarrow \quad \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\Rightarrow \quad \tan^2 x + 1 = \sec^2 x \qquad \left[\text{since } \frac{\sin x}{\cos x} = \tan x \text{ and } \frac{1}{\cos x} = \sec x\right]$$

$$\Rightarrow \quad 1 + \tan^2 x = \sec^2 x$$

(b) To find $\int \tan^2 x dx$, rearrange the identity in (a) to give an identity for $\tan^2 x$.

$$1 + \tan^2 x = \sec^2 x \quad \Rightarrow \quad \tan^2 x = \sec^2 x - 1$$

Hence $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$

22.(a) **Method 1**

$$\text{LHS} = 1+\tan^2 x = 1+\frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x = \text{RHS}$$

(since $\tan x = \frac{\sin x}{\cos x}$; since $\sin^2 x + \cos^2 x = 1$)

Method 2

Start with the trig identity $\sin^2 x + \cos^2 x = 1$ and divide all terms in this identity by $\cos^2 x$.

$$\sin^2 x + \cos^2 x = 1 \quad \Rightarrow \quad \frac{\sin^2 x}{\cos^2 x}+\frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\Rightarrow \quad \tan^2 x + 1 = \sec^2 x \qquad \left[\text{since } \frac{\sin x}{\cos x} = \tan x \text{ and } \frac{1}{\cos x} = \sec x\right]$$

$$\Rightarrow \quad 1+\tan^2 x = \sec^2 x$$

(b) Let $I = \int_0^1 \frac{dx}{(1+x^2)^{\frac{3}{2}}}$.

Using the substitution $x = \tan\theta$, rewrite the entire integral in terms of θ (including the limits).

$$x = \tan\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = \sec^2\theta \quad \Rightarrow \quad dx = \sec^2\theta d\theta$$

$$(1+x^2)^{\frac{3}{2}} = (1+\tan^2\theta)^{\frac{3}{2}} = (\sec^2\theta)^{\frac{3}{2}} = \sec^3\theta \qquad \text{[multiplying the indices]}$$

When $x = 0$: $\quad 0 = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1} 0 = 0$

When $x = 1$: $\quad 1 = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$$I = \int_0^1 \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{\sec^3\theta} = \int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta} d\theta = \int_0^{\frac{\pi}{4}} \cos\theta d\theta = [\sin\theta]_0^{\frac{\pi}{4}} = \left[\sin\frac{\pi}{4}\right] - [\sin 0]$$

(since $\sec\theta = \frac{1}{\cos\theta}$)

$$= \left[\frac{1}{\sqrt{2}}\right] - [0]$$

$$= \frac{1}{\sqrt{2}}$$

Note The new limits must be expressed in radians.