

Exam Questions on Differential Equations 1

Solve each first order differential equation by separating the variables.

1. Given that

$$x^2 e^y \frac{dy}{dx} = 1,$$

and $y = 0$ when $x = 1$, find y in terms of x .

2. A mathematical biologist believes that the differential equation $y \frac{dy}{dx} - 3x = x^4$ models a process.

Given that $y = 2$ when $x = 1$, obtain y in terms of x .

3. Solve the differential equation

$$\cos^2 x \frac{dy}{dx} = y,$$

given that $y > 0$ and that $y = 2$ when $x = 0$.

4. Given that $y > -1$ and $x > -1$, obtain the general solution of the differential equation

$$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$$

expressing your answer in the form $y = f(x)$.

5. Solve $\frac{dy}{dx} = e^{2x}(1+y^2)$ given that when $x = 0$, $y = 1$.

Express y in terms of x .

6. A chemical food loses effectiveness at a rate proportional to the amount present in the soil. The amount M grams of plant food effective after t days satisfies the differential equation

$$\frac{dM}{dt} = kM, \quad \text{where } k \text{ is a constant.}$$

- (a) Find the general solution for M in terms of t where the initial amount of plant food is 100 grams.
- (b) Find the value of k if, after 30 days, only half the initial amount of plant food is effective.
- (c) What percentage of the original amount of plant food is effective after 35 days?
- (d) The plant food has to be renewed when its effectiveness falls below 25%. Is the manufacturer justified in calling its product “sixty day super food”?
7. The number of strands of bacteria, $B(t)$, present in a culture after t days of growth is assumed to be increasing at a rate proportional to the number of strands present.

This can be represented by the differential equation

$$\frac{dB}{dt} = kB, \quad \text{where } k \text{ is a constant.}$$

Solve the differential equation to find an expression for B in terms of t .

Given that the number of strands observed after 1 day is 502 and after 4 days is 1833, find the number of strands initially present.

8. In a town with population 40 000, a ‘flu virus spread rapidly last winter. The **percentage** P of the population infected t days after the initial outbreak satisfies the differential equation

$$\frac{dP}{dt} = kP, \quad \text{where } k \text{ is a constant.}$$

- (a) If 100 people are infected initially, find, in terms of k , the percentage infected t days later.
- (b) Given that 500 people have ‘flu after 7 days, how many more are likely to have contracted the virus after 10 days?

9. Newton's law of cooling states that a body loses heat at a rate which is proportional to the difference in temperature between itself and its surroundings. So, in a room with constant temperature 22°C , the temperature $T^{\circ}\text{C}$ of a body after a time t minutes satisfies

$$\frac{dT}{dt} = k(T - 22)$$

where k is a negative constant.

- (a) Hence show that T can be expressed in the form $T = Ae^{kt} + 22$ for some arbitrary constant A .
- (b) In a restaurant where the temperature remains constant at 22°C , a freshly baked roll, with temperature 82°C , is placed on a cooling tray. After 5 minutes, the temperature of the roll has fallen by 20 degrees.

Calculate the values of A and k .

Write down an expression for the temperature of the roll after t minutes.

Supposing the roll remains uneaten after a further 5 minutes, what will its temperature be?

10. A turkey is taken from a refrigerator to be cooked. Its temperature is 4°C when it is placed in an oven preheated to 180°C .

Its temperature, $T^{\circ}\text{C}$, after a time of x hours in the oven satisfies the equation

$$\frac{dT}{dx} = k(180 - T).$$

- (a) Show that $T = 180 - 176e^{-kx}$.
- (b) After an hour in the oven the temperature of the turkey is 30°C . Calculate the value of k correct to 2 decimal places.
- (c) The turkey will be cooked when it reaches a temperature of 80°C . After how long will the turkey be cooked?

- 11.** When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water.

This can be represented by the differential equation

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{10}, \quad h \geq 0,$$

where h is the depth (in metres) of the water and t is the time (in minutes) elapsed since the valve was opened.

- (a) Express h as a function of t .
- (b) Find the solution of the equation, given that the pool was initially 4 m deep.
- (c) The next time the pool had to be drained, the water was initially 9 m deep. How long did it take to drain the pool on this occasion?
- 12.** A tank initially holds 20 litres of pure water. A solution of water containing salt flows into the tank at a uniform rate. The contents of the tank are stirred continually to maintain a uniform concentration and liquid flows out at the same rate. At time t minutes, the water in the tank contains x kg of salt.

At any time $t > 0$, the amount of salt, x kg, in the tank can be modelled by the differential equation

$$\frac{dx}{dt} = \frac{2-x}{5}.$$

- (a) Find a formula for x in terms of t .
- (b) How much salt is present after 20 minutes?
- (c) In the long term, what will be the amount of salt in the tank?

13. The volume $V(t)$ of a cell at time t changes according to the law

$$\frac{dV}{dt} = V(10 - V) \quad \text{for } 0 < V < 10.$$

Show that

$$\frac{1}{10} \ln V - \frac{1}{10} \ln(10 - V) = t + C$$

for some constant C .

Given that $V(0) = 5$, show that $V(t) = \frac{10e^{10t}}{1 + e^{10t}}$.

Obtain the limiting value of $V(t)$ as $t \rightarrow \infty$.

14. In a chemical reaction, two substances X and Y combine to form a third substance Z . Let $Q(t)$ denote the number of grams of Z formed t minutes after the reaction begins. The rate at which $Q(t)$ varies is governed by the differential equation

$$\frac{dQ}{dt} = \frac{(30 - Q)(15 - Q)}{900}.$$

- (a) Express $\frac{900}{(30 - Q)(15 - Q)}$ in partial fractions.
- (b) Use your answer to (a) to show that the general solution of the differential equation can be written in the form

$$A \ln \left(\frac{30 - Q}{15 - Q} \right) = t + C,$$

where A and C are constants.

State the value of A and, given that $Q(0) = 0$, find the value of C .

Find, correct to two decimal places:

- (i) the time taken to form 5 grams of Z ;
- (ii) the number of grams of Z formed 45 minutes after the reaction begins.

15. A flu-like virus starts to spread through the 20 000 inhabitants of Dumbarton. The situation can be modelled by the differential equation

$$\frac{dN}{dt} = \frac{N(20000 - N)}{10000},$$

where N is the number of people infected after t days and $0 < N < 20000$.

- (a) Express $\frac{10000}{N(20000 - N)}$ in partial fractions and show that

$$\ln\left(\frac{N}{20000 - N}\right) = 2t + C, \text{ for some constant } C.$$

- (b) Initially there were 100 people infected.

$$\text{Show that } N = \frac{20000e^{2t}}{199 + e^{2t}}.$$

16. In an environment without enough resources to support a population greater than 1000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{dP}{dt} = P(1000 - P).$$

Show that

$$\ln\left(\frac{P}{1000 - P}\right) = 1000t + C \text{ for some constant } C.$$

Hence show that

$$P(t) = \frac{1000K}{K + e^{-1000t}} \text{ for some constant } K.$$

Given that $P(0) = 200$, determine at what time t , $P(t) = 900$.

17. A beaker of liquid was placed in a fridge.
The rate of cooling is given by

$$\frac{dT}{dt} = -k(T - T_F), \quad k > 0,$$

where T_F is the constant temperature in the fridge and T is the temperature of the liquid at time t .

- The constant temperature of the fridge is 4°C .
- When first placed in the fridge, the temperature of the liquid was 25°C .
- At 12 noon, the temperature of the liquid was 9.8°C .
- At 12:15 pm, the temperature of the liquid had dropped to 6.5°C .

At what time, to the nearest minute, was the liquid placed in the fridge?