Solutions to Exam Questions on Differential Equations 1

1. The differential equation can be solved by separating the variables and then integrating.

$$x^2 e^y \frac{dy}{dx} = 1 \quad \Rightarrow \quad x^2 e^y dy = dx \quad \Rightarrow \quad \int e^y dy = \int \frac{1}{x^2} dx$$

$$\Rightarrow \quad \int e^y dy = \int x^{-2} dx$$

$$\Rightarrow \quad e^y = \frac{x^{-1}}{-1} + C$$

$$\Rightarrow \quad e^y = -\frac{1}{x} + C$$

$$\Rightarrow \quad e^y = C - \frac{1}{x}$$

$y = 0$ when $x = 1 \quad \Rightarrow \quad e^0 = C - \frac{1}{1} \quad \Rightarrow \quad 1 = C - 1 \quad \Rightarrow \quad C = 2$

Hence $\quad e^y = 2 - \frac{1}{x} \quad \Rightarrow \quad y = \ln\left(2 - \frac{1}{x}\right)$

Note

The solution does not have to be written in the form $y = f(x)$ before substituting the values of x and y to find C. You can substitute the values of x and y at any time after integrating.

2. The differential equation can be solved by separating the variables and then integrating.

$$y\frac{dy}{dx} - 3x = x^4 \quad \Rightarrow \quad y\frac{dy}{dx} = x^4 + 3x \quad \Rightarrow \quad \int y\,dy = \int (x^4 + 3x)\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = \frac{x^5}{5} + \frac{3x^2}{2} + C$$

$$y = 2 \text{ when } x = 1 \quad \Rightarrow \quad \frac{2^2}{2} = \frac{1^5}{5} + \frac{3(1)^2}{2} + C$$

$$\Rightarrow \quad 2 = \frac{1}{5} + \frac{3}{2} + C$$

$$\Rightarrow \quad 2 = \frac{17}{10} + C$$

$$\Rightarrow \quad C = \frac{3}{10}$$

Hence $\frac{y^2}{2} = \frac{x^5}{5} + \frac{3x^2}{2} + \frac{3}{10} \quad \Rightarrow \quad y^2 = 2\left(\frac{x^5}{5} + \frac{3x^2}{2} + \frac{3}{10}\right)$

$$\Rightarrow \quad y^2 = \frac{2x^5}{5} + 3x^2 + \frac{3}{5}$$

$$\Rightarrow \quad y = \pm\sqrt{\frac{2x^5}{5} + 3x^2 + \frac{3}{5}}$$

Note that when $x = 1$, $y > 0$, hence $y = \sqrt{\frac{2x^5}{5} + 3x^2 + \frac{3}{5}}$.

<u>Note</u>

The solution does not have to be written in the form $y = f(x)$ before substituting the values of *x* and *y* to find *C*. You can substitute the values of *x* and *y* at any time after integrating.

3. The differential equation can be solved by separating the variables and then integrating.

$$\cos^2 x\frac{dy}{dx} = y \quad \Rightarrow \quad \cos^2 x dy = y dx \quad \Rightarrow \quad \int \frac{1}{y}dy = \int \frac{1}{\cos^2 x}dx$$

$$\Rightarrow \quad \int \frac{1}{y}dy = \int \sec^2 x dx \qquad \left[\text{since } \sec x = \frac{1}{\cos x}\right]$$

$$\Rightarrow \quad \ln y = \tan x + C$$

$$\Rightarrow \quad y = e^{\tan x + C}$$

$$\Rightarrow \quad y = e^{\tan x}e^{C}$$

$$\Rightarrow \quad y = Ae^{\tan x} \qquad [\text{where } A = e^{C}]$$

$$y = 2 \text{ when } x = 0 \quad \Rightarrow \quad 2 = Ae^{\tan 0} \quad \Rightarrow \quad 2 = Ae^{0} \quad \Rightarrow \quad 2 = A(1) \quad \Rightarrow \quad A = 2$$

Hence $y = 2e^{\tan x}$.

<u>Note</u>

The question states that $y > 0$, so there is no need to include a modulus sign when finding $\int \frac{1}{y}dy$.

4. The differential equation can be solved by separating the variables and then integrating.

$$\frac{dy}{dx} = 3(1+y)\sqrt{1+x} \quad \Rightarrow \quad dy = 3(1+y)\sqrt{1+x}dx$$

$$\Rightarrow \quad \int \frac{1}{1+y} dy = \int 3\sqrt{1+x}dx$$

$$\Rightarrow \quad \int \frac{1}{1+y} dy = \int 3(1+x)^{\frac{1}{2}} dx$$

$$\Rightarrow \quad \ln(1+y) = \frac{3(1+x)^{\frac{3}{2}}}{\frac{3}{2}\times 1} + C$$

$$\Rightarrow \quad \ln(1+y) = \frac{6\sqrt{(1+x)^3}}{3} + C$$

$$\Rightarrow \quad \ln(1+y) = 2\sqrt{(1+x)^3} + C$$

$$\Rightarrow \quad 1+y = e^{2\sqrt{(1+x)^3}+C}$$

$$\Rightarrow \quad 1+y = e^{2\sqrt{(1+x)^3}}e^{C}$$

$$\Rightarrow \quad 1+y = Ae^{2\sqrt{(1+x)^3}} \qquad [\text{where } A = e^{C}]$$

$$\Rightarrow \quad y = Ae^{2\sqrt{(1+x)^3}} - 1$$

<u>Note</u>

The question states that $y > -1$, so $1+y > 0$ and there is no need to include a modulus sign when finding $\int \frac{1}{1+y}dy$.

5. The differential equation can be solved by separating the variables and then integrating.

$$\frac{dy}{dx} = e^{2x}(1+y^2) \quad \Rightarrow \quad dy = e^{2x}(1+y^2)dx$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = \int e^{2x}dx$$

$$\Rightarrow \quad \tan^{-1} y = \frac{1}{2}e^{2x} + C$$

When $x = 0$, $y = 1$ $\quad \Rightarrow \quad \tan^{-1} 1 = \frac{1}{2}e^{2(0)} + C$

$$\Rightarrow \quad \frac{\pi}{4} = \frac{1}{2}e^{0} + C$$

$$\Rightarrow \quad \frac{\pi}{4} = \frac{1}{2}(1) + C$$

$$\Rightarrow \quad C = \frac{\pi}{4} - \frac{1}{2}$$

Hence $\quad \tan^{-1} y = \frac{1}{2}e^{2x} + \frac{\pi}{4} - \frac{1}{2} \quad \Rightarrow \quad y = \tan\left(\frac{1}{2}e^{2x} + \frac{\pi}{4} - \frac{1}{2}\right)$.

Note

The solution does not have to be written in the form $y = f(x)$ before substituting the values of *x* and *y* to find *C*. You can substitute the values of *x* and *y* at any time after integrating.

6.(a) The differential equation can be solved by separating the variables and then integrating. Note that the two variables are *M* and *t* and that *k* is a constant.

$$\frac{dM}{dt} = kM \quad \Rightarrow \quad dM = kMdt \quad \Rightarrow \quad \int \frac{1}{M}\,dM = \int k\,dt$$
$$\Rightarrow \quad \ln M = kt + C$$
$$\Rightarrow \quad M = e^{kt+C}$$
$$\Rightarrow \quad M = e^{kt}e^{C}$$
$$\Rightarrow \quad M = Ae^{kt} \quad [\text{where } A = e^{C}]$$

When $t = 0$, $M = 100$ $\Rightarrow$ $100 = Ae^{k(0)}$ $\Rightarrow$ $100 = Ae^{0}$ $\Rightarrow$ $100 = A(1)$
$\Rightarrow$ $A = 100$

Hence $M = 100e^{kt}$.

Note

In the context of this question, $M > 0$ and there is no need to include a modulus sign when finding $\int \frac{1}{M}dM$.

(b) $M = 100e^{kt}$

When $t = 30$, $M = 50$ $\Rightarrow$ $50 = 100e^{k(30)}$ $\Rightarrow$ $100e^{30k} = 50$

$$\Rightarrow \quad e^{30k} = \frac{1}{2}$$
$$\Rightarrow \quad 30k = \ln\left(\frac{1}{2}\right)$$
$$\Rightarrow \quad k = \frac{\ln\left(\frac{1}{2}\right)}{30} = -0\cdot 0231 \quad \text{(to 4 dp)}$$

(c) $M = 100e^{-0\cdot 0231t}$

When $t = 35$: $M = 100e^{-0\cdot 0231(35)} = 100e^{-0\cdot 8085} = 44\cdot 55$ (to 2 dp)

Hence 44·55% of the plant food is still effective after 35 days.

(d) **Method 1**

Calculate the percentage of the plant food which is still effective after 60 days.

When $t = 60$: $M = 100e^{-0\cdot0231(60)} = 100e^{-1\cdot386} = 25\cdot01$ (to 2 dp)

25·01% of the plant food is still effective after 60 days, so more than 25% of the plant food is still effective after 60 days and the manufacturer is justified in calling its product "sixty day super food".

Method 2

Calculate after how many days the plant food will be 25% effective.

When $M = 25$: $25 = 100e^{-0\cdot0231t} \Rightarrow e^{-0\cdot0231t} = \frac{1}{4}$

$$\Rightarrow -0\cdot0231t = \ln\left(\frac{1}{4}\right)$$

$$\Rightarrow t = \frac{\ln\left(1/4\right)}{-0\cdot0231} = 60\cdot01 \quad \text{(to 2 dp)}$$

Since $60\cdot01 > 60$, the plant food will still be effective after 60 days and the manufacturer is justified in calling its product "sixty day super food".

7. The differential equation can be solved by separating the variables and then integrating. Note that the two variables are *B* and *t* and that *k* is a constant.

$$\frac{dB}{dt} = kB \quad \Rightarrow \quad dB = kBdt \quad \Rightarrow \quad \int \frac{1}{B}\, dB = \int k dt$$

$$\Rightarrow \quad \ln B = kt + C$$

$$\Rightarrow \quad B = e^{kt+C}$$

$$\Rightarrow \quad B = e^{kt}e^{C}$$

$$\Rightarrow \quad B = Ae^{kt} \qquad [\text{where } A = e^{C}]$$

When $t = 1,\ B = 502 \quad \Rightarrow \quad 502 = Ae^{k(1)} \quad \Rightarrow \quad Ae^{k} = 502 \qquad ...(1)$

When $t = 4,\ B = 1833 \quad \Rightarrow \quad 1833 = Ae^{k(4)} \quad \Rightarrow \quad Ae^{4k} = 1833 \qquad ...(2)$

To eliminate *A*, divide equation (2) by equation (1):

$$(2) \div (1) \quad \Rightarrow \quad \frac{Ae^{4k}}{Ae^{k}} = \frac{1833}{502} \quad \Rightarrow \quad e^{3k} = \frac{1833}{502} \quad \Rightarrow \quad 3k = \ln\left(\frac{1833}{502}\right)$$

$$\Rightarrow \quad k = \frac{\ln\left(1833/502\right)}{3} = 0\cdot4317 \quad (\text{to 4 dp})$$

Substitute in (1) $\quad \Rightarrow \quad Ae^{0\cdot4317} = 502 \quad \Rightarrow \quad A = \dfrac{502}{e^{0\cdot4317}} = 326$

Hence $\quad B = 326e^{0\cdot4317t}$.

When $t = 0: \quad B = 326e^{0\cdot4317(0)} = 326e^{0} = 326(1) = 326$

There are 326 strands of bacteria initially present.

Note

In the context of this question, $B > 0$ and there is no need to include a modulus sign when finding $\int \frac{1}{B} dB$.

8.(a) The differential equation can be solved by separating the variables and then integrating. Note that the two variables are *P* and *t* and that *k* is a constant.

$$\frac{dP}{dt} = kP \quad \Rightarrow \quad dP = kPdt \quad \Rightarrow \quad \int \frac{1}{P}\,dP = \int k\,dt$$

$$\Rightarrow \quad \ln P = kt + C$$

$$\Rightarrow \quad P = e^{kt+C}$$

$$\Rightarrow \quad P = e^{kt}e^{C}$$

$$\Rightarrow \quad P = Ae^{kt} \qquad [\text{where } A = e^{C}]$$

Note that *P* is the **percentage** of the population infected *t* days after the initial outbreak.

When $t = 0$, $P = \frac{100}{40000} \times 100 = 0 \cdot 25\,(\%) \quad \Rightarrow \quad 0 \cdot 25 = Ae^{k(0)} \quad \Rightarrow \quad 0 \cdot 25 = Ae^{0}$

$$\Rightarrow \quad 0 \cdot 25 = A(1)$$

$$\Rightarrow \quad A = 0 \cdot 25$$

Hence the percentage infected *t* days later is given by the formula $P = 0 \cdot 25e^{kt}$.

<u>Note</u>

In the context of this question, $P > 0$ and there is no need to include a modulus sign when finding $\int \frac{1}{P}dP$.

(b) $P = 0 \cdot 25e^{kt}$

When $t = 7$, $P = \frac{500}{40000} \times 100 = 1 \cdot 25\,(\%) \quad \Rightarrow \quad 1 \cdot 25 = 0 \cdot 25e^{k(7)}$

$$\Rightarrow \quad 0 \cdot 25e^{7k} = 1 \cdot 25$$

$$\Rightarrow \quad e^{7k} = 5$$

$$\Rightarrow \quad 7k = \ln 5$$

$$\Rightarrow \quad k = \frac{\ln 5}{7} = 0 \cdot 2299 \quad (\text{to 4 dp})$$

Hence $P = 0 \cdot 25e^{0 \cdot 2299t}$.

When $t = 10$: $P = 0 \cdot 25e^{0 \cdot 2299(10)} = 0 \cdot 25e^{2 \cdot 299} = 2 \cdot 49\,(\%)$

So 2·49% of the population is infected after 10 days and the number of people infected after 10 days will be $\frac{2 \cdot 49}{100} \times 40000 = 996$.

996 – 500 = 496, so 496 more people will be infected after 10 days (compared with after 7 days).

9.(a) The differential equation can be solved by separating the variables and then integrating. Note that the two variables are T and t and that k is a constant.

$$\frac{dT}{dt} = k(T-22) \quad \Rightarrow \quad dT = k(T-22)dt \quad \Rightarrow \quad \int \frac{1}{T-22}\,dT = \int k\,dt$$

$$\Rightarrow \quad \ln(T-22) = kt + C$$

$$\Rightarrow \quad T - 22 = e^{kt+C}$$

$$\Rightarrow \quad T - 22 = e^{kt}e^{C}$$

$$\Rightarrow \quad T - 22 = Ae^{kt} \quad [\text{where } A = e^{C}]$$

$$\Rightarrow \quad T = Ae^{kt} + 22$$

<u>Note</u>

In the context of this question, $T > 22$ and $T - 22 > 0$, so there is no need to include a modulus sign when finding $\int \frac{1}{T-22}\,dT$.

(b) $T = Ae^{kt} + 22$

When $t = 0$, $T = 82$ $\quad \Rightarrow \quad 82 = Ae^{k(0)} + 22 \quad \Rightarrow \quad 82 = Ae^{0} + 22$

$$\Rightarrow \quad 82 = A(1) + 22$$

$$\Rightarrow \quad A = 60$$

When $t = 5$, $T = 82 - 20 = 62$ $\quad \Rightarrow \quad 62 = Ae^{k(5)} + 22 \quad \Rightarrow \quad 62 = 60e^{5k} + 22$

$$\Rightarrow \quad 60e^{5k} = 40$$

$$\Rightarrow \quad e^{5k} = \frac{2}{3}$$

$$\Rightarrow \quad 5k = \ln\left(\frac{2}{3}\right)$$

$$\Rightarrow \quad k = \frac{\ln\left(2/3\right)}{5}$$

$$\Rightarrow \quad k = -0\cdot0811 \quad \text{(to 4 dp)}$$

$A = 60$ and $k = -0\cdot0811$.

The temperature of the roll after t minutes is given by the formula $T = 60e^{-0\cdot0811t} + 22$.

When $t = 10$: $\quad T = 60e^{-0\cdot0811(10)} + 22 = 60e^{-0\cdot811} + 22 = 48\cdot7 \quad$ (to 1 dp)

The temperature of the roll after a further 5 minutes will be 48·7°C.

10.(a) The differential equation can be solved by separating the variables and then integrating. Note that the two variables are T and x and that k is a constant.

$$\frac{dT}{dx} = k(180 - T) \quad \Rightarrow \quad dT = k(180 - T)dx$$

$$\Rightarrow \quad \int \frac{1}{180 - T} dT = \int k dx$$

$$\Rightarrow \quad -\ln(180 - T) = kx + C \qquad [\times(-1)]$$

$$\Rightarrow \quad \ln(180 - T) = -kx + D \qquad [\text{where } D = -C]$$

$$\Rightarrow \quad 180 - T = e^{-kx+D}$$

$$\Rightarrow \quad 180 - T = e^{-kx}e^{D}$$

$$\Rightarrow \quad 180 - T = Ae^{-kx} \qquad [\text{where } A = e^{D}]$$

When $x = 0$, $T = 4$ $\Rightarrow$ $180 - 4 = Ae^{-k(0)}$ $\Rightarrow$ $176 = Ae^{0}$ $\Rightarrow$ $176 = A(1)$

$\Rightarrow$ $A = 176$

Hence $180 - T = 176e^{-kx}$ $\Rightarrow$ $T = 180 - 176e^{-kx}$

Notes

(1) In the context of this question, $180 > T$ and $180 - T > 0$, so there is no need to include a modulus sign when finding $\int \frac{1}{180 - T} dT$.

(2) $\int \frac{1}{180 - T} dT = -\ln(180 - T) + C$ using $\int \frac{1}{ax + b} = \frac{1}{a}\ln|ax + b| + C$; remember to divide by the coefficient of T when integrating.

(b) $T = 180 - 176e^{-kx}$

When $x = 1$, $T = 30$ $\Rightarrow$ $30 = 180 - 176e^{-k(1)}$

$$\Rightarrow \quad 30 = 180 - 176e^{-k}$$

$$\Rightarrow \quad 176e^{-k} = 150$$

$$\Rightarrow \quad e^{-k} = \frac{150}{176} = \frac{75}{88}$$

$$\Rightarrow \quad -k = \ln\left(\frac{75}{88}\right)$$

$$\Rightarrow \quad k = -\ln\left(\frac{75}{88}\right) = 0 \cdot 16 \quad \text{(to 2 dp)}$$

(c) $T = 180 - 176e^{-0{\cdot}16x}$

When $T = 80$: $80 = 180 - 176e^{-0{\cdot}16x} \Rightarrow 176e^{-0{\cdot}16x} = 100$

$$\Rightarrow e^{-0{\cdot}16x} = \frac{100}{176} = \frac{25}{44}$$

$$\Rightarrow -0\cdot 16x = \ln\left(\frac{25}{44}\right)$$

$$\Rightarrow x = \frac{\ln\left({}^{25}\!/\!_{44}\right)}{-0\cdot 16} = 3\cdot 53 \quad \text{(to 2 dp)}$$

The turkey will be cooked after 3·53 hours.

<u>Note</u>

Although not necessary, the answer can also be given in hours and minutes: $0\cdot 53 \times 60 = 31\cdot 8$, so the turkey will be cooked after about 3 hours 32 minutes.

11.(a) The differential equation can be solved by separating the variables and then integrating.

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{10} \quad \Rightarrow \quad 10dh = -\sqrt{h}dt \quad \Rightarrow \quad \int \frac{10}{\sqrt{h}} dh = \int -dt$$

$$\Rightarrow \quad \int 10h^{-\frac{1}{2}} dh = \int -1dt$$

$$\Rightarrow \quad \frac{10h^{\frac{1}{2}}}{1/2} = -t + C$$

$$\Rightarrow \quad 20\sqrt{h} = C - t$$

$$\Rightarrow \quad \sqrt{h} = D - \frac{t}{20} \quad \text{[where } D = \frac{C}{20}\text{]}$$

$$\Rightarrow \quad h = \left(D - \frac{t}{20}\right)^2$$

The general solution is $h = \left(D - \frac{t}{20}\right)^2$ for some constant *D*.

(b) When $t = 0$, $h = 4$.

To find *D*, substitute $t = 0$ and $h = 4$ into $\sqrt{h} = D - \frac{t}{20} \Rightarrow \sqrt{4} = D - \frac{0}{20}$

$$\Rightarrow \quad D = 2$$

Hence $h = \left(2 - \frac{t}{20}\right)^2$.

(c) Now when $t = 0$, $h = 9$ and the value of *D* must be calculated again.

To find *D*, substitute $t = 0$ and $h = 9$ into $\sqrt{h} = D - \frac{t}{20} \Rightarrow \sqrt{9} = D - \frac{0}{20}$

$$\Rightarrow \quad D = 3$$

Hence $h = \left(3 - \frac{t}{20}\right)^2$.

The pool is drained when $h = 0 \quad \Rightarrow \quad 0 = \left(3 - \frac{t}{20}\right)^2 \quad \Rightarrow \quad 3 - \frac{t}{20} = 0$

$$\Rightarrow \quad \frac{t}{20} = 3$$

$$\Rightarrow \quad t = 60$$

It will take 60 minutes (1 hour) to drain the pool on this occasion.

12.(a) The differential equation can be solved by separating the variables and then integrating.

$$\frac{dx}{dt} = \frac{2-x}{5} \quad \Rightarrow \quad 5dx = (2-x)dt \quad \Rightarrow \quad \int \frac{5}{2-x}\,dx = \int 1dt$$

$$\Rightarrow \quad -5\ln(2-x) = t + C \qquad [\div(-5)]$$

$$\Rightarrow \quad \ln(2-x) = -\frac{t}{5} + D \qquad [\text{where } D = -\frac{C}{5}]$$

$$\Rightarrow \quad 2-x = e^{-\frac{t}{5}+D}$$

$$\Rightarrow \quad 2-x = e^{-\frac{t}{5}}e^{D}$$

$$\Rightarrow \quad 2-x = Ae^{-\frac{t}{5}} \qquad [\text{where } A = e^{D}]$$

The tank initially holds 20 litres of pure water, so when $t = 0$, $x = 0$ (remember that x kg is the amount of salt in the tank at time t minutes).

When $t = 0$, $x = 0 \quad \Rightarrow \quad 2 - 0 = Ae^{0} \quad \Rightarrow \quad 2 = A(1) \quad \Rightarrow \quad A = 2$

Hence $\quad 2 - x = 2e^{-\frac{t}{5}} \quad \Rightarrow \quad x = 2 - 2e^{-\frac{t}{5}} \quad$ *or* $\quad x = 2\left(1 - e^{-\frac{t}{5}}\right)$

<u>Note</u>

$\int \frac{5}{2-x}\,dx = -5\ln(2-x) + C \quad$ using $\quad \int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$;
remember to divide by the coefficient of x when integrating.

(b) $x = 2\left(1 - e^{-\frac{t}{5}}\right)$

When $t = 20$: $\quad x = 2\left(1 - e^{-\frac{20}{5}}\right) = 2(1 - e^{-4}) = 1\cdot 96 \quad$ (to 2 dp)

There will be 1·96 kg of salt present after 20 minutes.

(c) $x = 2\left(1 - e^{-\frac{t}{5}}\right)$

As $t \to \infty$, $e^{-\frac{t}{5}} \to 0 \quad$ and $\quad x \to 2(1 - 0) = 2$.

In the long term, the limiting amount of salt in the tank will be 2 kg.

13. The differential equation can be solved by separating the variables and then integrating.

$$\frac{dV}{dt} = V(10-V) \quad \Rightarrow \quad dV = V(10-V)dt \quad \Rightarrow \quad \int \frac{1}{V(10-V)}dV = \int dt \qquad ...(*)$$

To find $\int \frac{1}{V(10-V)}dV$, first express $\frac{1}{V(10-V)}$ in partial fractions.
Note that the denominator contains distinct linear factors.

$$\frac{1}{V(10-V)} = \frac{A}{V} + \frac{B}{10-V}$$
$$= \frac{A(10-V)+BV}{V(10-V)}$$

$$1 = A(10-V) + BV$$

Let $V = 10 \quad \Rightarrow \quad 1 = A(0) + B(10) \quad \Rightarrow \quad 1 = 10B \quad \Rightarrow \quad B = \frac{1}{10}$

Let $V = 0 \quad \Rightarrow \quad 1 = A(10) + B(0) \quad \Rightarrow \quad 1 = 10A \quad \Rightarrow \quad A = \frac{1}{10}$

Hence $\quad \frac{1}{V(10-V)} = \frac{{}^{1}\!/_{10}}{V} + \frac{{}^{1}\!/_{10}}{10-V}$.

From (*): $\quad \int \frac{1}{V(10-V)}dV = \int dt \quad \Rightarrow \quad \int \left(\frac{{}^{1}\!/_{10}}{V} + \frac{{}^{1}\!/_{10}}{10-V} \right) dV = \int 1dt$

$$\Rightarrow \quad \frac{1}{10}\ln V - \frac{1}{10}\ln(10-V) = t + C$$

$V(0) = 5$, so when $t = 0$, $V = 5 \quad \Rightarrow \quad \frac{1}{10}\ln 5 - \frac{1}{10}\ln(10-5) = 0 + C$

$$\Rightarrow \quad \frac{1}{10}\ln 5 - \frac{1}{10}\ln 5 = C$$

$$\Rightarrow \quad C = 0$$

Hence $\frac{1}{10}\ln V - \frac{1}{10}\ln(10-V) = t \quad \Rightarrow \quad \frac{1}{10}(\ln V - \ln(10-V)) = t$

$$\Rightarrow \quad \frac{1}{10}\ln\left(\frac{V}{10-V}\right) = t$$

$$\Rightarrow \quad \ln\left(\frac{V}{10-V}\right) = 10t$$

$$\Rightarrow \quad \frac{V}{10-V} = e^{10t}$$

$$\Rightarrow \quad V = e^{10t}(10-V)$$

$$\Rightarrow \quad V = 10e^{10t} - Ve^{10t}$$

$$\Rightarrow \quad V + Ve^{10t} = 10e^{10t}$$

$$\Rightarrow \quad V(1+e^{10t}) = 10e^{10t}$$

$$\Rightarrow \quad V = \frac{10e^{10t}}{1+e^{10t}}$$

So $V(t) = \frac{10e^{10t}}{1+e^{10t}}$.

As $t \to \infty$, $e^{10t} \approx 1 + e^{10t}$ and $V(t) = 10\left(\frac{e^{10t}}{1+e^{10t}}\right) \approx 10(1) = 10$.

Hence $V(t) \to 10$ as $t \to \infty$ and the limiting value of $V(t)$ as $t \to \infty$ is 10.

Notes

(1) The partial fractions can be written as $\frac{1}{10V} + \frac{1}{10(10-V)}$ but there is no need to do this.

(2) Since $0 < V < 10$, $V > 0$ and $10 - V > 0$, so there is no need to include modulus signs when finding $\int \frac{1/10}{V}dV$ and $\int \frac{1/10}{10-V}dV$.

(3) $\int \frac{1/10}{10-V}dV = -\frac{1}{10}\ln(10-V) + C$ using $\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$; remember to divide by the coefficient of V when integrating.

14.(a) The denominator contains distinct linear factors.

$$\frac{900}{(30-Q)(15-Q)} = \frac{A}{30-Q} + \frac{B}{15-Q}$$
$$= \frac{A(15-Q)+B(30-Q)}{(30-Q)(15-Q)}$$

$$900 = A(15-Q)+B(30-Q)$$

Let $Q=15 \Rightarrow 900 = A(0)+B(15) \Rightarrow 900 = 15B \Rightarrow B = 60$
Let $Q=30 \Rightarrow 900 = A(-15)+B(0) \Rightarrow 900 = -15A \Rightarrow A = -60$

Hence $\frac{900}{(30-Q)(15-Q)} = \frac{-60}{30-Q} + \frac{60}{15-Q}$ *or* $\frac{900}{(30-Q)(15-Q)} = \frac{60}{15-Q} - \frac{60}{30-Q}$.

(b) The differential equation can be solved by separating the variables and then integrating.

$$\frac{dQ}{dt} = \frac{(30-Q)(15-Q)}{900} \Rightarrow 900\,dQ = (30-Q)(15-Q)\,dt$$
$$\Rightarrow \int \frac{900}{(30-Q)(15-Q)}\,dQ = \int dt$$
$$\Rightarrow \int \left(\frac{60}{15-Q} - \frac{60}{30-Q}\right) dQ = \int 1\,dt$$
$$\Rightarrow -60\ln(15-Q) + 60\ln(30-Q) = t + C$$
$$\Rightarrow 60\ln(30-Q) - 60\ln(15-Q) = t + C$$
$$\Rightarrow 60(\ln(30-Q) - \ln(15-Q)) = t + C$$
$$\Rightarrow 60\ln\left(\frac{30-Q}{15-Q}\right) = t + C$$

$A = 60$

$Q(0)=0$, so when $t=0$, $Q=0 \Rightarrow 60\ln\left(\frac{30-0}{15-0}\right) = 0 + C$
$$\Rightarrow 60\ln 2 = C$$

Hence $A = 60$ and $C = 60\ln 2$.

<u>Note</u>

$\int \frac{60}{15-Q}\,dQ = -60\ln(15-Q) + C$ and $\int \frac{60}{30-Q}\,dQ = -60\ln(30-Q) + C$ using $\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$; remember to divide by the coefficient of Q when integrating.

(i) $$60\ln\left(\frac{30-Q}{15-Q}\right)=t+60\ln 2$$

When $Q=5$: $$60\ln\left(\frac{30-5}{15-5}\right)=t+60\ln 2$$

$$\Rightarrow \quad 60\ln\left(\frac{25}{10}\right)=t+60\ln 2$$

$$\Rightarrow \quad 60\ln\left(\frac{5}{2}\right)=t+60\ln 2$$

$$\Rightarrow \quad t=60\ln\left(\frac{5}{2}\right)-60\ln 2=13\cdot 39 \quad \text{(to 2 dp)}$$

It takes 13·39 minutes to form 5 grams of *Z*.

Note

Although not necessary, the answer can also be given in minutes and seconds:
$0\cdot 39\times 60=23\cdot 4$, so it takes about 13 minutes 23 seconds to form 5 grams of *Z*.

(ii) $$60\ln\left(\frac{30-Q}{15-Q}\right)=t+60\ln 2$$

When $t=45$: $$60\ln\left(\frac{30-Q}{15-Q}\right)=45+60\ln 2 \qquad \text{[divide all terms by 60]}$$

$$\Rightarrow \quad \ln\left(\frac{30-Q}{15-Q}\right)=\frac{3}{4}+\ln 2$$

$$\Rightarrow \quad \ln\left(\frac{30-Q}{15-Q}\right)=1\cdot 443 \quad \text{(to 3 dp)}$$

$$\Rightarrow \quad \frac{30-Q}{15-Q}=e^{1\cdot 443}$$

$$\Rightarrow \quad 30-Q=e^{1\cdot 443}(15-Q)$$

$$\Rightarrow \quad 30-Q=15e^{1\cdot 443}-Qe^{1\cdot 443}$$

$$\Rightarrow \quad Qe^{1\cdot 443}-Q=15e^{1\cdot 443}-30$$

$$\Rightarrow \quad Q(e^{1\cdot 443}-1)=15e^{1\cdot 443}-30$$

$$\Rightarrow \quad Q=\frac{15e^{1\cdot 443}-30}{e^{1\cdot 443}-1}=10\cdot 36 \quad \text{(to 2 dp)}$$

10·36 grams of *Z* will be formed 45 minutes after the reaction begins.

15.(a) The denominator contains distinct linear factors.

$$\frac{10000}{N(20000-N)} = \frac{A}{N} + \frac{B}{20000-N}$$
$$= \frac{A(20000-N)+BN}{N(20000-N)}$$

$$10000 = A(20000-N)+BN$$

Let $N = 20000 \quad \Rightarrow \quad 10000 = A(0)+B(20000) \quad \Rightarrow \quad 10000 = 20000B \quad \Rightarrow \quad B = \frac{1}{2}$

Let $N = 0 \quad \Rightarrow \quad 10000 = A(20000)+B(0) \quad \Rightarrow \quad 10000 = 20000A \quad \Rightarrow \quad A = \frac{1}{2}$

Hence $\dfrac{10000}{N(20000-N)} = \dfrac{1/2}{N} + \dfrac{1/2}{20000-N}$.

The differential equation can be solved by separating the variables and then integrating.

$$\frac{dN}{dt} = \frac{N(20000-N)}{10000} \quad \Rightarrow \quad 10000\,dN = N(20000-N)dt$$
$$\Rightarrow \quad \int \frac{10000}{N(20000-N)}\,dN = \int dt$$
$$\Rightarrow \quad \int \left(\frac{1/2}{N} + \frac{1/2}{20000-N} \right) dN = \int 1dt$$
$$\Rightarrow \quad \frac{1}{2}\ln N - \frac{1}{2}\ln(20000-N) = t + C$$
$$\Rightarrow \quad \frac{1}{2}(\ln N - \ln(20000-N)) = t + C$$
$$\Rightarrow \quad \frac{1}{2}\ln\left(\frac{N}{20000-N}\right) = t + C$$
$$\Rightarrow \quad \ln\left(\frac{N}{20000-N}\right) = 2t + D \qquad \text{[where } D = 2C\text{]}$$

Hence $\ln\left(\dfrac{N}{20000-N}\right) = 2t + C$ for some constant C.

Notes

(1) The partial fractions can be written as $\frac{10000}{N(20000-N)} = \frac{1}{2N} + \frac{1}{2(20000-N)}$ but there is no need to do this.

(2) Since $0 < N < 20000$, $N > 0$ and $20000 - N > 0$, so there is no need to include modulus signs when finding $\int \frac{1/2}{N} dN$ and $\int \frac{1/2}{20000-N} dN$.

(3) $\int \frac{1/2}{20000-N} dN = -\frac{1}{2}\ln(20000-N) + C$ using $\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$; remember to divide by the coefficient of N when integrating.

(b) $\ln\left(\frac{N}{20000-N}\right) = 2t + C$

When $t = 0$, $N = 100$ $\Rightarrow$ $\ln\left(\frac{100}{20000-100}\right) = 2(0) + C$

$$\Rightarrow \ln\left(\frac{100}{19900}\right) = C$$

$$\Rightarrow C = \ln\left(\frac{1}{199}\right)$$

Hence $\ln\left(\frac{N}{20000-N}\right) = 2t + \ln\left(\frac{1}{199}\right)$ $\Rightarrow$ $\ln\left(\frac{N}{20000-N}\right) - \ln\left(\frac{1}{199}\right) = 2t$

$$\Rightarrow \ln\left(\frac{\frac{N}{20000-N}}{\frac{1}{199}}\right) = 2t$$

$$\Rightarrow \ln\left(\frac{199N}{20000-N}\right) = 2t$$

$$\Rightarrow \frac{199N}{20000-N} = e^{2t}$$

$$\Rightarrow 199N = e^{2t}(20000 - N)$$

$$\Rightarrow 199N = 20000e^{2t} - Ne^{2t}$$

$$\Rightarrow 199N + Ne^{2t} = 20000e^{2t}$$

$$\Rightarrow N(199 + e^{2t}) = 20000e^{2t}$$

$$\Rightarrow N = \frac{20000e^{2t}}{199 + e^{2t}}$$

16. The differential equation can be solved by separating the variables and then integrating.

$$\frac{dP}{dt} = P(1000-P) \quad \Rightarrow \quad dP = P(1000-P)dt \quad \Rightarrow \quad \int \frac{1}{P(1000-P)}dP = \int dt \qquad ...(*)$$

To find $\int \frac{1}{P(1000-P)}dP$, first express $\frac{1}{P(1000-P)}$ in partial fractions.
Note that the denominator contains distinct linear factors.

$$\frac{1}{P(1000-P)} = \frac{A}{P} + \frac{B}{1000-P}$$
$$= \frac{A(1000-P)+BP}{P(1000-P)}$$

$$1 = A(1000-P) + BP$$

Let $P = 1000 \quad \Rightarrow \quad 1 = A(0) + B(1000) \quad \Rightarrow \quad 1 = 1000B \quad \Rightarrow \quad B = \frac{1}{1000}$

Let $P = 0 \quad \Rightarrow \quad 1 = A(1000) + B(0) \quad \Rightarrow \quad 1 = 1000A \quad \Rightarrow \quad A = \frac{1}{1000}$

Hence $\frac{1}{P(1000-P)} = \frac{1/1000}{P} + \frac{1/1000}{1000-P}$.

From (*): $\int \frac{1}{P(1000-P)}dP = \int dt \quad \Rightarrow \quad \int \left(\frac{1/1000}{P} + \frac{1/1000}{1000-P} \right) dV = \int 1dt$

$$\Rightarrow \quad \frac{1}{1000}\ln P - \frac{1}{1000}\ln(1000-P) = t + C$$
$$\Rightarrow \quad \frac{1}{1000}(\ln P - \ln(1000-P)) = t + C$$
$$\Rightarrow \quad \frac{1}{1000}\ln\left(\frac{P}{1000-P}\right) = t + C$$
$$\Rightarrow \quad \ln\left(\frac{P}{1000-P}\right) = 1000t + D \qquad [\text{where } D = 1000C]$$

Hence $\ln\left(\frac{P}{1000-P}\right) = 1000t + C$ for some constant C.

$$\ln\left(\frac{P}{1000-P}\right)=1000t+C \quad \Rightarrow \quad \frac{P}{1000-P}=e^{1000t+C}$$

$$\Rightarrow \quad \frac{P}{1000-P}=e^{1000t}e^{C}$$

$$\Rightarrow \quad \frac{P}{1000-P}=Ae^{1000t} \qquad \text{[where } A=e^{C}\text{]}$$

$$\Rightarrow \quad P=Ae^{1000t}(1000-P)$$

$$\Rightarrow \quad P=1000Ae^{1000t}-APe^{1000t}$$

$$\Rightarrow \quad P+APe^{1000t}=1000Ae^{1000t}$$

$$\Rightarrow \quad P(1+Ae^{1000t})=1000Ae^{1000t}$$

$$\Rightarrow \quad P=\frac{1000Ae^{1000t}}{1+Ae^{1000t}} \qquad \text{[divide all terms by } e^{1000t}\text{]}$$

$$\Rightarrow \quad P=\frac{1000A}{\frac{1}{e^{1000t}}+A}=\frac{1000A}{e^{-1000t}+A}$$

Hence $P(t)=\dfrac{1000K}{K+e^{-1000t}}$ for some constant K.

$P(0)=200$, so when $t=0$, $P=200$ $\quad \Rightarrow \quad 200=\dfrac{1000K}{K+e^{-1000(0)}}$

$$\Rightarrow \quad 200=\frac{1000K}{K+e^{0}}$$

$$\Rightarrow \quad 200=\frac{1000K}{K+1}$$

$$\Rightarrow \quad 200(K+1)=1000K$$

$$\Rightarrow \quad 200K+200=1000K$$

$$\Rightarrow \quad 200=800K$$

$$\Rightarrow \quad K=\frac{1}{4}$$

Hence $P(t)=\dfrac{1000\left(\frac{1}{4}\right)}{\frac{1}{4}+e^{-1000t}}=\dfrac{250}{\frac{1}{4}+e^{-1000t}}=\dfrac{1000}{1+4e^{-1000t}}$ [multiplying all terms in the fraction by 4 to simplify]

When $P(t) = 900$: $\quad 900 = \dfrac{1000}{1+4e^{-1000t}} \quad \Rightarrow \quad 900(1+4e^{-1000t}) = 1000$

$$\Rightarrow \quad 900 + 3600e^{-1000t} = 1000$$
$$\Rightarrow \quad 3600e^{-1000t} = 100$$
$$\Rightarrow \quad e^{-1000t} = \frac{1}{36}$$
$$\Rightarrow \quad 900 + 3600e^{-1000t} = 1000$$
$$\Rightarrow \quad 3600e^{-1000t} = 100$$
$$\Rightarrow \quad e^{-1000t} = \frac{1}{36}$$
$$\Rightarrow \quad -1000t = \ln\left(\frac{1}{36}\right)$$
$$\Rightarrow \quad t = \frac{\ln\left(\frac{1}{36}\right)}{-1000} = 0 \cdot 0036 \quad \text{(to 4 dp)}$$

$P(t) = 900$ when $t = 0 \cdot 0036$.

<u>Notes</u>

(1) The partial fractions can be written as $\dfrac{1}{1000P} + \dfrac{1}{1000(1000-P)}$ but there is no need to do this.

(2) Since $0 < P < 1000$, $P > 0$ and $1000 - P > 0$, so there is no need to include modulus signs when finding $\int \dfrac{1/1000}{P} dP$ and $\int \dfrac{1/1000}{1000-P} dP$.

(3) $\int \dfrac{1/1000}{1000-P} dP = -\dfrac{1}{1000}\ln(1000-P) + C$ using $\int \dfrac{1}{ax+b} = \dfrac{1}{a}\ln|ax+b| + C$; remember to divide by the coefficient of P when integrating.

17. Measure time in minutes from the time the beaker of liquid was placed in the fridge and at 12 noon, let $t = n$.
To determine the time the liquid was placed in the fridge, we must find the value of n.

Before starting to solve the differential equation, interpret the information given mathematically.

- The constant temperature of the fridge is 4°C.
  This means that $T_F = 4$.

- When first placed in the fridge, the temperature of the liquid was 25°C.
  This means that when $t = 0$, $T = 25$.

- At 12 noon, the temperature of the liquid was 9·8°C.
  This means that when $t = n$, $T = 9 \cdot 8$.

- At 12·15 pm, the temperature of the liquid had dropped to 6·5°C.
  This means that when $t = n + 15$, $T = 6 \cdot 5$.

The differential equation can be solved by separating the variables and then integrating.
Note that the two variables are T and t and that k is a constant.

$$\frac{dT}{dt} = -k(T - T_F) \quad \Rightarrow \quad \frac{dT}{dt} = -k(T - 4) \quad \Rightarrow \quad dT = -k(T - 4)dt$$

$$\Rightarrow \quad \int \frac{1}{T - 4} dT = \int -k dt$$

$$\Rightarrow \quad \ln(T - 4) = -kt + C$$

$$\Rightarrow \quad T - 4 = e^{-kt+C}$$

$$\Rightarrow \quad T - 4 = e^{-kt} e^{C}$$

$$\Rightarrow \quad T - 4 = Ae^{-kt} \qquad [\text{where } A = e^{C}]$$

$$\Rightarrow \quad T = 4 + Ae^{-kt}$$

When $t = 0$, $T = 25$ $\quad \Rightarrow \quad 25 = 4 + Ae^{-k(0)} \quad \Rightarrow \quad 25 = 4 + Ae^{(0)}$

$$\Rightarrow \quad 25 = 4 + A(1)$$

$$\Rightarrow \quad A = 21$$

Hence $\quad T = 4 + 21e^{-kt}$

When $t = n$, $T = 9 \cdot 8$ $\Rightarrow$ $9 \cdot 8 = 4 + 21e^{-kn}$ $\Rightarrow$ $21e^{-kn} = 5 \cdot 8$ ...(1)

When $t = n + 15$, $T = 6 \cdot 5$

$$\Rightarrow \quad 6 \cdot 5 = 4 + 21e^{-k(n+15)}$$
$$\Rightarrow \quad 21e^{-k(n+15)} = 2 \cdot 5$$
$$\Rightarrow \quad 21e^{-kn-15k} = 2 \cdot 5$$
$$\Rightarrow \quad 21e^{-kn}e^{-15k} = 2 \cdot 5$$
$$\Rightarrow \quad (21e^{-kn})e^{-15k} = 2 \cdot 5$$
$$\Rightarrow \quad 5 \cdot 8e^{-15k} = 2 \cdot 5 \qquad [\text{since } 21e^{-kn} = 5 \cdot 8]$$
$$\Rightarrow \quad e^{-15k} = \frac{2 \cdot 5}{5 \cdot 8}$$
$$\Rightarrow \quad -15k = \ln\left(\frac{2 \cdot 5}{5 \cdot 8}\right)$$
$$\Rightarrow \quad k = \frac{\ln\left(2 \cdot 5 / 5 \cdot 8\right)}{-15} = 0 \cdot 0561 \quad \text{(to 4 dp)}$$

Substitute $k = 0 \cdot 0561$ into equation (1)

$$\Rightarrow \quad 21e^{-kn} = 5 \cdot 8$$
$$\Rightarrow \quad 21e^{-0 \cdot 0561n} = 5 \cdot 8$$
$$\Rightarrow \quad e^{-0 \cdot 0561n} = \frac{5 \cdot 8}{21}$$
$$\Rightarrow \quad -0 \cdot 0561n = \ln\left(\frac{5 \cdot 8}{21}\right)$$
$$\Rightarrow \quad n = \frac{\ln\left(5 \cdot 8 / 21\right)}{-0 \cdot 0561} = 22 \cdot 9 \quad \text{(to 1 dp)}$$

Hence 12 noon occurs when $t = 22 \cdot 9$ meaning that the liquid was first placed in the fridge 22·9 minutes before 12 noon.

So the liquid was first placed in the fridge at approximately 11·37 am.

Note

In the context of this question, $T > 4$, so $T - 4 > 0$ and there is no need to include a modulus sign when finding $\int \frac{1}{T-4}\,dT$.