Solutions to Exam Questions on Differential Equations 2

1. $\frac{dy}{dx}+\frac{y}{x}=x \quad \Rightarrow \quad \frac{dy}{dx}+\frac{1}{x}y=x$

The differential equation is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$ and is solved by using an **integrating factor**.

$P(x)=\frac{1}{x}$ and $Q(x)=x$

Integrating Factor: $I(x)=e^{\int P(x)dx}=e^{\int \frac{1}{x}dx}=e^{\ln x}=x$

General Solution:
$$I(x)y=\int I(x)Q(x)dx$$
$$\Rightarrow \quad xy=\int x(x)dx$$
$$\Rightarrow \quad xy=\int x^2dx$$
$$\Rightarrow \quad xy=\frac{x^3}{3}+C$$
$$\Rightarrow \quad y=\frac{x^2}{3}+\frac{C}{x}$$

Notes

(1) The question states that $x>0$, so there is no need to include a modulus sign when finding $\int \frac{1}{x}dx$.

(2) The integrating factor must be simplified before it can be used to solve the differential equation.

2.(a) Let $I = \int x \sin 3x dx$.

Use integration by parts with $f(x) = x$ and $g'(x) = \sin 3x$.

$$f(x) = x \quad \text{and} \quad g'(x) = \sin 3x$$

$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = -\frac{1}{3}\cos 3x$$

$$I = f(x)g(x) - \int f'(x)g(x)dx$$
$$= x\left(-\frac{1}{3}\cos 3x\right) - \int 1\left(-\frac{1}{3}\cos 3x\right)dx$$
$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x dx$$
$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\left(\frac{1}{3}\sin 3x\right) + C$$
$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

(b) $\frac{dy}{dx} - \frac{2}{x}y = x^3 \sin 3x$

The differential equation is a first order linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ and is solved by using an **integrating factor**.

$$P(x) = -\frac{2}{x} \quad \text{and} \quad Q(x) = x^3 \sin 3x$$

Integrating Factor: $I(x) = e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

General Solution: $I(x)y = \int I(x)Q(x)dx$

$$\Rightarrow \quad \frac{1}{x^2}y = \int \frac{1}{x^2}(x^3 \sin 3x)dx$$

$$\Rightarrow \quad \frac{y}{x^2} = \int x \sin 3x dx$$

$$\Rightarrow \quad \frac{y}{x^2} = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$ [using the answer to (a)]

$y = 0$ when $x = \pi \quad \Rightarrow \quad \frac{0}{\pi^2} = -\frac{1}{3}\pi\cos 3\pi + \frac{1}{9}\sin 3\pi + C$

$$\Rightarrow \quad 0 = -\frac{1}{3}\pi(-1) + \frac{1}{9}(0) + C$$

$$\Rightarrow \quad 0 = \frac{\pi}{3} + C$$

$$\Rightarrow \quad C = -\frac{\pi}{3}$$

Hence $\frac{y}{x^2} = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x - \frac{\pi}{3} \quad \Rightarrow \quad y = x^2\left(-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x - \frac{\pi}{3}\right)$

Notes

(1) The integrating factor must be simplified before it can be used to solve the differential equation.

(2) The solution does not have to be written in the form $y = f(x)$ before substituting the values of *x* and *y* to find *C*. You can substitute the values of *x* and *y* at any time after you are finished integrating.

3. $\dfrac{dy}{dx} = 4 - \dfrac{3y}{x}$

The differential equation can be written in the form $\dfrac{dy}{dx} + P(x)y = Q(x)$ and is then solved by using an **integrating factor**.

$$\frac{dy}{dx} = 4 - \frac{3y}{x} \quad \Rightarrow \quad \frac{dy}{dx} + \frac{3y}{x} = 4 \quad \Rightarrow \quad \frac{dy}{dx} + \frac{3}{x}y = 4$$

$$P(x) = \frac{3}{x} \quad \text{and} \quad Q(x) = 4$$

Integrating Factor: $I(x) = e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

General Solution: $I(x)y = \int I(x)Q(x)dx$

$$\Rightarrow \quad x^3 y = \int x^3 (4) dx$$

$$\Rightarrow \quad x^3 y = \int 4x^3 dx$$

$$\Rightarrow \quad x^3 y = \frac{4x^4}{4} + C$$

$$\Rightarrow \quad x^3 y = x^4 + C$$

At the point (1, 3) where $x = 1$ and $y = 3$ $\Rightarrow \quad 1^3(3) = 1^4 + C$

$$\Rightarrow \quad 3 = 1 + C$$

$$\Rightarrow \quad C = 2$$

Hence $x^3 y = x^4 + 2 \quad \Rightarrow \quad y = x + \dfrac{2}{x^3}$

Notes

(1) The integrating factor must be simplified before it can be used to solve the differential equation.

(2) The solution does not have to be written in the form $y = f(x)$ before substituting the values of x and y to find C. You can substitute the values of x and y at any time after you are finished integrating.

4. $x\dfrac{dy}{dx} - y = x^2 e^x$

The differential equation can be written in the form $\dfrac{dy}{dx} + P(x)y = Q(x)$ by dividing all the terms in the equation by *x* and is then solved by using an **integrating factor**.

$$x\frac{dy}{dx} - y = x^2 e^x \quad \Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = xe^x \quad \Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = xe^x$$

$P(x) = -\dfrac{1}{x}$ and $Q(x) = xe^x$

Integrating Factor: $I(x) = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \dfrac{1}{x}$

General Solution: $I(x)y = \int I(x)Q(x)dx$

$$\Rightarrow \quad \frac{1}{x}y = \int \frac{1}{x}(xe^x)dx$$

$$\Rightarrow \quad \frac{y}{x} = \int e^x dx$$

$$\Rightarrow \quad \frac{y}{x} = e^x + C$$

$y = 2$ when $x = 1 \quad \Rightarrow \quad \dfrac{2}{1} = e^1 + C \quad \Rightarrow \quad 2 = e + C \quad \Rightarrow \quad C = 2 - e$

Hence $\dfrac{y}{x} = e^x + 2 - e \quad \Rightarrow \quad y = x(e^x + 2 - e)$

<u>Notes</u>

(1) The integrating factor must be simplified before it can be used to solve the differential equation.

(2) The solution does not have to be written in the form $y = f(x)$ before substituting the values of *x* and *y* to find *C*. You can substitute the values of *x* and *y* at any time after you are finished integrating.

5. $x\frac{dy}{dx} - 3y = x^4$

The differential equation can be written in the form $\frac{dy}{dx} + P(x)y = Q(x)$ by dividing all the terms in the equation by x and is then solved by using an **integrating factor**.

$$x\frac{dy}{dx} - 3y = x^4 \quad \Rightarrow \quad \frac{dy}{dx} - \frac{3y}{x} = x^3 \quad \Rightarrow \quad \frac{dy}{dx} - \frac{3}{x}y = x^3$$

$P(x) = -\frac{3}{x}$ and $Q(x) = x^3$

Integrating Factor: $I(x) = e^{\int P(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3} = \frac{1}{x^3}$

General Solution: $I(x)y = \int I(x)Q(x)dx$

$$\Rightarrow \quad \frac{1}{x^3}y = \int \frac{1}{x^3}(x^3)dx$$

$$\Rightarrow \quad \frac{y}{x^3} = \int 1dx$$

$$\Rightarrow \quad \frac{y}{x^3} = x + C$$

$$\Rightarrow \quad y = x^3(x + C) \quad or \quad y = x^4 + Cx^3$$

$y = 2$ when $x = 1 \quad \Rightarrow \quad 2 = 1^4 + C(1)^3 \quad \Rightarrow \quad 2 = 1 + C \quad \Rightarrow \quad C = 1$

Hence the particular solution is $y = x^4 + x^3$.

<u>Note</u>

The integrating factor must be simplified before it can be used to solve the differential equation.

6. $\frac{1}{x}\frac{dy}{dx} + 2y = 6$

The differential equation can be written in the form $\frac{dy}{dx} + P(x)y = Q(x)$ by multiplying all the terms in the equation by x and is then solved by using an **integrating factor**.

$$\frac{1}{x}\frac{dy}{dx} + 2y = 6 \quad \Rightarrow \quad \frac{dy}{dx} + 2xy = 6x \quad \Rightarrow \quad \frac{dy}{dx} + (2x)y = 6x$$

$P(x) = 2x$ and $Q(x) = 6x$

Integrating Factor: $I(x) = e^{\int P(x)dx} = e^{\int 2xdx} = e^{\frac{2x^2}{2}} = e^{x^2}$

General Solution: $I(x)y = \int I(x)Q(x)dx$

$$\Rightarrow \quad e^{x^2} y = \int e^{x^2} (6x)dx$$

$$\Rightarrow \quad e^{x^2} y = \int 6xe^{x^2} dx \qquad ...(*)$$

To find $\int 6xe^{x^2} dx$, use the substitution $u = x^2$ as the term $6x$ is related to the derivative of u.

Let $I = \int 6xe^{x^2} dx$.

$$u = x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2xdx \quad \Rightarrow \quad \frac{1}{2}du = xdx$$

$$I = \int 6xe^{x^2} dx = \int 6e^{x^2} (xdx) = \int 6e^u \left(\frac{1}{2}du\right) = \int 3e^u du = 3e^u + C = 3e^{x^2} + C$$

Hence from (*): $e^{x^2} y = 3e^{x^2} + C \quad \Rightarrow \quad y = 3 + \frac{C}{e^{x^2}} \quad or \quad y = 3 + Ce^{-x^2}$

<u>**Note**</u>

$\int 6xe^{x^2} dx$ could also be found by inspection by realising that $\frac{d}{dx}e^{x^2} = 2xe^{x^2}$.