ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Differential Equations 3

1. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$

Note that this is a homogeneous differential equation (RHS = 0), so it is not necessary to consider a particular integral.

Auxiliary Equation: $m^2 + 4m + 5 = 0$

$a = 1,\ b = 4,\ c = 5 \quad \Rightarrow \quad b^2 - 4ac = 4^2 - 4(1)(5) = -4$

$b^2 - 4ac < 0$, so the equation has non-real roots and the roots can be found using the quadratic formula.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{-4}}{2(1)} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Hence the general solution is $y = e^{-2x}(A\sin x + B\cos x)$.

$$\begin{aligned}
y = 3 \text{ when } x = 0 \quad &\Rightarrow \quad 3 = e^{-2(0)}(A\sin 0 + B\cos 0) \\
&\Rightarrow \quad 3 = e^{0}(A(0) + B(1)) \\
&\Rightarrow \quad 3 = 1(B) \\
&\Rightarrow \quad B = 3
\end{aligned}$$

$$\begin{aligned}
y = e^{-\pi} \text{ when } x = \frac{\pi}{2} \quad &\Rightarrow \quad e^{-\pi} = e^{-2\left(\frac{\pi}{2}\right)}\left(A\sin\frac{\pi}{2} + B\cos\frac{\pi}{2}\right) \\
&\Rightarrow \quad e^{-\pi} = e^{-\pi}\left(A(1) + B(0)\right) \\
&\Rightarrow \quad e^{-\pi} = Ae^{-\pi} \\
&\Rightarrow \quad A = 1
\end{aligned}$$

Hence the particular solution is $y = e^{-2x}(\sin x + 3\cos x)$.

2. $$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

Note that this is a homogeneous differential equation (RHS = 0), so it is not necessary to consider a particular integral.

Auxiliary Equation: $4m^2 - 4m + 1 = 0$

$a = 4$, $b = -4$, $c = 1 \quad \Rightarrow \quad b^2 - 4ac = (-4)^2 - 4(4)(1) = 0$

$b^2 - 4ac = 0$, so the equation has two real and equal roots.

$$4m^2 - 4m + 1 = 0 \quad \Rightarrow \quad (2m-1)(2m-1) = 0 \quad \Rightarrow \quad m = \frac{1}{2},\ m = \frac{1}{2}$$

Hence the general solution is $y = (Ax + B)e^{\frac{1}{2}x}$.

$$y = 4 \text{ when } x = 0 \quad \Rightarrow \quad 4 = (A(0) + B)e^{\frac{1}{2}(0)} \quad \Rightarrow \quad 4 = Be^0 \quad \Rightarrow \quad 4 = B(1) \quad \Rightarrow \quad B = 4$$

To differentiate $y = (Ax + B)e^{\frac{1}{2}x}$, use the **product rule** as y is the product of two functions of x.

$$\frac{dy}{dx} = (Ax + B)\frac{1}{2}e^{\frac{1}{2}x} + e^{\frac{1}{2}x}(A) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x}(Ax + B) + Ae^{\frac{1}{2}x}$$

$$\frac{dy}{dx} = 3 \text{ when } x = 0 \quad \Rightarrow \quad 3 = \frac{1}{2}e^{\frac{1}{2}(0)}(A(0) + B) + Ae^{\frac{1}{2}(0)}$$

$$\Rightarrow \quad 3 = \frac{1}{2}e^0(B) + Ae^0$$

$$\Rightarrow \quad 3 = \frac{1}{2}(1)(B) + A(1)$$

$$\Rightarrow \quad 3 = \frac{1}{2}B + A$$

$$\Rightarrow \quad 3 = \frac{1}{2}(4) + A$$

$$\Rightarrow \quad 3 = 2 + A$$

$$\Rightarrow \quad A = 1$$

Hence the particular solution is $y = (x + 4)e^{\frac{1}{2}x}$.

3. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$

Note that this is a homogeneous differential equation (RHS = 0), so it is not necessary to consider a particular integral.

Auxiliary Equation: $m^2 + 2m + 2 = 0$

$a = 1,\ b = 2,\ c = 2 \quad \Rightarrow \quad b^2 - 4ac = 2^2 - 4(1)(2) = -4$

$b^2 - 4ac < 0$, so the equation has non-real roots and the roots can be found using the quadratic formula.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{-4}}{2(1)} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Hence the general solution is $y = e^{-x}(A\sin x + B\cos x)$.

When $x = 0,\ y = 0 \quad \Rightarrow \quad 0 = e^0(A\sin 0 + B\cos 0)$

$$\Rightarrow \quad 0 = 1(A(0) + B(1))$$
$$\Rightarrow \quad 0 = B$$

To differentiate $y = e^{-x}(A\sin x + B\cos x)$, use the **product rule** as *y* is the product of two functions of *x*.

$$\frac{dy}{dx} = e^{-x}(A\cos x - B\sin x) + (A\sin x + B\cos x)(-e^{-x})$$
$$\frac{dy}{dx} = e^{-x}(A\cos x - B\sin x) - e^{-x}(A\sin x + B\cos x)$$

When $x = 0,\ \dfrac{dy}{dx} = 2 \quad \Rightarrow \quad 2 = e^0(A\cos 0 - B\sin 0) - e^0(A\sin 0 + B\cos 0)$

$$\Rightarrow \quad 2 = 1(A(1) - B(0)) - 1(A(0) + B(1))$$
$$\Rightarrow \quad 2 = 1(A) - 1(B)$$
$$\Rightarrow \quad 2 = A - B$$
$$\Rightarrow \quad 2 = A - 0$$
$$\Rightarrow \quad A = 2$$

Hence the particular solution is $y = e^{-x}(2\sin x + 0\cos x) \quad \Rightarrow \quad y = 2e^{-x}\sin x$

4. $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=e^{2x}$

CF Auxiliary Equation: $m^2+6m+9=0$

$a=1,\ b=6,\ c=9 \quad \Rightarrow \quad b^2-4ac=6^2-4(1)(9)=0$

$b^2-4ac=0$, so the equation has two real and equal roots.

$m^2+6m+9=0 \quad \Rightarrow \quad (m+3)(m+3)=0 \quad \Rightarrow \quad m=-3,\ m=-3$

Hence the CF is $y=(Ax+B)e^{-3x}$.

PI Let $y=ae^{2x} \quad \Rightarrow \quad \frac{dy}{dx}=2ae^{2x}$ and $\frac{d^2y}{dx^2}=4ae^{2x}$

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=e^{2x} \quad \Rightarrow \quad 4ae^{2x}+6(2ae^{2x})+9(ae^{2x})=e^{2x}$$

$$\Rightarrow \quad 4ae^{2x}+12ae^{2x}+9ae^{2x}=e^{2x}$$

$$\Rightarrow \quad 25ae^{2x}=e^{2x}$$

Equating coefficients of $e^{2x} \quad \Rightarrow \quad 25a=1 \quad \Rightarrow \quad a=\frac{1}{25}$

Hence the PI is $y=\frac{1}{25}e^{2x}$.

GS The general solution is $y=(Ax+B)e^{-3x}+\frac{1}{25}e^{2x}$.

5. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 6x - 1$

CF Auxiliary Equation: $m^2 + 2m - 3 = 0$

$a = 1,\ b = 2,\ c = -3 \quad \Rightarrow \quad b^2 - 4ac = 2^2 - 4(1)(-3) = 16$

$b^2 - 4ac > 0$, so the equation has two real and distinct roots.

$m^2 + 2m - 3 = 0 \quad \Rightarrow \quad (m+3)(m-1) = 0 \quad \Rightarrow \quad m = -3,\ m = 1$

Hence the CF is $y = Ae^{-3x} + Be^{x}$.

PI Let $y = ax + b \quad \Rightarrow \quad \dfrac{dy}{dx} = a$ and $\dfrac{d^2y}{dx^2} = 0$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x - 1 \quad \Rightarrow \quad 0 + 2(a) - 3(ax + b) = 6x - 1$$
$$\Rightarrow \quad 2a - 3ax - 3b = 6x - 1$$

Equating coefficients of $x \quad \Rightarrow \quad -3a = 6 \quad \Rightarrow \quad a = -2$

Equating constants $\Rightarrow \quad 2a - 3b = -1$
$\Rightarrow \quad 2(-2) - 3b = -1$
$\Rightarrow \quad -4 - 3b = -1$
$\Rightarrow \quad -3b = 3$
$\Rightarrow \quad b = -1$

Hence the PI is $y = -2x - 1$.

GS The general solution is $y = Ae^{-3x} + Be^{x} - 2x - 1$.

6. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2x^2$

CF Auxiliary Equation: $m^2 - 3m + 2 = 0$

$a = 1,\ b = -3,\ c = 2 \quad \Rightarrow \quad b^2 - 4ac = (-3)^2 - 4(1)(2) = 1$

$b^2 - 4ac > 0$, so the equation has two real and distinct roots.

$m^2 - 3m + 2 = 0 \quad \Rightarrow \quad (m-1)(m-2) = 0 \quad \Rightarrow \quad m = 1,\ m = 2$

Hence the CF is $y = Ae^x + Be^{2x}$.

PI Note that the RHS of the differential equation is a quadratic which suggests using $y = ax^2 + bx + c$ as the PI.

Let $y = ax^2 + bx + c \quad \Rightarrow \quad \dfrac{dy}{dx} = 2ax + b$ and $\dfrac{d^2y}{dx^2} = 2a$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 \quad \Rightarrow \quad 2a - 3(2ax + b) + 2(ax^2 + bx + c) = 2x^2$$

$$\Rightarrow \quad 2a - 6ax - 3b + 2ax^2 + 2bx + 2c = 2x^2$$

Equating coefficients of x^2 $\quad \Rightarrow \quad 2a = 2 \quad \Rightarrow \quad a = 1$

Equating coefficients of x $\quad \Rightarrow \quad -6a + 2b = 0$

$\Rightarrow \quad -6(1) + 2b = 0$

$\Rightarrow \quad -6 + 2b = 0$

$\Rightarrow \quad 2b = 6$

$\Rightarrow \quad b = 3$

Equating constants $\quad \Rightarrow \quad 2a - 3b + 2c = 0$

$\Rightarrow \quad 2(1) - 3(3) + 2c = 0$

$\Rightarrow \quad 2 - 9 + 2c = 0$

$\Rightarrow \quad -7 + 2c = 0 \quad \Rightarrow \quad 2c = 7 \quad \Rightarrow \quad c = \dfrac{7}{2}$

Hence the PI is $y = x^2 + 3x + \dfrac{7}{2}$.

GS The general solution is $y = Ae^x + Be^{2x} + x^2 + 3x + \dfrac{7}{2}$.

$$y = Ae^{x} + Be^{2x} + x^2 + 3x + \frac{7}{2}$$

$$y = \frac{1}{2} \text{ when } x = 0 \quad \Rightarrow \quad \frac{1}{2} = Ae^{0} + Be^{2(0)} + 0^2 + 3(0) + \frac{7}{2}$$

$$\Rightarrow \quad \frac{1}{2} = Ae^{0} + Be^{0} + \frac{7}{2}$$

$$\Rightarrow \quad \frac{1}{2} = A(1) + B(1) + \frac{7}{2}$$

$$\Rightarrow \quad A + B = -3 \qquad ...(1)$$

$$y = Ae^{x} + Be^{2x} + x^2 + 3x + \frac{7}{2} \quad \Rightarrow \quad \frac{dy}{dx} = Ae^{x} + 2Be^{2x} + 2x + 3$$

$$\frac{dy}{dx} = 1 \text{ when } x = 0 \quad \Rightarrow \quad 1 = Ae^{0} + 2Be^{2(0)} + 2(0) + 3$$

$$\Rightarrow \quad 1 = Ae^{0} + 2Be^{0} + 3$$

$$\Rightarrow \quad 1 = A(1) + 2B(1) + 3$$

$$\Rightarrow \quad A + 2B = -2 \qquad ...(2)$$

Solving equations (1) and (2) gives $A = -4$ and $B = 1$.

Hence the particular solution is $y = -4e^{x} + e^{2x} + x^2 + 3x + \frac{7}{2}$.

<u>Note</u>

A common error in this question is to incorrectly use $y = ax^2$ as the PI.

7. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$

CF Auxiliary Equation: $m^2 - 4m + 4 = 0$

$a = 1,\ b = -4,\ c = 4 \quad \Rightarrow \quad b^2 - 4ac = (-4)^2 - 4(1)(4) = 0$

$b^2 - 4ac = 0$, so the equation has two real and equal roots.

$m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m-2)(m-2) = 0 \quad \Rightarrow \quad m = 2,\ m = 2$

Hence the CF is $y = (Ax + B)e^{2x}$.

PI Let $y = ae^x \quad \Rightarrow \quad \frac{dy}{dx} = ae^x$ and $\frac{d^2y}{dx^2} = ae^x$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x \quad \Rightarrow \quad ae^x - 4(ae^x) + 4(ae^x) = e^x$$
$$\Rightarrow \quad ae^x - 4ae^x + 4ae^x = e^x$$
$$\Rightarrow \quad ae^x = e^x$$

Equating coefficients of $e^x \quad \Rightarrow \quad a = 1$

Hence the PI is $y = e^x$.

GS The general solution is $y = (Ax + B)e^{2x} + e^x$.

$$y = 2 \text{ when } x = 0 \quad \Rightarrow \quad 2 = (A(0) + B)e^{2(0)} + e^0$$
$$\Rightarrow \quad 2 = Be^0 + e^0$$
$$\Rightarrow \quad 2 = B(1) + 1$$
$$\Rightarrow \quad B = 1$$

To differentiate $y = (Ax + B)e^{2x} + e^x$, use the **product rule** to differentiate $(Ax + B)e^{2x}$ as this term is the product of two functions of *x*. Remember to also differentiate the term e^x.

$$\frac{dy}{dx} = (Ax + B)2e^{2x} + e^{2x}(A) + e^x \quad \Rightarrow \quad \frac{dy}{dx} = 2e^{2x}(Ax + B) + Ae^{2x} + e^x$$

$$\frac{dy}{dx} = 1 \text{ when } x = 0 \quad \Rightarrow \quad 1 = 2e^{2(0)}(A(0) + B) + Ae^{2(0)} + e^0$$

$$\Rightarrow \quad 1 = 2e^0(B) + Ae^0 + e^0$$

$$\Rightarrow \quad 1 = 2(1)(B) + A(1) + 1$$

$$\Rightarrow \quad 1 = 2B + A + 1$$

$$\Rightarrow \quad 1 = 2(1) + A + 1 \qquad \text{[substituting } B = 1\text{]}$$

$$\Rightarrow \quad 1 = 3 + A$$

$$\Rightarrow \quad A = -2$$

Hence the particular solution is $y = (-2x + 1)e^{2x} + e^x$ *or* $y = (1 - 2x)e^{2x} + e^x$.

8. $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+6y=12x^2+2x-5$

CF Auxiliary Equation: $m^2+5m+6=0$

$a=1,\ b=5,\ c=6 \quad\Rightarrow\quad b^2-4ac=5^2-4(1)(6)=1$

$b^2-4ac>0$, so the equation has two real and distinct roots.

$m^2+5m+6=0 \quad\Rightarrow\quad (m+3)(m+2)=0 \quad\Rightarrow\quad m=-3,\ m=-2$

Hence the CF is $y=Ae^{-3x}+Be^{-2x}$.

PI Let $y=ax^2+bx+c \quad\Rightarrow\quad \dfrac{dy}{dx}=2ax+b$ and $\dfrac{d^2y}{dx^2}=2a$

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12x^2+2x-5 \quad\Rightarrow\quad 2a+5(2ax+b)+6(ax^2+bx+c)=12x^2+2x-5$$
$$\Rightarrow\quad 2a+10ax+5b+6ax^2+6bx+6c=12x^2+2x-5$$

Equating coefficients of x^2 $\quad\Rightarrow\quad 6a=12 \quad\Rightarrow\quad a=2$

Equating coefficients of x $\quad\Rightarrow\quad 10a+6b=2$
$\Rightarrow\quad 10(2)+6b=2$
$\Rightarrow\quad 20+6b=2$
$\Rightarrow\quad 6b=-18$
$\Rightarrow\quad b=-3$

Equating constants $\quad\Rightarrow\quad 2a+5b+6c=-5$
$\Rightarrow\quad 2(2)+5(-3)+6c=-5$
$\Rightarrow\quad 4-15+6c=-5$
$\Rightarrow\quad -11+6c=-5$
$\Rightarrow\quad 6c=6$
$\Rightarrow\quad c=1$

Hence the PI is $y=2x^2-3x+1$.

GS The general solution is $y=Ae^{-3x}+Be^{-2x}+2x^2-3x+1$.

$y = Ae^{-3x} + Be^{-2x} + 2x^2 - 3x + 1$

$$
\begin{aligned}
y = -6 \text{ when } x = 0 \quad &\Rightarrow \quad -6 = Ae^{-3(0)} + Be^{-2(0)} + 2(0)^2 - 3(0) + 1 \\
&\Rightarrow \quad -6 = Ae^{0} + Be^{0} + 1 \\
&\Rightarrow \quad -6 = A(1) + B(1) + 1 \\
&\Rightarrow \quad A + B = -7 \qquad\qquad ...(1)
\end{aligned}
$$

$$y = Ae^{-3x} + Be^{-2x} + 2x^2 - 3x + 1 \quad \Rightarrow \quad \frac{dy}{dx} = -3Ae^{-3x} - 2Be^{-2x} + 4x - 3$$

$$
\begin{aligned}
\frac{dy}{dx} = 3 \text{ when } x = 0 \quad &\Rightarrow \quad 3 = -3Ae^{-3(0)} - 2Be^{-2(0)} + 4(0) - 3 \\
&\Rightarrow \quad 3 = -3Ae^{0} - 2Be^{0} - 3 \\
&\Rightarrow \quad 3 = -3A(1) - 2B(1) - 3 \\
&\Rightarrow \quad -3A - 2B = 6 \qquad\qquad ...(2)
\end{aligned}
$$

Solving equations (1) and (2) gives $A = 8$ and $B = -15$.

Hence the particular solution is $y = 8e^{-3x} + Be^{-2x} + 2x^2 - 3x + 1$.

9. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x}$

CF Auxiliary Equation: $m^2 + 2m + 10 = 0$

$a = 1,\ b = 2,\ c = 10 \quad \Rightarrow \quad b^2 - 4ac = 2^2 - 4(1)(10) = -36$

$b^2 - 4ac < 0$, so the equation has non-real roots and the roots can be found using the quadratic formula.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{-36}}{2(1)} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

Hence the CF is $y = e^{-x}(A\sin 3x + B\cos 3x)$.

PI Let $y = ae^{2x} \quad \Rightarrow \quad \frac{dy}{dx} = 2ae^{2x}$ and $\frac{d^2y}{dx^2} = 4ae^{2x}$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x} \quad \Rightarrow \quad 4ae^{2x} + 2(2ae^{2x}) + 10(ae^{2x}) = 3e^{2x}$$

$$\Rightarrow \quad 4ae^{2x} + 4ae^{2x} + 10ae^{2x} = 3e^{2x}$$

$$\Rightarrow \quad 18ae^{2x} = 3e^{2x}$$

Equating coefficients of $e^{2x} \quad \Rightarrow \quad 18a = 3 \quad \Rightarrow \quad a = \frac{1}{6}$

Hence the PI is $y = \frac{1}{6}e^{2x}$.

GS The general solution is $y = e^{-x}(A\sin 3x + B\cos 3x) + \frac{1}{6}e^{2x}$.

When $x = 0,\ y = 1 \quad \Rightarrow \quad 1 = e^0(A\sin 3(0) + B\cos 3(0)) + \frac{1}{6}e^{2(0)}$

$$\Rightarrow \quad 1 = 1(A\sin 0 + B\cos 0) + \frac{1}{6}e^0$$

$$\Rightarrow \quad 1 = A(0) + B(1) + \frac{1}{6}(1)$$

$$\Rightarrow \quad 1 = B + \frac{1}{6}$$

$$\Rightarrow \quad B = \frac{5}{6}$$

To differentiate $y = e^{-x}(A\sin 3x + B\cos 3x) + \frac{1}{6}e^{2x}$, use the **product rule** to differentiate $e^{-x}(A\sin 3x + B\cos 3x)$ as this term is the product of two functions of *x*. Remember to also differentiate the term $\frac{1}{6}e^{2x}$.

$$\frac{dy}{dx} = e^{-x}(3A\cos 3x - 3B\sin 3x) + (A\sin 3x + B\cos 3x)(-e^{-x}) + \frac{1}{6}(2e^{2x})$$
$$\frac{dy}{dx} = 3e^{-x}(A\cos 3x - B\sin 3x) - e^{-x}(A\sin 3x + B\cos 3x) + \frac{1}{3}e^{2x}$$

When $x = 0$, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 0 = 3e^{0}(A\cos 3(0) - B\sin 3(0)) - e^{0}(A\sin 3(0) + B\cos 3(0)) + \frac{1}{3}e^{2(0)}$$

$$\Rightarrow \quad 0 = 3(1)(A\cos 0 - B\sin 0) - 1(A\sin 0 + B\cos 0) + \frac{1}{3}(1)$$

$$\Rightarrow \quad 0 = 3(A(1) - B(0)) - 1(A(0) + B(1)) + \frac{1}{3}$$

$$\Rightarrow \quad 0 = 3(A) - 1(B) + \frac{1}{3}$$

$$\Rightarrow \quad 3A - B + \frac{1}{3} = 0$$

$$\Rightarrow \quad 3A - \frac{5}{6} + \frac{1}{3} = 0 \qquad \text{[substituting } B = \tfrac{5}{6}\text{]}$$

$$\Rightarrow \quad 3A - \frac{1}{2} = 0$$

$$\Rightarrow \quad 3A = \frac{1}{2}$$

$$\Rightarrow \quad A = \frac{1}{6}$$

Hence the particular solution is $y = e^{-x}\left(\frac{1}{6}\sin 3x + \frac{5}{6}\cos 3x\right) + \frac{1}{6}e^{2x}$.

<u>Note</u>

The particular solution can also be written as $y = \frac{1}{6}e^{-x}(\sin 3x + 5\cos 3x) + \frac{1}{6}e^{2x}$

or $y = \frac{1}{6}\left(e^{-x}(\sin 3x + 5\cos 3x) + e^{2x}\right)$ but there is no need to simplify the particular solution.

10. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 8\sin x + 19\cos x$

CF Auxiliary Equation: $m^2 - 6m + 9 = 0$

$a = 1,\ b = -6,\ c = 9 \quad \Rightarrow \quad b^2 - 4ac = (-6)^2 - 4(1)(9) = 0$

$b^2 - 4ac = 0$, so the equation has two real and equal roots.

$m^2 - 6m + 9 = 0 \quad \Rightarrow \quad (m-3)(m-3) = 0 \quad \Rightarrow \quad m = 3,\ m = 3$

Hence the CF is $y = (Ax + B)e^{3x}$.

PI Let $y = a\sin x + b\cos x \quad \Rightarrow \quad \dfrac{dy}{dx} = a\cos x - b\sin x$ and $\dfrac{d^2y}{dx^2} = -a\sin x - b\cos x$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 8\sin x + 19\cos x$$

$\Rightarrow \quad -a\sin x - b\cos x - 6(a\cos x - b\sin x) + 9(a\sin x + b\cos x) = 8\sin x + 19\cos x$

$\Rightarrow \quad -a\sin x - b\cos x - 6a\cos x + 6b\sin x + 9a\sin x + 9b\cos x = 8\sin x + 19\cos x$

Equating coefficients of $\sin x \quad \Rightarrow \quad -a + 6b + 9a = 8 \quad \Rightarrow \quad 8a + 6b = 8 \qquad$ ...(1)

Equating coefficients of $\cos x \quad \Rightarrow \quad -b - 6a + 9b = 19 \quad \Rightarrow \quad -6a + 8b = 19 \qquad$ ...(2)

Solving equations (1) and (2) gives $a = -\dfrac{1}{2}$ and $b = 2$.

Hence the PI is $y = -\dfrac{1}{2}\sin x + 2\cos x$.

GS The general solution is $y = (Ax + B)e^{3x} - \dfrac{1}{2}\sin x + 2\cos x$.

$y = 7$ when $x = 0 \quad \Rightarrow \quad 7 = (A(0) + B)e^{3(0)} - \dfrac{1}{2}\sin 0 + 2\cos 0$

$\Rightarrow \quad 7 = (B)e^0 - \dfrac{1}{2}(0) + 2(1)$

$\Rightarrow \quad 7 = (B)(1) + 2$

$\Rightarrow \quad B = 5$

To differentiate $y = (Ax + B)e^{3x} - \frac{1}{2}\sin x + 2\cos x$, use the **product rule** to differentiate $(Ax + B)e^{3x}$ as this term is the product of two functions of *x*. Remember to also differentiate the other two terms.

$$\frac{dy}{dx} = (Ax + B)(3e^{3x}) + e^{3x}(A) - \frac{1}{2}\cos x - 2\sin x$$
$$\frac{dy}{dx} = 3e^{3x}(Ax + B) + Ae^{3x} - \frac{1}{2}\cos x - 2\sin x$$

$$\frac{dy}{dx} = \frac{1}{2} \text{ when } x = 0 \quad \Rightarrow \quad \frac{1}{2} = 3e^{3(0)}(A(0) + B) + Ae^{(0)} - \frac{1}{2}\cos 0 - 2\sin 0$$
$$\Rightarrow \quad \frac{1}{2} = 3e^{0}(B) + Ae^{0} - \frac{1}{2}(1) - 2(0)$$
$$\Rightarrow \quad \frac{1}{2} = 3(1)(B) + A(1) - \frac{1}{2}$$
$$\Rightarrow \quad \frac{1}{2} = 3(5) + A - \frac{1}{2} \qquad \text{[substituting } B = 5\text{]}$$
$$\Rightarrow \quad 1 = 15 + A$$
$$\Rightarrow \quad A = -14$$

Hence the particular solution is $y = (-14x + 5)e^{3x} - \frac{1}{2}\sin x + 2\cos x$

or $y = (5 - 14x)e^{3x} - \frac{1}{2}\sin x + 2\cos x$.

11. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 20\sin x$

CF Auxiliary Equation: $m^2 - 3m + 2 = 0$

$a = 1,\ b = -3,\ c = 2 \quad \Rightarrow \quad b^2 - 4ac = (-3)^2 - 4(1)(2) = 1$

$b^2 - 4ac > 0$, so the equation has two real and distinct roots.

$m^2 - 3m + 2 = 0 \quad \Rightarrow \quad (m-1)(m-2) = 0 \quad \Rightarrow \quad m = 1,\ m = 2$

Hence the CF is $y = Ae^x + Be^{2x}$.

PI Let $y = a\sin x + b\cos x \quad \Rightarrow \quad \dfrac{dy}{dx} = a\cos x - b\sin x$ and $\dfrac{d^2y}{dx^2} = -a\sin x - b\cos x$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin x$$

$\Rightarrow \quad -a\sin x - b\cos x - 3(a\cos x - b\sin x) + 2(a\sin x + b\cos x) = 20\sin x$
$\Rightarrow \quad -a\sin x - b\cos x - 3a\cos x + 3b\sin x + 2a\sin x + 2b\cos x = 20\sin x$

Equating coefficients of $\sin x \quad \Rightarrow \quad -a + 3b + 2a = 20 \quad \Rightarrow \quad a + 3b = 20$ ...(1)

Equating coefficients of $\cos x \quad \Rightarrow \quad -b - 3a + 2b = 0 \quad \Rightarrow \quad -3a + b = 0$ ...(2)

Solving equations (1) and (2) gives $a = 2$ and $b = 6$.

Hence the PI is $y = 2\sin x + 6\cos x$.

GS The general solution is $y = Ae^x + Be^{2x} + 2\sin x + 6\cos x$.

$y = 0$ when $x = 0 \quad \Rightarrow \quad 0 = Ae^0 + Be^{2(0)} + 2\sin 0 + 6\cos 0$
$\Rightarrow \quad 0 = Ae^0 + Be^0 + 2(0) + 6(1)$
$\Rightarrow \quad 0 = A(1) + B(1) + 6$
$\Rightarrow \quad A + B = -6$ ...(1)

$$y = Ae^{x} + Be^{2x} + 2\sin x + 6\cos x \quad \Rightarrow \quad \frac{dy}{dx} = Ae^{x} + 2Be^{2x} + 2\cos x - 6\sin x$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0 \quad \Rightarrow \quad 0 = Ae^{0} + 2Be^{2(0)} + 2\cos 0 - 6\sin 0$$

$$\Rightarrow \quad 0 = Ae^{0} + 2Be^{0} + 2(1) - 6(0)$$

$$\Rightarrow \quad 0 = A(1) + 2B(1) + 2$$

$$\Rightarrow \quad A + 2B = -2 \qquad ...(2)$$

Solving equations (1) and (2) gives $A = -10$ and $B = 4$.

Hence the particular solution is $y = -10e^{x} + 4e^{2x} + 2\sin x + 6\cos x$

or $y = 4e^{2x} - 10e^{x} + 2\sin x + 6\cos x$.

Note

A common error in this question is to incorrectly use $y = a\sin x$ as the PI.

12. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x + 12$

CF Auxiliary Equation: $m^2 - m - 2 = 0$

$a = 1,\ b = -1,\ c = -2 \quad \Rightarrow \quad b^2 - 4ac = (-1)^2 - 4(1)(-2) = 9$

$b^2 - 4ac > 0$, so the equation has two real and distinct roots.

$m^2 - m - 2 = 0 \quad \Rightarrow \quad (m+1)(m-2) = 0 \quad \Rightarrow \quad m = -1,\ m = 2$

Hence the CF is $y = Ae^{-x} + Be^{2x}$.

PI Let $y = ae^x + b \quad \Rightarrow \quad \frac{dy}{dx} = ae^x$ and $\frac{d^2y}{dx^2} = ae^x$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x + 12 \quad \Rightarrow \quad ae^x - ae^x - 2(ae^x + b) = e^x + 12$$
$$\Rightarrow \quad -2ae^x - 2b = e^x + 12$$

Equating coefficients of $e^x \quad \Rightarrow \quad -2a = 1 \quad \Rightarrow \quad a = -\frac{1}{2}$

Equating constants $\quad \Rightarrow \quad -2b = 12 \quad \Rightarrow \quad b = -6$

Hence the PI is $y = -\frac{1}{2}e^x - 6$.

GS The general solution is $y = Ae^{-x} + Be^{2x} - \frac{1}{2}e^x - 6$.

$y = -\frac{3}{2}$ when $x = 0 \quad \Rightarrow \quad -\frac{3}{2} = Ae^0 + Be^{2(0)} - \frac{1}{2}e^0 - 6$

$$\Rightarrow \quad -\frac{3}{2} = Ae^0 + Be^0 - \frac{1}{2}e^0 - 6$$
$$\Rightarrow \quad -\frac{3}{2} = A(1) + B(1) - \frac{1}{2}(1) - 6$$
$$\Rightarrow \quad -1 = A + B - 6$$
$$\Rightarrow \quad A + B = 5 \qquad \text{...(1)}$$

$$y = Ae^{-x} + Be^{2x} - \frac{1}{2}e^{x} - 6 \quad \Rightarrow \quad \frac{dy}{dx} = -Ae^{-x} + 2Be^{2x} - \frac{1}{2}e^{x}$$

$$\frac{dy}{dx} = \frac{1}{2} \text{ when } x = 0 \quad \Rightarrow \quad \frac{1}{2} = -Ae^{0} + 2Be^{2(0)} - \frac{1}{2}e^{0}$$

$$\Rightarrow \quad \frac{1}{2} = -Ae^{0} + 2Be^{0} - \frac{1}{2}e^{0}$$

$$\Rightarrow \quad \frac{1}{2} = -A(1) + 2B(1) - \frac{1}{2}(1)$$

$$\Rightarrow \quad -A + 2B = 1 \qquad \ldots(2)$$

Solving equations (1) and (2) gives $A = 3$ and $B = 2$.

Hence the particular solution is $y = 3e^{-x} + 2e^{2x} - \frac{1}{2}e^{x} - 6$.

13. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 4\cos x$

CF Auxiliary Equation: $m^2 + 2m + 5 = 0$

$a = 1,\ b = 2,\ c = 5 \quad \Rightarrow \quad b^2 - 4ac = 2^2 - 4(1)(5) = -16$

$b^2 - 4ac < 0$, so the equation has non-real roots and the roots can be found using the quadratic formula.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{-16}}{2(1)} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Hence the CF is $y = e^{-x}(A\sin 2x + B\cos 2x)$.

PI Let $y = a\sin x + b\cos x \quad \Rightarrow \quad \dfrac{dy}{dx} = a\cos x - b\sin x$ and $\dfrac{d^2y}{dx^2} = -a\sin x - b\cos x$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4\cos x$$

$\Rightarrow \quad -a\sin x - b\cos x + 2(a\cos x - b\sin x) + 5(a\sin x + b\cos x) = 4\cos x$

$\Rightarrow \quad -a\sin x - b\cos x + 2a\cos x - 2b\sin x + 5a\sin x + 5b\cos x = 4\cos x$

Equating coefficients of $\sin x \quad \Rightarrow \quad -a - 2b + 5a = 0 \quad \Rightarrow \quad 4a - 2b = 0$ ..(1)

Equating coefficients of $\cos x \quad \Rightarrow \quad -b + 2a + 5b = 4 \quad \Rightarrow \quad 2a + 4b = 4$ ...(2)

Solving equations (1) and (2) gives $a = \dfrac{2}{5}$ and $b = \dfrac{4}{5}$.

Hence the PI is $y = \dfrac{2}{5}\sin x + \dfrac{4}{5}\cos x$.

GS The general solution is $y = e^{-x}(A\sin 2x + B\cos 2x) + \dfrac{2}{5}\sin x + \dfrac{4}{5}\cos x$.

$y(0) = 0$, so when $x = 0$, $y = 0$ $\Rightarrow$ $0 = e^0(A\sin 2(0) + B\cos 2(0)) + \frac{2}{5}\sin 0 + \frac{4}{5}\cos 0$

$$\Rightarrow \quad 0 = 1(A\sin 0 + B\cos 0) + \frac{2}{5}\sin 0 + \frac{4}{5}\cos 0$$

$$\Rightarrow \quad 0 = A(0) + B(1) + \frac{2}{5}(0) + \frac{4}{5}(1)$$

$$\Rightarrow \quad 0 = B + \frac{4}{5}$$

$$\Rightarrow \quad B = -\frac{4}{5}$$

To differentiate $y = e^{-x}(A\sin 2x + B\cos 2x) + \frac{2}{5}\sin x + \frac{4}{5}\cos x$, use the **product rule** to differentiate $e^{-x}(A\sin 2x + B\cos 2x)$ as this term is a product of two functions of *x*. Remember to also differentiate the other two terms.

$$\frac{dy}{dx} = e^{-x}(2A\cos 2x - 2B\sin 2x) + (A\sin 2x + B\cos 2x)(-e^{-x}) + \frac{2}{5}\cos x - \frac{4}{5}\sin x$$
$$\frac{dy}{dx} = 2e^{-x}(A\cos 2x - B\sin 2x) - e^{-x}(A\sin 2x + B\cos 2x) + \frac{2}{5}\cos x - \frac{4}{5}\sin x$$

$y'(0) = 1$, so when $x = 0$, $\frac{dy}{dx} = 1$

$$\Rightarrow \quad 1 = 2e^0(A\cos 2(0) - B\sin 2(0)) - e^0(A\sin 2(0) + B\cos 2(0)) + \frac{2}{5}\cos 0 - \frac{4}{5}\sin 0$$

$$\Rightarrow \quad 1 = 2(1)(A\cos 0 - B\sin 0) - 1(A\sin 0 + B\cos 0) + \frac{2}{5}\cos 0 - \frac{4}{5}\sin 0$$

$$\Rightarrow \quad 1 = 2(A(1) - B(0)) - 1(A(0) + B(1)) + \frac{2}{5}(1) - \frac{4}{5}(0)$$

$$\Rightarrow \quad 1 = 2(A) - 1(B) + \frac{2}{5}$$

$\Rightarrow \quad 1 = 2A - \left(-\frac{4}{5}\right) + \frac{2}{5}$ [substituting $B = -\frac{4}{5}$]

$$\Rightarrow \quad 1 = 2A + \frac{6}{5}$$

$$\Rightarrow \quad 2A = -\frac{1}{5} \quad \Rightarrow \quad A = -\frac{1}{10}$$

Hence the particular solution is $y = e^{-x}\left(-\frac{1}{10}\sin 2x - \frac{4}{5}\cos 2x\right) + \frac{2}{5}\sin x + \frac{4}{5}\cos x$.

<u>**Note**</u>

A common error in this question is to incorrectly use $y = a\cos x$ as the PI.