

**Exam Questions on Further Differentiation**

1. Given  $f(x) = \sin^{-1} 3x$ , find  $f'(x)$ .
2. Differentiate  $f(x) = \cos^{-1}(3x)$ , where  $-\frac{1}{3} < x < \frac{1}{3}$ .
3. Differentiate  $y = \tan^{-1}(3x^2)$ .
4. Differentiate the function  $\tan^{-1}(2\cos x)$ .
5. Differentiate the function  $f(x) = \tan^{-1}(\sqrt{x-1})$  with respect to  $x$ .
6. Differentiate  $y = x^2 \tan^{-1} x$  with respect to  $x$ .
7. Differentiate  $y = x \tan^{-1} 2x$ .
8. Differentiate  $g(x) = \frac{\tan^{-1} 2x}{1+4x^2}$ .
9. The function  $g$  is defined by

$$g(x) = x \tan^{-1} x, \quad x \in \mathbf{R}.$$

Verify that the second derivative of  $g$  is given by

$$g''(x) = \frac{C}{(1+x^2)^2},$$

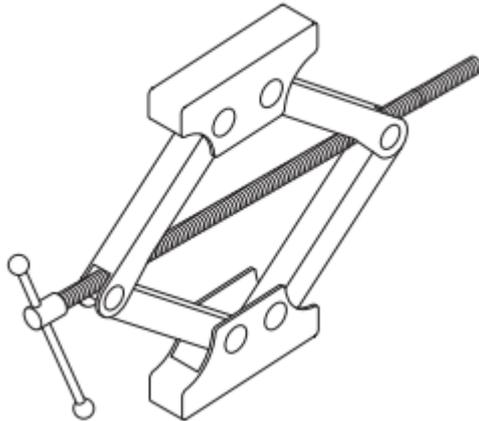
where  $C$  is a constant. State the value of  $C$ .

Explain why the graph of  $g$  has no points of inflexion.

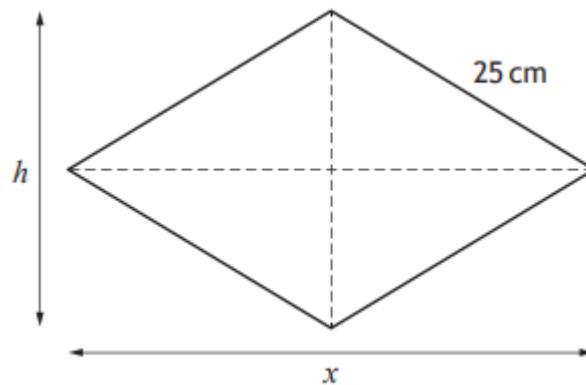
10. Obtain  $\frac{dy}{dx}$  when  $y$  is defined as a function of  $x$  by the equation
 
$$y + e^y = x^2.$$
11. For  $y \cos x + y^2 = 6x$ , use implicit differentiation to find  $\frac{dy}{dx}$ .

12. The equation  $x^4 + y^4 + 9x - 6y = 14$  defines a curve passing through the point A(1, 2). Obtain the equation of the tangent to the curve at A.
13. A curve has equation  $xy + y^2 = 2$ .
- (a) Use implicit differentiation to find  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ .
- (b) Hence find the equation of the tangent to the curve at the point (1, 1).
14. The equation  $y^3 + 3xy = 3x^2 - 5$  defines a curve passing through the point A(2, 1). Obtain the equation for the tangent to the curve at A.
15. Calculate the gradient of the curve defined by  $\frac{x^2}{y} + x = y - 5$  at the point (3, -1).
16. A curve is defined by the equation  $xy^2 + 3x^2y = 4$  for  $x > 0$  and  $y > 0$ .
- (a) Use implicit differentiation to find  $\frac{dy}{dx}$ .
- (b) Hence find the equation of the tangent to the curve where  $x = 1$ .
17. A curve is defined by  $3y^2 - x^2y = 4$ ,  $x \geq 0$ ,  $y \geq \frac{2}{\sqrt{3}}$ .  
Use implicit differentiation to find the gradient of the tangent when  $x = 2$ .
18. Given the equation  $2y^2 - 2xy - 4y + x^2 = 0$  of a curve, obtain the  $x$ -coordinate of each point at which the curve has a horizontal tangent.
19. Given  $xy - x = 4$ , use implicit differentiation to obtain  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ .  
Hence obtain  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ .
20. A curve has equation
- $$x^2 + 4xy + y^2 + 11 = 0.$$
- Find the values of  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at the point (-2, 3).
21. The height of a cube is increasing at the rate of  $5 \text{ cms}^{-1}$ .  
Find the rate of increase of the volume when the height of the cube is 3 cm.

22. An engineer has designed a lifting device. The handle turns a screw which shortens the horizontal length and increases the vertical height.



The device is modelled by a rhombus, with each side 25 cm. The horizontal length is  $x$  cm, and the vertical height is  $h$  cm as shown.



- (a) Show that  $h = \sqrt{2500 - x^2}$ .
- (b) The horizontal length decreases at a rate of  $0.3$  cm per second as the handle is turned. Find the rate of change of the vertical height when  $x = 30$ .
23. A spherical balloon is being inflated. When the radius is  $10$  cm the surface area is increasing at a rate of  $120\pi$   $\text{cm}^2\text{s}^{-1}$ .

Find the rate at which the volume is increasing at this moment.

(Volume of sphere =  $\frac{4}{3}\pi r^3$ , surface area =  $4\pi r^2$ )

24. The radius of a cylindrical column of liquid is decreasing at the rate of  $0.02 \text{ ms}^{-1}$ , while the height is increasing at the rate of  $0.01 \text{ ms}^{-1}$ .

Find the rate of change of the volume when the radius is  $0.6$  metres and the height is  $2$  metres.

[Recall that the volume of a cylinder is given by  $V = \pi r^2 h$ .]

25. Given  $y = 3^x$ , use logarithmic differentiation to obtain  $\frac{dy}{dx}$  in terms of  $x$ .

26. For  $y = 3^{x^2}$ , obtain  $\frac{dy}{dx}$ .

27. Obtain the derivative of the function  $y = 4^{(x^2+1)}$ .

28. Given  $y = x^{2x^3+1}$ , use logarithmic differentiation to find  $\frac{dy}{dx}$ .

Write your answer in terms of  $x$ .

29. A curve is defined by the equation  $y = \frac{e^{\sin x} (2+x)^3}{\sqrt{1-x}}$  for  $x < 1$ .

Use logarithmic differentiation to find  $\frac{dy}{dx}$  in terms of  $x$ .

Hence calculate the gradient of the curve when  $x = 0$ .

30. Given  $e^y = x^3 \cos^2 x$ ,  $x > 0$ , show that

$$\frac{dy}{dx} = \frac{a}{x} + b \tan x, \quad \text{for some constants } a \text{ and } b.$$

State the values of  $a$  and  $b$ .

31. Find  $\frac{dy}{dx}$  given that  $e^y = \frac{(3x+2)e^{2x}}{(2x-1)^2}$ ,  $x > \frac{1}{2}$ .

32. A curve is given by the parametric equations

$$x = 6t \quad \text{and} \quad y = 1 - \cos t.$$

Find  $\frac{dy}{dx}$  in terms of  $t$ .

33. Given  $x = 2\sec \theta$ ,  $y = 3\sin \theta$ , use parametric differentiation to find  $\frac{dy}{dx}$  in terms of  $\theta$ .
34. Given  $x = \ln(1+t^2)$ ,  $y = \ln(1+2t^2)$ , use parametric differentiation to find  $\frac{dy}{dx}$  in terms of  $t$ .
35. Given  $x = \sqrt{t+1}$  and  $y = \cot t$ ,  $0 < t < \pi$ , obtain  $\frac{dy}{dx}$  in terms of  $t$ .

36. A curve is defined parametrically by  $x = \frac{t}{t^2+1}$ ,  $y = \frac{t-1}{t^2+1}$ .

Obtain  $\frac{dy}{dx}$  as a function of  $t$ .

37. A curve is defined by the equations

$$x = 5 \cos t \quad \text{and} \quad y = 3 \sin t \quad (0 \leq t \leq 2\pi).$$

Find the gradient of the curve when  $t = \frac{\pi}{6}$ .

38. A particle moves along a curve in the  $x$ - $y$  plane. The curve is defined by the parametric equations

$$x = t^2 + 1, \quad y = 1 - 3t^3,$$

where  $t$  is the time elapsed since the start.

Find  $\frac{dy}{dx}$  in terms of  $t$  and hence obtain the equation of the tangent to the curve when  $t = 2$ .

39. A curve is defined by the equations

$$x = 5 \cos \theta, \quad y = 5 \sin \theta \quad (0 \leq \theta \leq 2\pi).$$

Use parametric differentiation to find  $\frac{dy}{dx}$  in terms of  $\theta$ .

Find the equation of the tangent to the curve at the point where  $\theta = \frac{\pi}{4}$ .

40. On a suitable domain, a curve is defined parametrically by  $x = t^2 + 1$  and  $y = \ln(3t + 2)$ .

Find the equation of the tangent to the curve where  $t = -\frac{1}{3}$ .

41. A curve is defined by the parametric equations  $x = 5t^2 - 5$ ,  $y = 3t^3$ .

Find the value of  $t$  corresponding to the point  $(0, -3)$  and calculate the gradient of the curve at this point.

42. A curve is defined by the parametric equations

$$x = t^2 + t - 1, \quad y = 2t^2 - t + 2$$

for all  $t$ . Show that the point  $A(-1, 5)$  lies on the curve and obtain the equation of the tangent to the curve at the point  $A$ .

43. The motion of a particle is defined by the equations

$$x = t(t + 4) \quad \text{and} \quad y = t(1 - t)^3$$

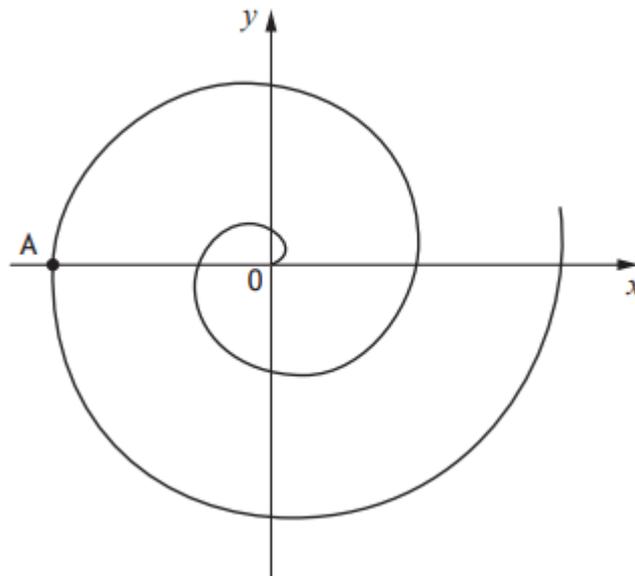
where  $t$  is the time elapsed since the start of motion.

Find the instantaneous speed of the particle when  $t = 3$ .

44. The position of a particle at time  $t$  is given by the parametric equations

$$x = t \cos t, \quad y = t \sin t, \quad t \geq 0.$$

- (a) Find an expression for the instantaneous speed of the particle at time  $t$ .  
(b) The diagram below shows the path that the particle takes.



Calculate the instantaneous speed of the particle at point  $A$ .

45. A cycloid curve is defined by the parametric equations

$$x = t - \sin t, \quad y = 1 - \cos t.$$

Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  in terms of  $t$ .

46. A curve is defined parametrically, for all  $t$ , by the equations

$$x = 2t + \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3 - 3t.$$

Obtain  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  as functions of  $t$ .

Find the values of  $t$  at which the curve has stationary points and determine their nature.

Show that the curve has exactly two points of inflexion.

47. Given  $x = \sqrt{t}$  and  $y = t^3 - \frac{5}{2}t^2$  for  $t > 0$ , use parametric differentiation to express  $\frac{dy}{dx}$  in terms of  $t$  in simplified form.

Show that  $\frac{d^2y}{dx^2} = at^2 + bt$ , determining the values of the constants  $a$  and  $b$ .

Obtain the equation of the tangent to the curve which passes through the point of inflexion.

48. A curve is defined by the parametric equations  $x = \cos 2t$ ,  $y = \sin 2t$ ,  $0 < t < \frac{\pi}{2}$ .

- (a) Use parametric differentiation to find  $\frac{dy}{dx}$ .

Hence find the equation of the tangent when  $t = \frac{\pi}{8}$ .

- (b) Obtain an expression for  $\frac{d^2y}{dx^2}$  and hence show that  $\sin 2t \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = k$ , where  $k$  is an integer. State the value of  $k$ .