Solutions to Exam Questions on Further Differentiation

1. $f(x) = \sin^{-1} 3x$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = \frac{1}{\sqrt{1-(3x)^2}} \times 3 = \frac{3}{\sqrt{1-9x^2}}$$

2. $f(x) = \cos^{-1}(3x)$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = -\frac{1}{\sqrt{1-(3x)^2}} \times 3 = -\frac{3}{\sqrt{1-9x^2}}$$

3. $y = \tan^{-1}(3x^2)$

To differentiate *y*, use the **chain rule** as *y* is a function of a function.

$$\frac{dy}{dx} = \left(\frac{1}{1+(3x^2)^2}\right) \times 6x = \frac{6x}{1+9x^4}$$

4. Let $f(x) = \tan^{-1}(2\cos x)$.

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = \left(\frac{1}{1+(2\cos x)^2}\right) \times (-2\sin x) = \frac{-2\sin x}{1+4\cos^2 x}$$

5. $f(x) = \tan^{-1}(\sqrt{x-1})$

To differentiate $f(x)$, use the **chain rule** as $f(x)$ is a function of a function.

$$f'(x) = \left(\frac{1}{1+(\sqrt{x-1})^2}\right) \times \frac{d}{dx}\sqrt{x-1}$$

To differentiate $\sqrt{x-1}$, use the chain rule as $\sqrt{x-1}$ is a function of a function.

$$\frac{d}{dx}\sqrt{x-1} = \frac{d}{dx}(x-1)^{\frac{1}{2}} = \frac{1}{2}(x-1)^{-\frac{1}{2}} \times 1 = \frac{1}{2\sqrt{x-1}}$$

Hence $$f'(x) = \left(\frac{1}{1+(x-1)}\right) \times \frac{1}{2\sqrt{x-1}}$$

$$= \frac{1}{x} \times \frac{1}{2\sqrt{x-1}}$$

$$= \frac{1}{2x\sqrt{x-1}}$$

6. $y = x^2 \tan^{-1} x$

To differentiate *y*, use the **product rule** as *y* is the product of two functions.

$$\frac{dy}{dx} = x^2\left(\frac{1}{1+x^2}\right) + \tan^{-1} x \times 2x$$

$$= \frac{x^2}{1+x^2} + 2x\tan^{-1} x$$

7. $y = x\tan^{-1} 2x$

To differentiate *y*, use the **product rule** as *y* is the product of two functions and use the **chain rule** to differentiate $\tan^{-1} 2x$.

$$\frac{d}{dx}\tan^{-1} 2x = \left(\frac{1}{1+(2x)^2}\right) \times 2 = \frac{2}{1+4x^2}$$

Hence $$\frac{dy}{dx} = x\left(\frac{2}{1+4x^2}\right) + \tan^{-1} 2x \times 1$$

$$= \frac{2x}{1+4x^2} + \tan^{-1} 2x$$

8. $g(x) = \dfrac{\tan^{-1} 2x}{1+4x^2}$

To differentiate $g(x)$, use the **quotient rule** as $g(x)$ is the quotient of two functions and use the **chain rule** to differentiate $\tan^{-1} 2x$.

$$\frac{d}{dx}\tan^{-1} 2x = \left(\frac{1}{1+(2x)^2}\right)\times 2 = \frac{2}{1+4x^2}$$

Hence $g'(x) = \dfrac{(1+4x^2)\left(\dfrac{2}{1+4x^2}\right) - \tan^{-1} 2x \times 8x}{(1+4x^2)^2}$

$$= \frac{2-8x\tan^{-1} 2x}{(1+4x^2)^2}$$

$$= \frac{2(1-4x\tan^{-1} 2x)}{(1+4x^2)^2}$$ [removing a common factor of 2 in the numerator]

9. $g(x) = x\tan^{-1} x$

To differentiate $g(x)$, use the **product rule** as $g(x)$ is the product of two functions.

$$g'(x) = x\left(\frac{1}{1+x^2}\right) + \tan^{-1} x \times 1 = \frac{x}{1+x^2} + \tan^{-1} x$$

To differentiate $g'(x)$ to find the second derivative $g''(x)$, use the quotient rule to differentiate $\dfrac{x}{1+x^2}$ and remember to also differentiate $\tan^{-1} x$.

$$g''(x) = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} + \frac{1}{1+x^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2} + \frac{1}{1+x^2}$$

$$= \frac{1-x^2}{(1+x^2)^2} + \frac{1}{1+x^2}$$

$$= \frac{1-x^2}{(1+x^2)^2} + \frac{1\times(1+x^2)}{(1+x^2)\times(1+x^2)}$$ [making a common denominator]

$$= \frac{1-x^2}{(1+x^2)^2} + \frac{1+x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2+1+x^2}{(1+x^2)^2} = \frac{2}{(1+x^2)^2} \quad\Rightarrow\quad C = 2$$

The condition for points of inflexion to occur is $g''(x) = 0$.

$g''(x) = \dfrac{2}{(1+x^2)^2}$ and $(1+x^2)^2 > 0$ for all *x*, hence $g''(x) > 0$ for all *x*.

So $g''(x)$ is never equal to zero, hence the graph of *g* has no points of inflexion.

10. $y + e^y = x^2$

Differentiate all terms in the equation with respect to *x*, using the **chain rule** to differentiate e^y as this term is a function of a function.

$$\frac{d}{dx} \quad \Rightarrow \quad \frac{dy}{dx} + e^y \frac{dy}{dx} = 2x$$

$$\Rightarrow \quad \frac{dy}{dx}(1+e^y) = 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{1+e^y}$$

11. $y\cos x + y^2 = 6x$

Differentiate all terms in the equation with respect to *x*, using the **product rule** to differentiate $y\cos x$ (as this term is a product of two functions of *x*) and the **chain rule** to differentiate y^2 (as this term is a function of a function).

$$\frac{d}{dx} \quad \Rightarrow \quad \left(y(-\sin x) + \cos x \frac{dy}{dx} \right) + 2y\frac{dy}{dx} = 6$$

$$\Rightarrow \quad -y\sin x + \cos x \frac{dy}{dx} + 2y\frac{dy}{dx} = 6$$

$$\Rightarrow \quad \cos x \frac{dy}{dx} + 2y\frac{dy}{dx} = 6 + y\sin x$$

$$\Rightarrow \quad \frac{dy}{dx}(\cos x + 2y) = 6 + y\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6 + y\sin x}{\cos x + 2y}$$

12. $x^4 + y^4 + 9x - 6y = 14$

Differentiate all terms in the equation with respect to *x*, using the **chain rule** to differentiate y^4 as this term is a function of a function.

$$\frac{d}{dx} \quad \Rightarrow \quad 4x^3 + 4y^3\frac{dy}{dx} + 9 - 6\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 4y^3\frac{dy}{dx} - 6\frac{dy}{dx} = -4x^3 - 9$$

$$\Rightarrow \quad \frac{dy}{dx}(4y^3 - 6) = -4x^3 - 9$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-4x^3 - 9}{4y^3 - 6}$$

At the point A(1, 2) where $x = 1$ and $y = 2$: $\quad m = \frac{dy}{dx} = \frac{-4(1)^3 - 9}{4(2)^3 - 6} = \frac{-13}{26} = -\frac{1}{2}$

Equation of tangent at A: $\quad y - b = m(x - a) \quad \Rightarrow \quad y - 2 = -\frac{1}{2}(x - 1)$

$$\Rightarrow \quad 2(y - 2) = -1(x - 1)$$

$$\Rightarrow \quad 2y - 4 = -x + 1$$

$$\Rightarrow \quad 2y = -x + 5 \quad or \quad 2y + x = 5$$

13.(a) $xy + y^2 = 2$

Differentiate all terms in the equation with respect to *x*, using the **product rule** to differentiate xy (as this term is a product of two functions of *x*) and the **chain rule** to differentiate y^2 (as this term is a function of a function).

$$\frac{d}{dx} \quad \Rightarrow \quad \left(x\frac{dy}{dx} + y(1)\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + 2y\frac{dy}{dx} = -y$$

$$\Rightarrow \quad \frac{dy}{dx}(x + 2y) = -y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x + 2y}$$

(b) At the point (1, 1) where $x = 1$ and $y = 1$: $m = \frac{dy}{dx} = \frac{-1}{1 + 2(1)} = -\frac{1}{3}$

Equation of tangent: $y - b = m(x - a) \quad \Rightarrow \quad y - 1 = -\frac{1}{3}(x - 1)$

$$\Rightarrow \quad 3(y - 1) = -1(x - 1)$$

$$\Rightarrow \quad 3y - 3 = -x + 1$$

$$\Rightarrow \quad 3y = -x + 4 \quad or \quad 3y + x = 4$$

14. $y^3 + 3xy = 3x^2 - 5$

Differentiate all terms in the equation with respect to *x*, using the and the **chain rule** to differentiate y^3 (as this term is a function of a function) and the **product rule** to differentiate $3xy$.

$$\frac{d}{dx} \quad \Rightarrow \quad 3y^2\frac{dy}{dx} + \left(3x\frac{dy}{dx} + y(3)\right) = 6x$$

$$\Rightarrow \quad 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 6x$$

$$\Rightarrow \quad 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} = 6x - 3y$$

$$\Rightarrow \quad \frac{dy}{dx}(3y^2 + 3x) = 6x - 3y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x - 3y}{3y^2 + 3x} = \frac{2x - y}{y^2 + x}$$ [dividing all terms in the fraction by 3 to simplify]

At the point A(2, 1) where $x = 2$ and $y = 1$: $m = \frac{dy}{dx} = \frac{2(2) - 1}{1^2 + 2} = \frac{3}{3} = 1$

Equation of tangent at A: $y - b = m(x - a) \quad \Rightarrow \quad y - 1 = 1(x - 2)$

$$\Rightarrow \quad y - 1 = x - 2$$

$$\Rightarrow \quad y = x - 1$$

15. **<u>Method 1</u>** (*Recommended*)

$$\frac{x^2}{y} + x = y - 5$$

First multiply all terms in the equation by y to remove the fraction.

$$\frac{x^2}{y} + x = y - 5 \quad \Rightarrow \quad x^2 + xy = y^2 - 5y$$

Now differentiate all terms in the equation with respect to *x*, using the **product rule** to differentiate xy (as this term is a product of two functions of *x*) and the **chain rule** to differentiate y^2 (as this term is a function of a function).

$$\frac{d}{dx} \quad \Rightarrow \quad 2x + \left(x\frac{dy}{dx} + y(1) \right) = 2y\frac{dy}{dx} - 5\frac{dy}{dx}$$

$$\Rightarrow \quad 2x + x\frac{dy}{dx} + y = 2y\frac{dy}{dx} - 5\frac{dy}{dx}$$

$$\Rightarrow \quad x\frac{dy}{dx} - 2y\frac{dy}{dx} + 5\frac{dy}{dx} = -2x - y$$

$$\Rightarrow \quad \frac{dy}{dx}(x - 2y + 5) = -2x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x - y}{x - 2y + 5}$$

At the point (3, −1) where $x = 3$ and $y = -1$: $m = \frac{dy}{dx} = \frac{-2(3) - (-1)}{3 - 2(-1) + 5} = \frac{-5}{10} = -\frac{1}{2}$

The gradient of the curve at the point (3, –1) is $-\frac{1}{2}$.

Method 2

$$\frac{x^2}{y} + x = y - 5$$

Differentiate all terms in the equation with respect to *x*, using the **quotent rule** to differentiate $\frac{x^2}{y}$ as this term is a quotient of two functions of *x*.

$$\frac{d}{dx} \quad \Rightarrow \quad \frac{y(2x) - x^2 \frac{dy}{dx}}{y^2} + 1 = \frac{dy}{dx}$$

There is no need to rearrange this equation to find an expression for $\frac{dy}{dx}$.

You can substitute the values of *x* and *y* into this equation to find the value of $\frac{dy}{dx}$.

At the point (3, −1) where $x = 3$ and $y = -1$:

$$\frac{2xy - x^2 \frac{dy}{dx}}{y^2} + 1 = \frac{dy}{dx} \quad \Rightarrow \quad \frac{2(3)(-1) - 3^2 \frac{dy}{dx}}{(-1)^2} + 1 = \frac{dy}{dx}$$

$$\Rightarrow \quad -6 - 9\frac{dy}{dx} + 1 = \frac{dy}{dx}$$

$$\Rightarrow \quad -10\frac{dy}{dx} = 5$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2}$$

Hence the gradient of the curve at the point (3, −1) is $-\frac{1}{2}$.

16.(a) $xy^2 + 3x^2y = 4$ for $x > 0$ and $y > 0$

Differentiate all terms in the equation with respect to *x*, using the **product rule** to differentiate both xy^2 and $3x^2y$ and the **chain rule** to differentiate y^2.

$$\frac{d}{dx} \Rightarrow \left(x\left(2y\frac{dy}{dx}\right) + y^2(1)\right) + \left(3x^2\frac{dy}{dx} + y(6x)\right) = 0$$

$$\Rightarrow 2xy\frac{dy}{dx} + y^2 + 3x^2\frac{dy}{dx} + 6xy = 0$$

$$\Rightarrow 2xy\frac{dy}{dx} + 3x^2\frac{dy}{dx} = -y^2 - 6xy$$

$$\Rightarrow \frac{dy}{dx}(2xy + 3x^2) = -y^2 - 6xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2 - 6xy}{2xy + 3x^2}$$

(b) To find the value of $\frac{dy}{dx}$ when $x = 1$, we need to find the value of *y* first.
To find the value of *y* when $x = 1$, substitute $x = 1$ into the equation of the curve.

When $x = 1$: $xy^2 + 3x^2y = 4 \Rightarrow 1y^2 + 3(1)^2y = 4$

$$\Rightarrow y^2 + 3y = 4$$

$$\Rightarrow y^2 + 3y - 4 = 0$$

$$\Rightarrow (y+4)(y-1) = 0$$

$$\Rightarrow y = -4,\ y = 1$$

Hence $y = 1$ since $y > 0$.

When $x = 1$ and $y = 1$: $m = \frac{dy}{dx} = \frac{-1^2 - 6(1)(1)}{2(1)(1) + 3(1)^2} = \frac{-7}{5}$

Equation of tangent: $y - b = m(x - a) \Rightarrow y - 1 = -\frac{7}{5}(x - 1)$

$$\Rightarrow 5(y-1) = -7(x-1)$$

$$\Rightarrow 5y - 5 = -7x + 7$$

$$\Rightarrow 5y = -7x + 12 \quad or \quad 5y + 7x = 12$$

17. $3y^2 - x^2 y = 4$, $x \geq 0$, $y \geq \frac{2}{\sqrt{3}}$

Differentiate all terms in the equation with respect to *x*, using the **chain rule** to differentiate $3y^2$ and the **product rule** to differentiate $x^2 y$.

$$\frac{d}{dx} \quad \Rightarrow \quad 6y\frac{dy}{dx} - \left(x^2\frac{dy}{dx} + y(2x)\right) = 0$$

$$\Rightarrow \quad 6y\frac{dy}{dx} - x^2\frac{dy}{dx} - 2xy = 0$$

$$\Rightarrow \quad 6y\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy$$

$$\Rightarrow \quad \frac{dy}{dx}(6y - x^2) = 2xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{6y - x^2}$$

To find the value of $\frac{dy}{dx}$ when $x = 2$, we need to find the value of *y* first.

To find the value of *y* when $x = 2$, substitute $x = 2$ into the equation of the curve.

When $x = 2$: $3y^2 - x^2 y = 4 \quad \Rightarrow \quad 3y^2 - 2^2 y = 4$

$$\Rightarrow \quad 3y^2 - 4y = 4$$

$$\Rightarrow \quad 3y^2 - 4y - 4 = 0$$

$$\Rightarrow \quad (3y + 2)(y - 2) = 0$$

$$\Rightarrow \quad y = -\frac{2}{3}, \; y = 2$$

Hence $y = 2$ since $y \geq \frac{2}{\sqrt{3}} = 1{\cdot}154...$

When $x = 2$ and $y = 2$: $m = \frac{dy}{dx} = \frac{2(2)(2)}{6(2) - 2^2} = \frac{8}{8} = 1$

The gradient of the tangent when $x = 2$ is 1.

18. $2y^2 - 2xy - 4y + x^2 = 0$

Differentiate all terms in the equation with respect to *x*, using the **chain rule** to differentiate $2y^2$ (as this term is a function of a function) and the **product rule** to differentiate $2xy$ (as this term is a product of two functions of *x*).

$$\frac{d}{dx} \quad \Rightarrow \quad 4y\frac{dy}{dx} - \left(2x\frac{dy}{dx} + y(2)\right) - 4\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad 4y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y - 4\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad 4y\frac{dy}{dx} - 2x\frac{dy}{dx} - 4\frac{dy}{dx} = 2y - 2x$$

$$\Rightarrow \quad \frac{dy}{dx}(4y - 2x - 4) = 2y - 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y - 2x}{4y - 2x - 4} = \frac{y - x}{2y - x - 2}$$ [dividing all terms in the fraction by 2 to simplify]

The curve has a horizontal tangent when $\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{y - x}{2y - x - 2} = 0$

$$\Rightarrow \quad y - x = 0$$

$$\Rightarrow \quad y = x$$

We know that the curve has a horizontal tangent at the points where $y = x$.
To find the values of *x*, substitute $y = x$ into the equation of the curve.

When $y = x$: $2y^2 - 2xy - 4y + x^2 = 0 \quad \Rightarrow \quad 2x^2 - 2x(x) - 4x + x^2 = 0$

$$\Rightarrow \quad 2x^2 - 2x^2 - 4x + x^2 = 0$$

$$\Rightarrow \quad x^2 - 4x = 0$$

$$\Rightarrow \quad x(x - 4) = 0$$

$$\Rightarrow \quad x = 0, \; x = 4$$

Hence the curve has a horizontal tangent at the points where $x = 0$ and $x = 4$.

19. $xy - x = 4$

Differentiate all terms in the equation with respect to *x*, using the **product rule** to differentiate xy as this term is a product of two functions of *x*.

$$\frac{d}{dx} \quad \Rightarrow \quad \left(x\frac{dy}{dx} + y(1)\right) - 1 = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + y - 1 = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} = 1 - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-y}{x}$$

To differentiate $\frac{dy}{dx}$ to find the second derivative $\frac{d^2y}{dx^2}$, use the quotient rule as $\frac{1-y}{x}$ is the quotient of two functions of *x*.

$$\frac{d^2y}{dx^2} = \frac{x\left(-\frac{dy}{dx}\right) - (1-y)1}{x^2} = \frac{-x\frac{dy}{dx} - 1 + y}{x^2}$$

To find an expression for $\frac{d^2y}{dx^2}$ in terms of *x* and *y*, replace $\frac{dy}{dx}$.

$$\frac{d^2y}{dx^2} = \frac{-x\left(\frac{1-y}{x}\right) - 1 + y}{x^2} = \frac{-(1-y) - 1 + y}{x^2} = \frac{-1 + y - 1 + y}{x^2} = \frac{2y - 2}{x^2} = \frac{2(y-1)}{x^2}$$

20. $x^2 + 4xy + y^2 + 11 = 0$

Differentiate all terms in the equation with respect to *x*, using the **product rule** to differentiate $4xy$ (as this term is a product of two functions of *x*) and the **chain rule** to differentiate y^2 (as this term is a function of a function).

$$\frac{d}{dx} \quad \Rightarrow \quad 2x + \left(4x\frac{dy}{dx} + y(4)\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2x + 4x\frac{dy}{dx} + 4y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - 4y$$

$$\Rightarrow \quad \frac{dy}{dx}(4x + 2y) = -2x - 4y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x - 4y}{4x + 2y} = \frac{-x - 2y}{2x + y}$$ [dividing all terms in the fraction by 2 to simplify]

At the point (–2, 3) where $x = -2$ and $y = 3$: $\frac{dy}{dx} = \frac{-(-2) - 2(3)}{2(-2) + 3} = \frac{-4}{-1} = 4$

To differentiate $\frac{dy}{dx}$ to find the second derivative $\frac{d^2y}{dx^2}$, use the quotient rule as $\frac{-x - 2y}{2x + y}$ is the quotient of two functions of *x*.

$$\frac{d^2y}{dx^2} = \frac{(2x + y)\left(-1 - 2\frac{dy}{dx}\right) - (-x - 2y)\left(2 + \frac{dy}{dx}\right)}{(2x + y)^2}$$

There is no need to attempt to simplify this expression, as we only need to find the **value** of $\frac{d^2y}{dx^2}$ at the point (–2, 3).

At the point (–2, 3) where $x = -2$, $y = 3$ and $\frac{dy}{dx} = 4$:

$$\frac{d^2y}{dx^2} = \frac{(2(-2) + 3)\big(-1 - 2(4)\big) - (-(-2) - 2(3))\big(2 + 4\big)}{(2(-2) + 3)^2}$$

$$= \frac{(-1)(-9) - (-4)(6)}{(-1)^2}$$

$$= \frac{9 - (-24)}{1}$$

$$= 33$$

21. The formula for the volume of the cube is $V = h^3$, where *h* is the height of the cube and *h* is a function of time *t*.

The rate of increase of the volume at time *t* is given by the derivative $\frac{dV}{dt}$.

To differentiate *V* with respect to *t*, use the **chain rule** as *V* is a function of *h* and *h* is a function of *t*.

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

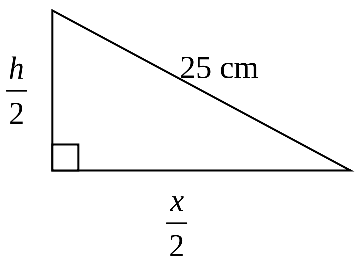
Now $V = h^3 \quad \Rightarrow \quad \frac{dV}{dh} = 3h^2$ and $\frac{dh}{dt} = 5$

Hence $\frac{dV}{dt} = 3h^2 \times 5 = 15h^2$

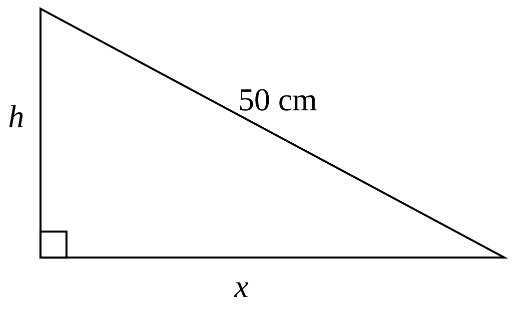
When $h = 3$: $\frac{dV}{dt} = 15(3)^2 = 135$

The rate of increase of the volume is 135 cm^3s^{-1}.

22.(a) You can find a formula for *h* in terms of *x* using Pythagoras' Theorem.
Consider one of the right-angled triangles in the diagram.



Enlarging this right-angled triangle with a scale factor of 2 gives the equivalent right-angled triangle below.



By Pythagoras' Theorem:

$$h^2 + x^2 = 50^2$$
$$\Rightarrow \quad h^2 = 2500 - x^2$$
$$\Rightarrow \quad h = \sqrt{2500 - x^2}$$

(b) $h = \sqrt{2500 - x^2}$ where x is a function of time t.

The rate of change of the vertical height at time t is given by the derivative $\frac{dh}{dt}$.

To differentiate h with respect to t, use the **chain rule** as h is a function of x and x is a function of t.

$$\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt}$$

$$h = \sqrt{2500 - x^2} = (2500 - x^2)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dh}{dx} = \frac{1}{2}(2500 - x^2)^{-\frac{1}{2}} \times (-2x)$$

$$= \frac{-x}{\sqrt{2500 - x^2}}$$

$\frac{dx}{dt} = -0 \cdot 3$ since the horizontal length x **decreases** at a rate of 0·3 cm per second.

Hence $\frac{dh}{dt} = \frac{-x}{\sqrt{2500 - x^2}} \times (-0 \cdot 3) = \frac{0 \cdot 3x}{\sqrt{2500 - x^2}}$

When $x = 30$: $\frac{dh}{dt} = \frac{0 \cdot 3(30)}{\sqrt{2500 - 30^2}} = \frac{9}{40}$

The rate of change of the vertical height when $x = 30$ is $\frac{9}{40}$ cm per second.

23. The surface area of the spherical balloon is given by $A = 4\pi r^2$, where r is the radius of the balloon and r is a function of time t.

The rate of increase of the surface area at time t is given by the derivative $\frac{dA}{dt}$.

To differentiate A with respect to t, use the **chain rule** as A is a function of r and r is a function of t.

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

Now $A = 4\pi r^2 \quad \Rightarrow \quad \frac{dA}{dr} = 8\pi r$ and hence $\frac{dA}{dt} = 8\pi r \times \frac{dr}{dt}$

When the radius is 10 cm the surface area is increasing at a rate of 120π cm^2s^{-1}.
Thus when $r = 10$, $\frac{dA}{dt} = 120\pi$.

$$\frac{dA}{dt} = 8\pi r \times \frac{dr}{dt} \quad \Rightarrow \quad 120\pi = 8\pi(10) \times \frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{120\pi}{80\pi} = \frac{3}{2}$$

This means that the radius of the balloon is increasing at the rate of $\frac{3}{2}$ cms^{-1} at this moment.

The volume of the balloon is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the balloon and r is a function of time t.

The rate of increase of the volume at time t is given by the derivative $\frac{dV}{dt}$.

To differentiate V with respect to t, use the **chain rule** as V is a function of r and r is a function of t.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Now $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dr} = 4\pi r^2$ and hence $\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$

At the moment concerned, $r = 10$ and $\frac{dr}{dt} = \frac{3}{2}$.

Hence $\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt} = 4\pi(10)^2 \times \frac{3}{2} = 600\pi$

The rate at which the volume is increasing at this moment is 600π cm^{3}s^{-1}.

24. $V = \pi r^2 h$ where both *r* and *h* are functions of time *t*.

The rate of change of the volume at time *t* is given by the derivative $\frac{dV}{dt}$.

To differentiate *V* with respect to *t*, use the **product rule** as *V* is the product of two functions of *t* and use the **chain rule** to differentiate πr^2 with respect to *t*.

$$V = \pi r^2 h \quad \Rightarrow \quad \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + h\left(2\pi r \frac{dr}{dt}\right)$$
$$= \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$$
$$= \pi r^2 (0 \cdot 01) + 2\pi r h(-0 \cdot 02)$$
$$= 0 \cdot 01\pi r^2 - 0 \cdot 04\pi r h$$

When $r = 0 \cdot 6$ and $h = 2$: $\frac{dV}{dt} = 0 \cdot 01\pi(0 \cdot 6)^2 - 0 \cdot 04\pi(0 \cdot 6)(2)$
$$= 0 \cdot 0036\pi - 0 \cdot 048\pi$$
$$= -0 \cdot 0444\pi$$

The rate of change of the volume is $-0 \cdot 0444\pi$ m^{3}s^{-1}.

25. $y = 3^x$

Take logs and simplify: $\ln y = \ln 3^x \quad \Rightarrow \quad \ln y = x \ln 3$

Now differentiate all terms in this equation with respect to *x*, using the **chain rule** to differentiate $\ln y$ as this term is a function of a function.

$$\frac{d}{dx} \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln 3 \quad \Rightarrow \quad \frac{dy}{dx} = y \ln 3 = 3^x \ln 3$$

26. $y = 3^{x^2}$

To differentiate *y*, logarithmic differentiation must be used as the power is a function of *x*.

Take logs and simplify: $\ln y = \ln 3^{x^2} \quad \Rightarrow \quad \ln y = x^2 \ln 3$

Now differentiate all terms in this equation with respect to *x*, using the **chain rule** to differentiate $\ln y$ as this term is a function of a function.

$$\frac{d}{dx} \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 2x\ln 3 \quad \Rightarrow \quad \frac{dy}{dx} = 2xy\ln 3 = 2x \times 3^{x^2} \ln 3$$

27. $y = 4^{(x^2+1)}$

To differentiate *y*, logarithmic differentiation must be used as the power is a function of *x*.

Take logs and simplify: $\ln y = \ln 4^{(x^2+1)} \quad \Rightarrow \quad \ln y = (x^2+1)\ln 4$

Now differentiate all terms in this equation with respect to *x*, using the **chain rule** to differentiate $\ln y$ as this term is a function of a function.

$$\frac{d}{dx} \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 2x\ln 4 \quad \Rightarrow \quad \frac{dy}{dx} = 2xy\ln 4 = 2x \times 4^{(x^2+1)} \ln 4$$

28. $y = x^{2x^3+1}$

Take logs and simplify: $\ln y = \ln x^{2x^3+1} \quad \Rightarrow \quad \ln y = (2x^3+1)\ln x$

Now differentiate all terms in this equation with respect to *x*, using the **chain rule** to differentiate $\ln y$ and the **product rule** to differentiate $(2x^3+1)\ln x$.

$$\frac{d}{dx} \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (2x^3+1)\frac{1}{x} + \ln x \times 6x^2$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{2x^3+1}{x} + 6x^2 \ln x$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{2x^3+1}{x} + 6x^2 \ln x\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{2x^3+1}\left(\frac{2x^3+1}{x} + 6x^2 \ln x\right)$$

29. $y = \dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$

Take logs and simplify: $\ln y = \ln\left(\dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}\right)$

$$\Rightarrow \quad \ln y = \ln(e^{\sin x}(2+x)^3) - \ln\sqrt{1-x}$$

$$\Rightarrow \quad \ln y = \ln(e^{\sin x}) + \ln(2+x)^3 - \ln(1-x)^{\frac{1}{2}}$$

$$\Rightarrow \quad \ln y = \sin x + 3\ln(2+x) - \frac{1}{2}\ln(1-x)$$

Now differentiate all terms in this equation with respect to *x*, using the **chain rule** to differentiate $\ln y$, $\ln(2+x)$ and $\ln(1-x)$.

$$\frac{d}{dx} \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \cos x + 3\left(\frac{1}{(2+x)} \times 1\right) - \frac{1}{2}\left(\frac{1}{(1-x)} \times (-1)\right)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \cos x + \frac{3}{2+x} + \frac{1}{2(1-x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\cos x + \frac{3}{2+x} + \frac{1}{2(1-x)}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}\left(\cos x + \frac{3}{2+x} + \frac{1}{2(1-x)}\right)$$

When $x = 0$: $m = \dfrac{dy}{dx} = \dfrac{e^{\sin 0}(2)^3}{\sqrt{1}}\left(\cos 0 + \dfrac{3}{2} + \dfrac{1}{2(1)}\right)$

$$= \frac{e^0(8)}{1}\left(1 + \frac{3}{2} + \frac{1}{2}\right)$$

$$= 8(3)$$

$$= 24$$

The gradient of the curve when $x = 0$ is 24.

30. **Method 1**

$e^y = x^3 \cos^2 x$

First find an expression for y: $y = \ln(x^3 \cos^2 x)$

Before differentiating *y*, use the laws of logarithms to expand and simplify the right-hand side of the above equation.

$$y = \ln(x^3 \cos^2 x) \quad \Rightarrow \quad y = \ln(x^3) + \ln(\cos^2 x)$$
$$\Rightarrow \quad y = \ln(x^3) + \ln(\cos x)^2$$
$$\Rightarrow \quad y = 3\ln x + 2\ln(\cos x)$$

Now differentiate *y*, using the **chain rule** to differentiate $\ln(\cos x)$ as this term is a function of a function.

$$\frac{dy}{dx} = 3\left(\frac{1}{x}\right) + 2\left(\frac{1}{\cos x} \times (-\sin x)\right) = \frac{3}{x} - 2\tan x \qquad \left[\text{since } \frac{\sin x}{\cos x} = \tan x\right]$$

Hence $\dfrac{dy}{dx} = \dfrac{a}{x} + b\tan x$ where $a = 3$ and $b = -2$.

Method 2

$e^y = x^3 \cos^2 x$

Use **implicit differentiation** to differentiate all terms in the equation with respect to *x*.
Use the **chain rule** to differentiate e^y and the **product rule** to differentiate $x^3 \cos^2 x$.

Note that $\dfrac{d}{dx}\cos^2 x = \dfrac{d}{dx}(\cos x)^2 = 2\cos x(-\sin x) = -2\sin x\cos x$

$$\frac{d}{dx} \quad \Rightarrow \quad e^y \frac{dy}{dx} = x^3(-2\sin x\cos x) + \cos^2 x(3x^2)$$
$$\Rightarrow \quad e^y \frac{dy}{dx} = 3x^2\cos^2 x - 2x^3 \sin x\cos x$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2\cos^2 x - 2x^3\sin x\cos x}{e^y}$$
$$= \frac{3x^2\cos^2 x - 2x^3\sin x\cos x}{x^3\cos^2 x} \qquad [\text{replacing } e^y \text{ with } x^3\cos^2 x]$$
$$= \frac{3x^2\cos^2 x}{x^3\cos^2 x} - \frac{2x^3\sin x\cos x}{x^3\cos^2 x} \qquad [\text{dividing each term in the numerator by } x^3\cos^2 x]$$

$$= \frac{3}{x} - \frac{2\sin x}{\cos x}$$

$$= \frac{3}{x} - 2\tan x \qquad [\text{since } \frac{\sin x}{\cos x} = \tan x\,]$$

Hence $\frac{dy}{dx} = \frac{a}{x} + b\tan x$ where $a = 3$ and $b = -2$.

31. $e^y = \frac{(3x+2)e^{2x}}{(2x-1)^2}$

First find an expression for y: $y = \ln\left(\frac{(3x+2)e^{2x}}{(2x-1)^2}\right)$

Before differentiating y, use the laws of logarithms to expand and simplify the right-hand side of the above equation.

$$y = \ln\left(\frac{(3x+2)e^{2x}}{(2x-1)^2}\right) \quad \Rightarrow \quad y = \ln\left((3x+2)e^{2x}\right) - \ln(2x-1)^2$$

$$\Rightarrow \quad y = \ln(3x+2) + \ln(e^{2x}) - 2\ln(2x-1)$$

$$\Rightarrow \quad y = \ln(3x+2) + 2x - 2\ln(2x-1)$$

Now differentiate y, using the **chain rule** to differentiate $\ln(3x+2)$ and $\ln(2x-1)$.

$$\frac{dy}{dx} = \frac{1}{(3x+2)} \times 3 + 2 - 2\left(\frac{1}{(2x-1)} \times 2\right) = \frac{3}{3x+2} + 2 - \frac{4}{2x-1}$$

32. $x = 6t \quad \Rightarrow \quad \frac{dx}{dt} = 6$ and $y = 1 - \cos t \quad \Rightarrow \quad \frac{dy}{dt} = \sin t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{6}$$

33. $x = 2\sec\theta \quad \Rightarrow \quad \dfrac{dx}{d\theta} = 2\sec\theta\tan\theta \quad$ and $\quad y = 3\sin\theta \quad \Rightarrow \quad \dfrac{dy}{d\theta} = 3\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta}{2\sec\theta\tan\theta}$$

Note

Although not necessary, the derivative can be simplified as follows using the fact that $\sec\theta = \dfrac{1}{\cos\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$\frac{dy}{dx} = \frac{3\cos\theta}{2\sec\theta\tan\theta} = \frac{3\cos\theta}{2\left(\dfrac{1}{\cos\theta}\right)\left(\dfrac{\sin\theta}{\cos\theta}\right)} = \frac{3\cos^3\theta}{2\sin\theta}$$

or $$\frac{dy}{dx} = \frac{3\cos^3\theta}{2\sin\theta} = \frac{3}{2}\cos^2\theta\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{3}{2}\cos^2\theta\cot\theta$$

34. $x = \ln(1+t^2)\,, \quad y = \ln(1+2t^2)$

To differentiate *x* with respect to *t*, use the **chain rule** as *x* is a function of a function. Similarly for *y*.

$$x = \ln(1+t^2) \quad \Rightarrow \quad \frac{dx}{dt} = \frac{1}{(1+t^2)} \times 2t = \frac{2t}{1+t^2}$$

$$y = \ln(1+2t^2) \quad \Rightarrow \quad \frac{dy}{dt} = \frac{1}{(1+2t^2)} \times 4t = \frac{4t}{1+2t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \left(\frac{4t}{1+2t^2}\right) \div \left(\frac{2t}{1+t^2}\right) = \left(\frac{4t}{1+2t^2}\right) \times \left(\frac{1+t^2}{2t}\right) = \frac{4t(1+t^2)}{2t(1+2t^2)} = \frac{2(1+t^2)}{1+2t^2}$$

35. $x = \sqrt{t+1} = (t+1)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}(t+1)^{-\frac{1}{2}} \times 1 = \frac{1}{2\sqrt{t+1}}$

$y = \cot t \quad \Rightarrow \quad \frac{dy}{dt} = -\operatorname{cosec}^2 t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\operatorname{cosec}^2 t}{\frac{1}{2\sqrt{t+1}}} = -2\sqrt{t+1}\operatorname{cosec}^2 t$$

36. $x = \frac{t}{t^2+1}, \quad y = \frac{t-1}{t^2+1}$

To differentiate x with respect to t, use the **quotent rule** as x is a quotient of two functions of t. Similarly for y.

$$x = \frac{t}{t^2+1} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{(t^2+1)1 - t(2t)}{(t^2+1)^2} = \frac{t^2+1-2t^2}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}$$

$$y = \frac{t-1}{t^2+1} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{(t^2+1)1-(t-1)2t}{(t^2+1)^2} = \frac{t^2+1-2t(t-1)}{(t^2+1)^2} = \frac{t^2+1-2t^2+2t}{(t^2+1)^2} = \frac{1+2t-t^2}{(t^2+1)^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \left(\frac{1+2t-t^2}{(t^2+1)^2}\right) \div \left(\frac{1-t^2}{(t^2+1)^2}\right) = \left(\frac{1+2t-t^2}{(t^2+1)^2}\right) \times \left(\frac{(t^2+1)^2}{1-t^2}\right) = \frac{1+2t-t^2}{1-t^2}$$

37. $x = 5\cos t \quad \Rightarrow \quad \frac{dx}{dt} = -5\sin t \quad$ and $\quad y = 3\sin t \quad \Rightarrow \quad \frac{dy}{dt} = 3\cos t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t}{-5\sin t} = -\frac{3}{5}\cot t \qquad \left[\text{since } \frac{\cos t}{\sin t} = \cot t\right]$$

When $t = \frac{\pi}{6}$: $\quad m = \frac{dy}{dx} = -\frac{3}{5}\cot\frac{\pi}{6} = -\frac{3}{5}(\sqrt{3}) = -\frac{3\sqrt{3}}{5}$

[Note that $\cot\frac{\pi}{6} = \frac{1}{\tan\frac{\pi}{6}} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$]

The gradient of the curve when $t = \frac{\pi}{6}$ is $-\frac{3\sqrt{3}}{5}$.

38. $x = t^2 + 1 \Rightarrow \frac{dx}{dt} = 2t$ and $y = 1 - 3t^3 \Rightarrow \frac{dy}{dt} = -9t^2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-9t^2}{2t} = \frac{-9t}{2}$$

When $t = 2$: $x = 2^2 + 1 = 5$, $y = 1 - 3(2)^3 = -23 \Rightarrow$ point = (5, –23)

$$m = \frac{dy}{dx} = \frac{-9(2)}{2} = -9$$

Equation of tangent: $y - b = m(x - a) \Rightarrow y + 23 = -9(x - 5)$

$\Rightarrow y + 23 = -9x + 45$

$\Rightarrow y = -9x + 22 \quad or \quad y + 9x = 22$

39. $x = 5\cos\theta \Rightarrow \frac{dx}{d\theta} = -5\sin\theta$ and $y = 5\sin\theta \Rightarrow \frac{dy}{d\theta} = 5\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{5\cos\theta}{-5\sin\theta} = -\cot\theta \qquad \left[\text{since } \frac{\cos\theta}{\sin\theta} = \cot\theta\right]$$

When $\theta = \frac{\pi}{4}$: $x = 5\cos\frac{\pi}{4} = 5\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$, $y = 5\sin\frac{\pi}{4} = 5\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$

$\Rightarrow$ point $= \left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$

$$m = \frac{dy}{dx} = -\cot\frac{\pi}{4} = -1 \qquad \left[\text{Note that } \cot\frac{\pi}{4} = \frac{1}{\tan\frac{\pi}{4}} = \frac{1}{1} = 1\right]$$

Equation of tangent: $y - b = m(x - a) \Rightarrow y - \frac{5}{\sqrt{2}} = -1\left(x - \frac{5}{\sqrt{2}}\right)$

$\Rightarrow y - \frac{5}{\sqrt{2}} = -x + \frac{5}{\sqrt{2}}$

$\Rightarrow y = -x + \frac{10}{\sqrt{2}}$

$\Rightarrow y + x = \frac{10}{\sqrt{2}}$

$\Rightarrow \sqrt{2}y + \sqrt{2}x = 10$ (or equivalent)

40. $x = t^2 + 1 \quad \Rightarrow \quad \dfrac{dx}{dt} = 2t$

To differentiate y with respect to t, use the chain rule as $\ln(3t+2)$ is a function of a function.

$$y = \ln(3t+2) \quad \Rightarrow \quad \frac{dy}{dt} = \frac{1}{(3t+2)} \times 3 = \frac{3}{3t+2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{3}{3t+2}}{2t} = \frac{3}{2t(3t+2)}$$

When $t = -\dfrac{1}{3}$: $\quad x = \left(-\dfrac{1}{3}\right)^2 + 1 = \dfrac{1}{9} + 1 = \dfrac{10}{9}, \quad y = \ln\left(3\left(-\dfrac{1}{3}\right) + 2\right) = \ln 1 = 0$

$$\Rightarrow \quad \text{point} = \left(\frac{10}{9}, 0\right)$$

$$m = \frac{dy}{dx} = \frac{3}{2\left(-\frac{1}{3}\right)\left(3\left(-\frac{1}{3}\right)+2\right)} = \frac{3}{2\left(-\frac{1}{3}\right)(1)} = \frac{3}{-2/3} = -\frac{9}{2}$$

Equation of tangent: $\quad y - b = m(x-a) \quad \Rightarrow \quad y - 0 = -\dfrac{9}{2}\left(x - \dfrac{10}{9}\right)$

$$\Rightarrow \quad y = -\frac{9}{2}x + 5$$

$$\Rightarrow \quad 2y = -9x + 10 \quad \textit{or} \quad 2y + 9x = 10$$

41. $x = 5t^2 - 5, \quad y = 3t^3$

At the point (0, −3), $x = 0$ and $y = -3$.

When $x = 0$: $\quad 5t^2 - 5 = 0 \quad \Rightarrow \quad 5t^2 = 5 \quad \Rightarrow \quad t^2 = 1 \quad \Rightarrow \quad t = \pm 1$

When $y = -3$: $\quad 3t^3 = -3 \quad \Rightarrow \quad t^3 = -1 \quad \Rightarrow \quad t = -1$

Hence $t = -1$ gives $x = 0$ and $y = -3$, thus giving the point (0, −3).

$$x = 5t^2 - 5 \quad \Rightarrow \quad \frac{dx}{dt} = 10t \qquad \text{and} \qquad y = 3t^3 \quad \Rightarrow \quad \frac{dy}{dt} = 9t^2$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2}{10t} = \frac{9t}{10}$$

When $t = -1$: $\quad m = \frac{dy}{dx} = \frac{9(-1)}{10} = -\frac{9}{10}$

The gradient of the curve at the point (0, −3) is $-\frac{9}{10}$.

41. $x = t^2 + t - 1, \quad y = 2t^2 - t + 2$

At the point A(–1, 5), $x = -1$ and $y = 5$.

To show that the point A(–1, 5) lies on the curve, we must find a value of t which gives both $x = -1$ and $y = 5$.

When $x = -1$: $t^2 + t - 1 = -1 \Rightarrow t^2 + t = 0 \Rightarrow t(t+1) = 0 \Rightarrow t = 0,\ t = -1$

When $y = 5$: $2t^2 - t + 2 = 5 \Rightarrow 2t^2 - t - 3 = 0$

$$\Rightarrow (2t-3)(t+1) = 0$$

$$\Rightarrow t = \frac{3}{2},\ t = -1$$

Hence $t = -1$ gives $x = -1$ and $y = 5$, thus giving the point A(–1, 5).
So the point A(–1, 5) lies on the curve.

$x = t^2 + t - 1 \Rightarrow \dfrac{dx}{dt} = 2t + 1$ and $y = 2t^2 - t + 2 \Rightarrow \dfrac{dy}{dt} = 4t - 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-1}{2t+1}$$

When $t = -1$: $m = \dfrac{dy}{dx} = \dfrac{4(-1)-1}{2(-1)+1} = \dfrac{-5}{-1} = 5$

Equation of tangent at the point A(–1, 5): $y - b = m(x - a) \Rightarrow y - 5 = 5(x+1)$

$$\Rightarrow y - 5 = 5x + 5$$

$$\Rightarrow y = 5x + 10$$

43. The instantaneous speed, v, of the particle at time t is given by $v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

$$x = t(t+4) = t^2 + 4t \quad \Rightarrow \quad \frac{dx}{dt} = 2t + 4$$

To differentiate $y = t(1-t)^3$ with respect to t, use the **product rule** as y is the product of two functions of t and use the **chain rule** to differentiate $(1-t)^3$.

$$\frac{d}{dt}(1-t)^3 = 3(1-t)^2 \times (-1) = -3(1-t)^2$$

Hence $\frac{dy}{dt} = t\left(-3(1-t)^2\right) + (1-t)^3(1)$

$$= -3t(1-t)^2 + (1-t)^3$$
$$= (1-t)^3 - 3t(1-t)^2$$

When $t = 3$: $\frac{dx}{dt} = 2(3) + 4 = 10$, $\frac{dy}{dt} = (-2)^3 - 3(3)(-2)^2 = -44$

and $v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{10^2 + (-44)^2} = \sqrt{2036}$

The instantaneous speed of the particle when $t = 3$ is $\sqrt{2036}$.

<u>Notes</u>

(1) No units are given in this question.

(2) Although not necessary, the expression for $\frac{dy}{dt}$ can be simplified by factorising as follows.

$$\frac{dy}{dt} = (1-t)^3 - 3t(1-t)^2$$
$$= (1-t)^2\left((1-t) - 3t\right) \qquad \text{[removing a common factor of } (1-t)^2\text{]}$$
$$= (1-t)^2(1-4t)$$

44.(a) The instantaneous speed, v, of the particle at time t is given by $v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

$x = t\cos t$, $y = t\sin t$

To differentiate x with respect to t, use the **product rule** as x is a product of two functions of t. Similarly for y.

$$x = t\cos t \quad \Rightarrow \quad \frac{dx}{dt} = t(-\sin t) + \cos t(1) = -t\sin t + \cos t = \cos t - t\sin t$$

$$y = t\sin t \quad \Rightarrow \quad \frac{dy}{dt} = t\cos t + \sin t(1) = t\cos t + \sin t$$

Hence $v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}$

(b) To calculate the instantaneous speed of the particle at point A, we need to find the value of t at point A.

Point A is on the x-axis, so point A occurs when $y = 0$.

When $y = 0$: $y = t\sin t \quad \Rightarrow \quad t\sin t = 0$

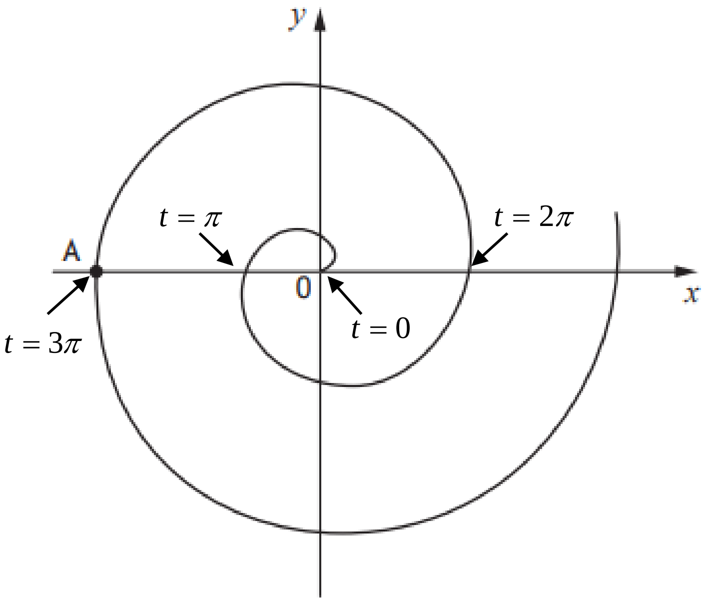
$$\Rightarrow \quad t = 0 \text{ or } \sin t = 0$$

$$\Rightarrow \quad t = 0 \text{ or } t = 0,\ \pi,\ 2\pi,\ 3\pi,\ ...$$

$$\Rightarrow \quad t = 0,\ \pi,\ 2\pi,\ 3\pi,\ ...$$

The particle starts from O at time $t = 0$ and the above solutions indicate the times when the particle crosses the x-axis.

Tracing the spiralling path of the particle anticlockwise from O shows that the point A occurs when $t = 3\pi$ (see diagram below).



When $t = 3\pi$: $v = \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}$

$$= \sqrt{(\cos 3\pi - 3\pi\sin 3\pi)^2 + (3\pi\cos 3\pi + \sin 3\pi)^2}$$

$$= \sqrt{\left(-1 - 3\pi(0)\right)^2 + \left(3\pi(-1) + 0\right)^2}$$

$$= \sqrt{(-1)^2 + (-3\pi)^2}$$

$$= \sqrt{1 + 9\pi^2}$$

The instantaneous speed of the particle at point A is $\sqrt{1 + 9\pi^2}$.

Notes

(1) No units are given in this question.

(2) Although not necessary, the expression for the instantaneous speed of the particle in (a) can be simplified as follows.

$$v = \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}$$

$$= \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t}$$

$$= \sqrt{\cos^2 t + t^2\sin^2 t + t^2\cos^2 t + \sin^2 t} \qquad [\text{since } -2t\sin t\cos t + 2t\sin t\cos t = 0]$$

$$= \sqrt{(\sin^2 t + \sin^2 t) + t^2(\sin^2 t + \cos^2 t)}$$

$$= \sqrt{1 + t^2} \qquad [\text{since } \sin^2 t + \cos^2 t = 1]$$

Then in (b), when $t = 3\pi$: $v = \sqrt{1 + t^2} = \sqrt{1 + (3\pi)^2} = \sqrt{1 + 9\pi^2}$

45. $x = t - \sin t \;\Rightarrow\; \dfrac{dx}{dt} = 1 - \cos t$ and $y = 1 - \cos t \;\Rightarrow\; \dfrac{dy}{dt} = \sin t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}$$

To differentiate $\dfrac{dy}{dx}$ with respect to x to find the second derivative $\dfrac{d^2y}{dx^2}$, you must use the chain rule as $\dfrac{dy}{dx}$ is a function of t and t is a function of x.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\sin t}{1 - \cos t}\right)$$

$$= \frac{(1 - \cos t)\cos t - \sin t(\sin t)}{(1 - \cos t)^2} \qquad \text{[using the quotient rule]}$$

$$= \frac{\cos t(1 - \cos t) - \sin^2 t}{(1 - \cos t)^2}$$

$$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2}$$

$$= \frac{\cos t - (\cos^2 t + \sin^2 t)}{(1 - \cos t)^2}$$

$$= \frac{\cos t - 1}{(1 - \cos t)^2} \qquad \text{[since } \sin^2 t + \cos^2 t = 1\text{]}$$

$$= \frac{-(1 - \cos t)}{(1 - \cos t)(1 - \cos t)}$$

$$= \frac{-1}{1 - \cos t} \qquad \text{[cancelling the factor of } (1 - \cos t)\text{]}$$

$$\frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{1 - \cos t}$$

Hence $\dfrac{d^2y}{dx^2} = \left(\dfrac{-1}{1 - \cos t}\right) \times \left(\dfrac{1}{1 - \cos t}\right) = \dfrac{-1}{(1 - \cos t)^2}$

46. $x = 2t + \frac{1}{2}t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2 + t$ and $y = \frac{1}{3}t^3 - 3t \quad \Rightarrow \quad \frac{dy}{dt} = t^2 - 3$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2 - 3}{2 + t} = \frac{t^2 - 3}{t + 2}$$

To differentiate $\frac{dy}{dx}$ with respect to x to find the second derivative $\frac{d^2y}{dx^2}$, you must use the chain rule as $\frac{dy}{dx}$ is a function of t and t is a function of x.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{t^2 - 3}{t + 2}\right)$$

$$= \frac{(t+2)2t - (t^2 - 3)1}{(t+2)^2} \qquad \text{[using the quotient rule]}$$

$$= \frac{2t(t+2) - (t^2 - 3)}{(t+2)^2}$$

$$= \frac{2t^2 + 4t - t^2 + 3}{(t+2)^2}$$

$$= \frac{t^2 + 4t + 3}{(t+2)^2}$$

$$\frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{2 + t} = \frac{1}{t + 2}$$

Hence $\frac{d^2y}{dx^2} = \left(\frac{t^2 + 4t + 3}{(t+2)^2}\right) \times \left(\frac{1}{t+2}\right) = \frac{t^2 + 4t + 3}{(t+2)^3}$ *or* $\frac{d^2y}{dx^2} = \frac{(t+1)(t+3)}{(t+2)^3}$

Stationary points occur when $\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{t^2 - 3}{t + 2} = 0$

$$\Rightarrow \quad t^2 - 3 = 0$$

$$\Rightarrow \quad t^2 = 3$$

$$\Rightarrow \quad t = \pm\sqrt{3}$$

To determine the nature of each stationary point, use the second derivative test.

When $t = -\sqrt{3}$: $\dfrac{d^2y}{dx^2} = \dfrac{(-\sqrt{3})^2 + 4(-\sqrt{3}) + 3}{(-\sqrt{3}+2)^3} = \dfrac{3 - 4\sqrt{3} + 3}{(2-\sqrt{3})^3} = \dfrac{6 - 4\sqrt{3}}{(2-\sqrt{3})^3} = -48 \cdot 2487... < 0$

Hence $t = -\sqrt{3}$ gives a maximum turning point.

When $t = \sqrt{3}$: $\dfrac{d^2y}{dx^2} = \dfrac{(\sqrt{3})^2 + 4(\sqrt{3}) + 3}{(\sqrt{3}+2)^3} = \dfrac{3 + 4\sqrt{3} + 3}{(2+\sqrt{3})^3} = \dfrac{6 + 4\sqrt{3}}{(2+\sqrt{3})^3} = 0 \cdot 2487... > 0$

Hence $t = \sqrt{3}$ gives a minimum turning point.

Points of inflexion occur when $\dfrac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \dfrac{t^2 + 4t + 3}{(t+2)^3} = 0$

$$\Rightarrow \quad t^2 + 4t + 3 = 0$$

$$\Rightarrow \quad (t+1)(t+3) = 0$$

$$\Rightarrow \quad t = -1,\ t = -3$$

There are two distinct values of *t*, hence the curve has exactly two points of inflexion.

47. $x = \sqrt{t} = t^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$ and $y = t^3 - \frac{5}{2}t^2 \quad \Rightarrow \quad \frac{dy}{dt} = 3t^2 - 5t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 5t}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}(3t^2 - 5t) = 2t^{\frac{1}{2}}(3t^2 - 5t) = 6t^{\frac{5}{2}} - 10t^{\frac{3}{2}}$$

To differentiate $\frac{dy}{dx}$ with respect to x to find the second derivative $\frac{d^2y}{dx^2}$, you must use the chain rule as $\frac{dy}{dx}$ is a function of t and t is a function of x.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(6t^{\frac{5}{2}} - 10t^{\frac{3}{2}}\right) = 15t^{\frac{3}{2}} - 15t^{\frac{1}{2}}$$

$$\frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}$$

Hence $\frac{d^2y}{dx^2} = \left(15t^{\frac{3}{2}} - 15t^{\frac{1}{2}}\right) \times 2\sqrt{t} = 2t^{\frac{1}{2}}\left(15t^{\frac{3}{2}} - 15t^{\frac{1}{2}}\right) = 30t^2 - 30t$

Hence $\frac{d^2y}{dx^2} = at^2 + bt$ where $a = 30$ and $b = -30$.

Points of inflexion occur when $\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad 30t^2 - 30t = 0$

$$\Rightarrow \quad 30t(t-1) = 0$$
$$\Rightarrow \quad t = 0,\ t = 1$$
$$\Rightarrow \quad t = 1 \text{ (since } t > 0\text{)}$$

The point of inflexion occurs when $t = 1$.

When $t = 1$: $x = \sqrt{1} = 1$, $y = 1^3 - \frac{5}{2}(1)^2 = 1 - \frac{5}{2} = -\frac{3}{2} \quad \Rightarrow \quad \text{point} = \left(1, -\frac{3}{2}\right)$

$$m = \frac{dy}{dx} = 6(1)^{\frac{5}{2}} - 10(1)^{\frac{3}{2}} = 6 - 10 = -4$$

Equation of tangent: $y - b = m(x - a)$

$$\Rightarrow \quad y + \frac{3}{2} = -4(x - 1)$$

$$\Rightarrow \quad y + \frac{3}{2} = -4x + 4$$

$$\Rightarrow \quad 2y + 3 = -8x + 8$$

$$\Rightarrow \quad 2y = -8x + 5 \quad or \quad 2y + 8x = 5$$

48.(a) $x = \cos 2t \Rightarrow \dfrac{dx}{dt} = -2\sin 2t$ and $y = \sin 2t \Rightarrow \dfrac{dy}{dt} = 2\cos 2t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos 2t}{-2\sin 2t} = -\cot 2t \qquad [\text{since } \frac{\cos 2t}{\sin 2t} = \cot 2t\,]$$

When $t = \dfrac{\pi}{8}$: $\quad x = \cos 2\left(\dfrac{\pi}{8}\right) = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}, \quad y = \sin 2\left(\dfrac{\pi}{8}\right) = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$$\Rightarrow \text{point} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$m = \frac{dy}{dx} = -\cot 2\left(\frac{\pi}{4}\right) = -\cot\frac{\pi}{4} = -1 \qquad [\text{Note that } \cot\frac{\pi}{4} = \frac{1}{\tan\frac{\pi}{4}} = \frac{1}{1} = 1]$$

Equation of tangent: $\quad y - b = m(x - a) \Rightarrow y - \dfrac{1}{\sqrt{2}} = -1\left(x - \dfrac{1}{\sqrt{2}}\right)$

$$\Rightarrow y - \frac{1}{\sqrt{2}} = -x + \frac{1}{\sqrt{2}}$$

$$\Rightarrow y = -x + \frac{2}{\sqrt{2}}$$

$$\Rightarrow y + x = \frac{2}{\sqrt{2}}$$

$$\Rightarrow \sqrt{2}y + \sqrt{2}x = 2 \quad \text{(or equivalent)}$$

(b) To differentiate $\dfrac{dy}{dx}$ with respect to x to find the second derivative $\dfrac{d^2y}{dx^2}$, you must use the chain rule as $\dfrac{dy}{dx}$ is a function of t and t is a function of x.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(-\cot 2t) = -(-2\operatorname{cosec}^2 2t) = 2\operatorname{cosec}^2 2t$$

$$\frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{-2\sin 2t} = -\frac{1}{2}\operatorname{cosec} 2t \qquad [\text{since } \operatorname{cosec} 2t = \frac{1}{\sin 2t}\,]$$

Hence $\quad \dfrac{d^2y}{dx^2} = 2\,\text{cosec}^2 2t \times \left(-\dfrac{1}{2}\text{cos}ec2t\right) = -\,\text{cosec}^3 2t$

$$\sin 2t\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \sin 2t(-\,\text{cosec}^3 2t) + (-\cot 2t)^2$$

$$= \sin 2t\left(-\frac{1}{\sin^3 2t}\right) + \cot^2 2t$$

$$= -\frac{1}{\sin^2 2t} + \frac{\cos^2 2t}{\sin^2 2t} \qquad [\text{since } \cot 2t = \frac{\cos 2t}{\sin 2t}]$$

$$= \frac{-1+\cos^2 2t}{\sin^2 2t}$$

$$= \frac{-1+(1-\sin^2 2t)}{\sin^2 2t} \qquad [\text{since } \cos^2 2t = 1-\sin^2 2t\,]$$

$$= \frac{-1+1-\sin^2 2t}{\sin^2 2t}$$

$$= \frac{-\sin^2 2t}{\sin^2 2t}$$

$$= -1$$

Hence $\quad \sin 2t\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = k \quad$ where $k = -1$.