ADVANCED HIGHER MATHEMATICS

Exam Questions on Further Integration

1. Evaluate $\int_0^2 \frac{1}{4+x^2}\,dx$.

2. Find $\int \frac{2}{\sqrt{9-16x^2}}\,dx$.

3. Use the substitution $u = 7x^2$ to obtain $\int \frac{x}{\sqrt{1-49x^4}}\,dx$.

4. Use the substitution $t = x^4$ to obtain $\int \frac{x^3}{1+x^8}\,dx$.

5. Use the substitution $u = 5x^2$ to find the exact value of $\int_0^{\frac{1}{\sqrt{10}}} \frac{x}{\sqrt{1-25x^4}}\,dx$.

6. By using the substitution $u = 2\sin x$, evaluate the definite integral

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+4\sin^2 x}\,dx.$$

7. Express $\frac{5x-4}{x^2-4x}$ in partial fractions.

Hence find the indefinite integral $\int \frac{5x-4}{x^2-4x}\,dx$.

8. Use partial fractions to find $\int \frac{3x-7}{x^2-2x-15}\,dx$.

9. Express $\frac{11-2x}{x^2+x-2}$ in partial fractions

Hence evaluate $\int_3^5 \frac{11-2x}{x^2+x-2}\,dx$.

10. Express $\dfrac{1}{x^2 - x - 6}$ in partial fractions.

Hence evaluate $\displaystyle\int_0^1 \frac{1}{x^2 - x - 6}\,dx$.

11. Express $\dfrac{3x+32}{(x+4)(6-x)}$ in partial fractions and hence evaluate

$$\int_3^4 \frac{3x+32}{(x+4)(6-x)}\,dx.$$

Give your answer in the form $\ln\left(\dfrac{p}{q}\right)$.

12. (a) Express $\dfrac{1}{1-y^2}$ in partial fractions.

(b) Use the substitution $u = \sqrt{1-x}$ to obtain $\displaystyle\int \frac{dx}{x\sqrt{1-x}}$, $0 < x < 1$.

13. Express $\dfrac{8}{x(x+2)(x+4)}$ in partial fractions.

Calculate the area under the curve

$$y = \frac{8}{x^3 + 6x^2 + 8x}$$

between $x = 1$ and $x = 2$. Express your answer in the form $\ln\dfrac{a}{b}$, where a and b are positive integers.

14. (a) Find partial fractions for $\dfrac{13+6x+5x^2}{(1+x)(2-x)(3+x)}$.

(b) Show that $\displaystyle\int_0^1 \frac{13+6x+5x^2}{(1+x)(2-x)(3+x)}\,dx = \ln\frac{a}{b}$ where a and b are positive integers.

15. Express $\dfrac{2x^2-9x-6}{x(x^2-x-6)}$ in partial fractions.

Given that

$$\int_4^6 \frac{2x^2-9x-6}{x(x^2-x-6)}\,dx = \ln\frac{m}{n},$$

determine values for the integers *m* and *n*.

16. (a) Find partial fractions for $\dfrac{4}{x^2-4}$.

(b) By using (a) obtain $\displaystyle\int \frac{x^2}{x^2-4}\,dx$.

17. (a) Obtain partial fractions for $\dfrac{x}{x^2-1}$.

(b) Use the result of (a) to find $\displaystyle\int \frac{x^3}{x^2-1}\,dx$.

18. Use partial fractions to find the exact value of $\displaystyle\int_1^2 \frac{x+4}{(x+1)^2(2x-1)}\,dx$.

19. (a) Find a real root of the cubic polynomial

$$c(x) = x^3 - x^2 - x - 2$$

and hence factorise it as the product of a linear term $l(x)$ and a quadratic term $q(x)$.

(b) Show that $c(x)$ cannot be written as the product of three linear factors with real coefficients.

(c) Use your factorisation to find values of *A*, *B* and *C* such that

$$\frac{5x+4}{x^3-x^2-x-2} = \frac{A}{l(x)} + \frac{Bx+C}{q(x)}.$$

Hence obtain the indefinite integral

$$\int \frac{5x+4}{x^3-x^2-x-2}\,dx.$$

20. Express $\dfrac{12x^2+20}{x(x^2+5)}$ in partial fractions.

Hence find $\displaystyle\int \frac{12x^2+20}{x(x^2+5)}dx$ and evaluate $\displaystyle\int_1^2 \frac{12x^2+20}{x(x^2+5)}dx$.

21. Express $\dfrac{x^2+3}{x(1+x^2)}$ in partial fractions.

Hence obtain $\displaystyle\int_{\frac{1}{2}}^{1} \frac{x^2+3}{x(1+x^2)}dx$.

22. Express $\dfrac{1}{x^3+x}$ in partial fractions.

Obtain a formula for $I(k)$, where $I(k)=\displaystyle\int_1^k \frac{1}{x^3+x}dx$, expressing it in the form $\ln\left(\dfrac{a}{b}\right)$, where *a* and *b* are expressions in terms of *k*.

Write down an expression for $e^{I(k)}$ and obtain the limiting value of $e^{I(k)}$ as $k\to\infty$.

23. Find $\displaystyle\int \frac{2x^3-x-1}{(x-3)(x^2+1)}dx$, $x>3$.

24. Use integration by parts to find $\int x\sin 3x\,dx$.

25. Use integration by parts to evaluate $\displaystyle\int_0^{\pi} x\sin x\,dx$.

26. Find the value of $\displaystyle\int_0^{\frac{\pi}{4}} 2x\sin 4x\,dx$.

27. Find the exact value of $\displaystyle\int_0^{\frac{\pi}{6}} x\sin 3x\,dx$.

28. Use the method of integration by parts to evaluate $\displaystyle\int_0^1 xe^{-2x}dx$.

29. Use integration by parts to obtain $\int_0^3 x\sqrt{x+1}\,dx$.

30. Use integration by parts to find $\int x^2 \ln x\,dx$.

31. Use integration by parts to obtain $\int \frac{\ln x}{x^3}\,dx$ where $x > 0$.

32. Use integration by parts to find the exact value of

$$\int_2^4 \ln x\,dx,$$

expressing your answer in terms of $\ln 2$.

33. Use integration by parts to find $\int x^2 \sin x\,dx$.

34. Use integration by parts to obtain $\int 8x^2 \sin 4x\,dx$.

35. Use integration by parts to obtain $\int x^2 \cos 3x\,dx$.

36. Use integration by parts to obtain $\int x^2 \sin 5x\,dx$.

37. Find $\int x^2 e^{3x}\,dx$.

38. Obtain the exact value of $\int_0^2 x^2 e^{4x}\,dx$.

39. Obtain $\int x^7 (\ln x)^2\,dx$.

40. (a) Write down the derivative of $\sin^{-1} x$.

(b) By using a substitution, or otherwise, find $\int \frac{x}{\sqrt{1-x^2}}\,dx$.

(c) Use integration by parts to obtain $\int \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}}\,dx$.

41. (a) By using a substitution, or otherwise, find $\int \frac{x}{\sqrt{1-x^2}} dx$.

(b) Use integration by parts and the result of (a) to evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x dx$.

42. Use integration by parts to obtain the exact value of $\int_0^1 x \tan^{-1}(x^2) dx$.

43. (a) Evaluate $\int_0^1 xe^{2x} dx$.

(b) Use part (a) to evaluate $\int_0^1 x^2 e^{2x} dx$.

(c) Hence obtain $\int_0^1 (3x^2 + 2x)e^{2x} dx$.

44. By integrating by parts **twice**, show that

$$\int e^x \cos x dx = \frac{1}{2} e^x (\cos x + \sin x) + C$$

where C denotes the constant of integration.

45. (a) Use integration by parts to obtain an expression for $\int e^x \cos x dx$.

(b) Similarly, given $I_n = \int e^x \cos nx dx$, where $n \neq 0$, obtain an expression for I_n.

(c) Hence evaluate $\int_0^{\frac{\pi}{2}} e^x \cos 8x dx$.

46. Define $I_n = \int_0^1 x^n e^{-x} dx$ for $n \geq 1$.

(a) Use integration by parts to obtain the value of $I_1 = \int_0^1 xe^{-x} dx$.

(b) Similarly, show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.

(c) Evaluate I_3.