ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Further Integration

1. $\int_0^2 \frac{1}{4+x^2}\,dx$ can be related to the standard integral $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$.

$$\int_0^2 \frac{1}{4+x^2}\,dx = \int_0^2 \frac{1}{2^2+x^2}\,dx = \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \left[\frac{1}{2}\tan^{-1}1\right]-\left[\frac{1}{2}\tan^{-1}0\right]$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right]-\left[\frac{1}{2}(0)\right]$$

$$= \frac{\pi}{8}$$

2. Let $I = \int \frac{2}{\sqrt{9-16x^2}}dx$.

Note that $\int \frac{2}{\sqrt{9-16x^2}}dx = \int \frac{2}{\sqrt{3^2-(4x)^2}}dx$ and this integral can be related to the standard integral $\int \frac{1}{\sqrt{a^2-u^2}}\,du$ by using the substitution $u = 4x$.

$$u = 4x \quad \Rightarrow \quad \frac{du}{dx} = 4 \quad \Rightarrow \quad du = 4dx \quad \Rightarrow \quad \frac{1}{4}du = dx$$

$$\sqrt{9-16x^2} = \sqrt{9-(4x)^2} = \sqrt{9-u^2}$$

$$I = \int \frac{2}{\sqrt{9-16x^2}}dx = \int \frac{2}{\sqrt{9-u^2}}\left(\frac{1}{4}du\right) = \frac{1}{2}\int \frac{1}{\sqrt{9-u^2}}du = \frac{1}{2}\int \frac{1}{\sqrt{3^2-u^2}}du$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{3}\right)+C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{4x}{3}\right)+C$$

3. Let $I = \int \frac{x}{\sqrt{1-49x^4}}dx$.

Using the substitution $u = 7x^2$, rewrite the entire integral in terms of *u*.

$$u = 7x^2 \quad \Rightarrow \quad \frac{du}{dx} = 14x \quad \Rightarrow \quad du = 14xdx \quad \Rightarrow \quad \frac{1}{14}du = xdx$$

$$\sqrt{1-49x^4} = \sqrt{1-(7x^2)^2} = \sqrt{1-u^2}$$

$$\begin{aligned} I = \int \frac{x}{\sqrt{1-49x^4}}dx = \int \frac{1}{\sqrt{1-49x^4}}(xdx) = \int \frac{1}{\sqrt{1-u^2}}\left(\frac{1}{14}du\right) &= \frac{1}{14}\int \frac{1}{\sqrt{1-u^2}}du \\ &= \frac{1}{14}\sin^{-1}u + C \\ &= \frac{1}{14}\sin^{-1}(7x^2) + C \end{aligned}$$

4. Let $I = \int \frac{x^3}{1+x^8}dx$.

Using the substitution $t = x^4$, rewrite the entire integral in terms of *t*.

$$t = x^4 \quad \Rightarrow \quad \frac{dt}{dx} = 4x^3 \quad \Rightarrow \quad dt = 4x^3dx \quad \Rightarrow \quad \frac{1}{4}dt = x^3dx$$

$$1+x^8 = 1+(x^4)^2 = 1+t^2$$

$$\begin{aligned} I = \int \frac{x^3}{1+x^8}dx = \int\left(\frac{1}{1+x^8}\right)x^3dx = \int\left(\frac{1}{1+t^2}\right)\frac{1}{4}dt &= \frac{1}{4}\int \frac{1}{1+t^2}dt \\ &= \frac{1}{4}\tan^{-1}t + C \\ &= \frac{1}{4}\tan^{-1}(x^4) + C \end{aligned}$$

5. Let $I = \int_0^{\frac{1}{\sqrt{10}}} \frac{x}{\sqrt{1-25x^4}}\,dx$.

Using the substitution $u = 5x^2$, rewrite the entire integral in terms of u (including the limits).

$$u = 5x^2 \quad \Rightarrow \quad \frac{du}{dx} = 10x \quad \Rightarrow \quad du = 10x\,dx \quad \Rightarrow \quad \frac{1}{10}du = x\,dx$$

$$\sqrt{1-25x^4} = \sqrt{1-(5x^2)^2} = \sqrt{1-u^2}$$

When $x = 0$: $u = 5(0)^2 = 0$ When $x = \frac{1}{\sqrt{10}}$: $u = 5\left(\frac{1}{\sqrt{10}}\right)^2 = 5\left(\frac{1}{10}\right) = \frac{1}{2}$

$$I = \int_0^{\frac{1}{\sqrt{10}}} \frac{x}{\sqrt{1-25x^4}}\,dx = \int_0^{\frac{1}{\sqrt{10}}} \frac{1}{\sqrt{1-25x^4}}(x\,dx) = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\left(\frac{1}{10}du\right)$$

$$= \left[\frac{1}{10}\sin^{-1}u\right]_0^{\frac{1}{2}}$$

$$= \left[\frac{1}{10}\sin^{-1}\left(\frac{1}{2}\right)\right] - \left[\frac{1}{10}\sin^{-1}0\right]$$

$$= \left[\frac{1}{10}\left(\frac{\pi}{6}\right)\right] - \left[\frac{1}{10}(0)\right] = \frac{\pi}{60}$$

6. Let $I = \int_0^{\frac{\pi}{6}} \frac{\cos x}{1+4\sin^2 x}\,dx$.

Using the substitution $u = 2\sin x$, rewrite the entire integral in terms of u (including the limits).

$$u = 2\sin x \quad \Rightarrow \quad \frac{du}{dx} = 2\cos x \quad \Rightarrow \quad du = 2\cos x dx \quad \Rightarrow \quad \frac{1}{2}du = \cos x dx$$

$$1+4\sin^2 x = 1+(2\sin x)^2 = 1+u^2$$

When $x = 0$: $\quad u = 2\sin 0 = 2(0) = 0$

When $x = \frac{\pi}{6}$: $\quad u = 2\sin\frac{\pi}{6} = 2\left(\frac{1}{2}\right) = 1$

$$I = \int_0^{\frac{\pi}{6}} \frac{\cos x}{1+4\sin^2 x}\,dx = \int_0^{\frac{\pi}{6}} \left(\frac{1}{1+4\sin^2 x}\right)\cos x dx = \int_0^1 \left(\frac{1}{1+u^2}\right)\frac{1}{2}du$$

$$= \left[\frac{1}{2}\tan^{-1} u\right]_0^1$$

$$= \left[\frac{1}{2}\tan^{-1} 1\right] - \left[\frac{1}{2}\tan^{-1} 0\right]$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right] - \left[\frac{1}{2}(0)\right]$$

$$= \frac{\pi}{8}$$

7. First factorise the denominator: $x^2 - 4x = x(x-4)$

The denominator contains distinct linear factors.

$$\frac{5x-4}{x^2-4x} = \frac{5x-4}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$$
$$= \frac{A(x-4)+Bx}{x(x-4)}$$

$$5x-4 = A(x-4)+Bx$$

Let $x=4 \Rightarrow 5(4)-4 = A(0)+B(4) \Rightarrow 16 = 4B \Rightarrow B = 4$

Let $x=0 \Rightarrow 5(0)-4 = A(-4)+B(0) \Rightarrow -4 = -4A \Rightarrow A = 1$

Hence $\dfrac{5x-4}{x^2-4x} = \dfrac{1}{x} + \dfrac{4}{x-4}$.

$$\int \frac{5x-4}{x^2-4x}dx = \int\left(\frac{1}{x}+\frac{4}{x-4}\right)dx = \int \frac{1}{x}dx + \int \frac{4}{x-4}dx = \int \frac{1}{x}dx + 4\int \frac{1}{x-4}dx$$
$$= \ln|x| + 4\ln|x-4| + C$$

8. First factorise the denominator: $x^2 - 2x - 15 = (x+3)(x-5)$

The denominator contains distinct linear factors.

$$\frac{3x-7}{x^2-2x-15} = \frac{3x-7}{(x+3)(x-5)} = \frac{A}{x+3} + \frac{B}{x-5}$$
$$= \frac{A(x-5)+B(x+3)}{(x+3)(x-5)}$$

$$3x-7 = A(x-5)+B(x+3)$$

Let $x=5 \Rightarrow 3(5)-7 = A(0)+B(8) \Rightarrow 8 = 8B \Rightarrow B = 1$

Let $x=-3 \Rightarrow 3(-3)-7 = A(-8)+B(0) \Rightarrow -16 = -8A \Rightarrow A = 2$

Hence $\dfrac{3x-7}{x^2-2x-15} = \dfrac{2}{x+3} + \dfrac{1}{x-5}$.

$$\int \frac{3x-7}{x^2-2x-15}dx = \int\left(\frac{2}{x+3}+\frac{1}{x-5}\right)dx = \int \frac{2}{x+3}dx + \int \frac{1}{x-5}dx$$
$$= 2\int \frac{1}{x+3}dx + \int \frac{1}{x-5}dx$$
$$= 2\ln|x+3| + \ln|x-5| + C$$

9. First factorise the denominator: $x^2 + x - 2 = (x+2)(x-1)$

The denominator contains distinct linear factors.

$$\frac{11-2x}{x^2+x-2} = \frac{11-2x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$
$$= \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$11 - 2x = A(x-1) + B(x+2)$

Let $x = 1 \quad \Rightarrow \quad 11 - 2(1) = A(0) + B(3) \quad \Rightarrow \quad 9 = 3B \quad \Rightarrow \quad B = 3$

Let $x = -2 \quad \Rightarrow \quad 11 - 2(-2) = A(-3) + B(0) \quad \Rightarrow \quad 15 = -3A \quad \Rightarrow \quad A = -5$

Hence $\dfrac{11-2x}{x^2+x-2} = \dfrac{-5}{x+2} + \dfrac{3}{x-1}$ *or* $\dfrac{11-2x}{x^2+x-2} = \dfrac{3}{x-1} - \dfrac{5}{x+2}$.

$$\int_3^5 \frac{11-2x}{x^2+x-2}\,dx = \int_3^5 \left(\frac{3}{x-1} - \frac{5}{x+2}\right)dx = \Big[3\ln|x-1| - 5\ln|x+2|\Big]_3^5$$
$$= \Big[3\ln|4| - 5\ln|7|\Big] - \Big[3\ln|2| - 5\ln|5|\Big]$$
$$= \Big[3\ln 4 - 5\ln 7\Big] - \Big[3\ln 2 - 5\ln 5\Big]$$
$$= 0\cdot 3971 \text{ (to 4 dp)}$$

Note

The exact value of the definite integral is not required. An approximate value can be given to any reasonable degree of accuracy.

10. First factorise the denominator: $x^2 - x - 6 = (x+2)(x-3)$

The denominator contains distinct linear factors.

$$\frac{1}{x^2 - x - 6} = \frac{1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

$$= \frac{A(x-3) + B(x+2)}{(x+2)(x-3)}$$

$1 = A(x-3) + B(x+2)$

Let $x = 3 \quad \Rightarrow \quad 1 = A(0) + B(5) \quad \Rightarrow \quad 1 = 5B \quad \Rightarrow \quad B = \frac{1}{5}$

Let $x = -2 \quad \Rightarrow \quad 1 = A(-5) + B(0) \quad \Rightarrow \quad 1 = -5A \quad \Rightarrow \quad A = -\frac{1}{5}$

Hence $\frac{1}{x^2 - x - 6} = \frac{-1/5}{x+2} + \frac{1/5}{x-3}$ *or* $\frac{1}{x^2 - x - 6} = \frac{1/5}{x-3} - \frac{1/5}{x+2}$.

$$\int_0^1 \frac{1}{x^2 - x - 6}\,dx = \int_0^1 \left(\frac{1/5}{x-3} - \frac{1/5}{x+2} \right) dx = \left[\frac{1}{5}\ln|x-3| - \frac{1}{5}\ln|x+2| \right]_0^1$$

$$= \left[\frac{1}{5}\ln|-2| - \frac{1}{5}\ln|3| \right] - \left[\frac{1}{5}\ln|-3| - \frac{1}{5}\ln|2| \right]$$

$$= \left[\frac{1}{5}\ln 2 - \frac{1}{5}\ln 3 \right] - \left[\frac{1}{5}\ln 3 - \frac{1}{5}\ln 2 \right]$$

$$= -0 \cdot 1622 \text{ (to 4 dp)}$$

<u>Notes</u>

(1) The partial fractions can also be written as $\frac{1}{5(x-3)} - \frac{1}{5(x+2)}$ but there is no need to do this.

(2) The inclusion of modulus signs is essential in this question.

(3) The exact value of the definite integral is not required. An approximate value can be given to any reasonable degree of accuracy.

11. The denominator contains distinct linear factors.

$$\frac{3x+32}{(x+4)(6-x)} = \frac{A}{x+4} + \frac{B}{6-x}$$
$$= \frac{A(6-x)+B(x+4)}{(x+4)(6-x)}$$

$3x+32 = A(6-x)+B(x+4)$

Let $x=6 \quad \Rightarrow \quad 3(6)+32 = A(0)+B(10) \quad \Rightarrow \quad 50 = 10B \quad \Rightarrow \quad B=5$

Let $x=-4 \quad \Rightarrow \quad 3(-4)+32 = A(10)+B(0) \quad \Rightarrow \quad 20 = 10A \quad \Rightarrow \quad A=2$

Hence $\quad \dfrac{3x+32}{(x+4)(6-x)} = \dfrac{2}{x+4} + \dfrac{5}{6-x}$.

$$\int_3^4 \frac{3x+32}{(x+4)(6-x)}\,dx = \int_3^4 \left(\frac{2}{x+4} + \frac{5}{6-x}\right)dx = \left[2\ln|x+4| - 5\ln|6-x|\right]_3^4$$
$$= \left[2\ln|8| - 5\ln|2|\right] - \left[2\ln|7| - 5\ln|3|\right]$$
$$= \left[2\ln 8 - 5\ln 2\right] - \left[2\ln 7 - 5\ln 3\right]$$
$$= \left[\ln 8^2 - \ln 2^5\right] - \left[\ln 7^2 - \ln 3^5\right]$$
$$= \left[\ln 64 - \ln 32\right] - \left[\ln 49 - \ln 243\right]$$
$$= \left[\ln\left(\frac{64}{32}\right)\right] - \left[\ln\left(\frac{49}{243}\right)\right]$$
$$= \ln 2 - \ln\left(\frac{49}{243}\right) = \ln\left(\frac{2}{49/243}\right) = \ln\left(\frac{486}{49}\right)$$

Notes

(1) $\displaystyle\int \frac{5}{6-x}\,dx = -5\ln|6-x| + C$ using $\displaystyle\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$;
remember to divide by the coefficient of *x* when integrating.

(2) There are other ways of combining the logarithms to complete the evaluation,
eg $\left[\ln 64 - \ln 32\right] - \left[\ln 49 - \ln 243\right] = \ln 64 - \ln 32 - \ln 49 + \ln 243$
$$= (\ln 64 + \ln 243) - (\ln 32 + \ln 49)$$
$$= \ln(64 \times 243) - \ln(32 \times 49)$$
$$= \ln 15552 - \ln 1568$$
$$= \ln\left(\frac{15552}{1568}\right) = \ln\left(\frac{468}{49}\right)$$

12.(a) First factorise the denominator: $1-y^2 = (1-y)(1+y)$

The denominator contains distinct linear factors.

$$\frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y}$$
$$= \frac{A(1+y)+B(1-y)}{(1-y)(1+y)}$$

$$1 = A(1+y)+B(1-y)$$

Let $y=-1 \quad \Rightarrow \quad 1 = A(0)+B(2) \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

Let $y=1 \quad \Rightarrow \quad 1 = A(2)+B(0) \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$

Hence $\dfrac{1}{1-y^2} = \dfrac{1/2}{1-y} + \dfrac{1/2}{1+y}$ *or* $\dfrac{1}{1-y^2} = \dfrac{1}{2(1-y)} + \dfrac{1}{2(1+y)}$.

<u>Note</u>

The partial fractions can also be written as $\dfrac{1}{2(1-y)} + \dfrac{1}{2(1+y)}$ but there is no need to do this.

(b) Let $I = \displaystyle\int \frac{dx}{x\sqrt{1-x}}$.

Using the substitution $u = \sqrt{1-x}$, rewrite the entire integral in terms of *u*.

$$u = \sqrt{1-x} = (1-x)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}} \times (-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{1-x}}$$
$$\Rightarrow \quad du = -\frac{1}{2\sqrt{1-x}}\,dx$$
$$\Rightarrow \quad -2du = \frac{dx}{\sqrt{1-x}}$$

$$u = \sqrt{1-x} \quad \Rightarrow \quad u^2 = 1-x \quad \Rightarrow \quad x = 1-u^2$$

$$I=\int\frac{dx}{x\sqrt{1-x}}=\int\frac{1}{x}\left(\frac{dx}{\sqrt{1-x}}\right)=\int\left(\frac{1}{1-u^2}\right)(-2du)=-2\int\frac{1}{1-u^2}du$$

$$=-2\int\left(\frac{1/2}{1-u}+\frac{1/2}{1+u}\right)du$$

$$=-2\left(-\frac{1}{2}\ln|1-u|+\frac{1}{2}\ln|1+u|\right)+C$$

$$=\ln|1-u|-\ln|1+u|+C$$

$$=\ln\left|1-\sqrt{1-x}\right|-\ln\left|1+\sqrt{1-x}\right|+C$$

Notes

(1) $\int\frac{1/2}{1-u}dx=-\frac{1}{2}\ln|1-u|+C$ using $\int\frac{1}{ax+b}=\frac{1}{a}\ln|ax+b|+C$;
remember to divide by the coefficient of x when integrating.

(2) The final answer can be written as $\ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right|+C$ but there is no need to do this.

13. The denominator contains distinct linear factors.

$$\frac{8}{x(x+2)(x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4}$$
$$= \frac{A(x+2)(x+4) + Bx(x+4) + Cx(x+2)}{x(x+2)(x+4)}$$

$$8 = A(x+2)(x+4) + Bx(x+4) + Cx(x+2)$$

Let $x = -2 \Rightarrow 8 = A(0)(2) + B(-2)(2) + C(-2)(0) \Rightarrow 8 = -4B \Rightarrow B = -2$

Let $x = -4 \Rightarrow 8 = A(-2)(0) + B(-4)(0) + C(-4)(-2) \Rightarrow 8 = 8C \Rightarrow C = 1$

Let $x = 0 \Rightarrow 8 = A(2)(4) + B(0)(4) + C(0)(2) \Rightarrow 8 = 8A \Rightarrow A = 1$

Hence $\frac{8}{x(x+2)(x+4)} = \frac{1}{x} + \frac{-2}{x+2} + \frac{1}{x+4}$ *or* $\frac{8}{x(x+2)(x+4)} = \frac{1}{x} - \frac{2}{x+2} + \frac{1}{x+4}$.

$$y = \frac{8}{x^3 + 6x^2 + 8x} = \frac{8}{x(x^2 + 6x + 8)} = \frac{8}{x(x+2)(x+4)}$$

$$\text{Area} = \int_1^2 \frac{8}{x^3 + 6x^2 + 8x}\,dx = \int_1^2 \frac{8}{x(x+2)(x+4)}\,dx$$
$$= \int_1^2 \left(\frac{1}{x} - \frac{2}{x+2} + \frac{1}{x+4}\right)dx$$
$$= \left[\ln|x| - 2\ln|x+2| + \ln|x+4|\right]_1^2$$
$$= \left[\ln|2| - 2\ln|4| + \ln|6|\right] - \left[\ln|1| - 2\ln|3| + \ln|5|\right]$$
$$= \left[\ln 2 - 2\ln 4 + \ln 6\right] - \left[\ln 1 - 2\ln 3 + \ln 5\right]$$
$$= \left[\ln 2 - \ln 4^2 + \ln 6\right] - \left[0 - \ln 3^2 + \ln 5\right]$$
$$= \left[\ln 2 - \ln 16 + \ln 6\right] - \left[-\ln 9 + \ln 5\right]$$
$$= \left[(\ln 2 + \ln 6) - \ln 16\right] - \left[\ln 5 - \ln 9\right]$$
$$= \left[\ln(2 \times 6) - \ln 16\right] - \left[\ln\left(\frac{5}{9}\right)\right]$$
$$= \left[\ln 12 - \ln 16\right] - \left[\ln\left(\frac{5}{9}\right)\right]$$
$$= \left[\ln\left(\frac{12}{16}\right)\right] - \left[\ln\left(\frac{5}{9}\right)\right]$$
$$= \ln\left(\frac{3}{4}\right) - \ln\left(\frac{5}{9}\right) = \ln\left(\frac{3/4}{5/9}\right) = \ln\left(\frac{27}{20}\right) \text{ units}^2$$

Note

There are other ways of combining the logarithms to complete the evaluation,

eg $\left[\ln 2 - \ln 16 + \ln 6\right] - \left[-\ln 9 + \ln 5\right] = \ln 2 - \ln 16 + \ln 6 + \ln 9 - \ln 5$

$$= (\ln 2 + \ln 6 + \ln 9) - (\ln 16 + \ln 5)$$

$$= \ln(2 \times 6 \times 9) - \ln(16 \times 5)$$

$$= \ln 108 - \ln 80$$

$$= \ln\left(\frac{108}{80}\right)$$

$$= \ln\left(\frac{27}{20}\right)$$

14.(a) The denominator contains distinct linear factors.

$$\frac{13+6x+5x^2}{(1+x)(2-x)(3+x)} = \frac{A}{1+x} + \frac{B}{2-x} + \frac{C}{3+x}$$
$$= \frac{A(2-x)(3+x)+B(1+x)(3+x)+C(1+x)(2-x)}{(1+x)(2-x)(3+x)}$$

$$13+6x+5x^2 = A(2-x)(3+x)+B(1+x)(3+x)+C(1+x)(2-x)$$

Let $x=2 \quad \Rightarrow \quad 13+6(2)+5(2)^2 = A(0)(5)+B(3)(5)+C(3)(0) \quad \Rightarrow \quad 45=15B \quad \Rightarrow \quad B=3$

Let $x=-3 \quad \Rightarrow \quad 13+6(-3)+5(-3)^2 = A(5)(0)+B(-2)(0)+C(-2)(5) \quad \Rightarrow \quad 40=-10C$
$\Rightarrow \quad C=-4$

Let $x=-1 \quad \Rightarrow \quad 13+6(-1)+5(-1)^2 = A(3)(2)+B(0)(2)+C(0)(3) \quad \Rightarrow \quad 12=6A$
$\Rightarrow \quad A=2$

Hence $\dfrac{13+6x+5x^2}{(1+x)(2-x)(3+x)} = \dfrac{2}{1+x} + \dfrac{3}{2-x} + \dfrac{-4}{3+x}$

or $\dfrac{13+6x+5x^2}{(1+x)(2-x)(3+x)} = \dfrac{2}{1+x} + \dfrac{3}{2-x} - \dfrac{4}{3+x}$.

(b) $$\int_0^1 \frac{13+6x+5x^2}{(1+x)(2-x)(3+x)}\,dx = \int_0^1 \left(\frac{2}{1+x} + \frac{3}{2-x} - \frac{4}{3+x}\right)dx$$
$$= \Big[2\ln|1+x| - 3\ln|2-x| - 4\ln|3+x|\Big]_0^1$$
$$= \Big[2\ln|2| - 3\ln|1| - 4\ln|4|\Big] - \Big[2\ln|1| - 3\ln|2| - 4\ln|3|\Big]$$
$$= \big[2\ln 2 - 3\ln 1 - 4\ln 4\big] - \big[2\ln 1 - 3\ln 2 - 4\ln 3\big]$$
$$= \big[\ln 2^2 - 3(0) - \ln 4^4\big] - \big[2(0) - \ln 2^3 - \ln 3^4\big]$$
$$= \big[\ln 4 - \ln 256\big] - \big[-\ln 8 - \ln 81\big]$$
$$= \ln 4 - \ln 256 + \ln 8 + \ln 81$$
$$= (\ln 4 + \ln 8 + \ln 81) - \ln 256$$
$$= \ln(4\times 8\times 81) - \ln 256$$
$$= \ln 2592 - \ln 256 = \ln\left(\frac{2592}{256}\right) = \ln\left(\frac{81}{8}\right)$$

Hence $\displaystyle\int_0^1 \frac{13+6x+5x^2}{(1+x)(2-x)(3+x)}\,dx = \ln\frac{a}{b}$ where $a=81$ and $b=8$.

Note

$\int \frac{3}{2-x}dx = -3\ln|2-x| + C$ using $\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$;
remember to divide by the coefficient of *x* when integrating.

15. First check whether $x^2 - x - 6$ is irreducible:

$a = 1$, $b = -1$, $c = -6 \quad \Rightarrow \quad b^2 - 4ac = (-1)^2 - 4(1)(-6) = 25$

$b^2 - 4ac > 0$, so $x^2 - x - 6$ has two real and distinct roots and can be written as the product of two distinct linear factors, ie $x^2 - x - 6 = (x+2)(x-3)$.

This means that the denominator can be written as $x(x+2)(x-3)$ and contains distinct linear factors.

$$\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \frac{2x^2 - 9x - 6}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$
$$= \frac{A(x+2)(x-3) + Bx(x-3) + Cx(x+2)}{x(x+2)(x-3)}$$

$$2x^2 - 9x - 6 = A(x+2)(x-3) + Bx(x-3) + Cx(x+2)$$

Let $x = -2 \quad \Rightarrow \quad 2(-2)^2 - 9(-2) - 6 = A(0)(-5) + B(-2)(-5) + C(-2)(0)$
$\Rightarrow \quad 20 = 10B$
$\Rightarrow \quad B = 2$

Let $x = 3 \quad \Rightarrow \quad 2(3)^2 - 9(3) - 6 = A(5)(0) + B(3)(0) + C(3)(5)$
$\Rightarrow \quad -15 = 15C$
$\Rightarrow \quad C = -1$

Let $x = 0 \quad \Rightarrow \quad 2(0)^2 - 9(0) - 6 = A(2)(-3) + B(0)(-3) + C(0)(2)$
$\Rightarrow \quad -6 = -6A$
$\Rightarrow \quad A = 1$

Hence $\quad \dfrac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \dfrac{1}{x} + \dfrac{2}{x+2} + \dfrac{-1}{x-3} \quad$ *or* $\quad \dfrac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \dfrac{1}{x} + \dfrac{2}{x+2} - \dfrac{1}{x-3}$.

$$\int_4^6 \frac{2x^2-9x-6}{x(x^2-x-6)}\,dx = \int_4^6 \left(\frac{1}{x}+\frac{2}{x+2}-\frac{1}{x-3}\right)dx$$

$$= \left[\ln|x| + 2\ln|x+2| - \ln|x-3|\right]_4^6$$

$$= \left[\ln|6| + 2\ln|8| - \ln|3|\right] - \left[\ln|4| + 2\ln|6| - \ln|1|\right]$$

$$= \left[\ln 6 + 2\ln 8 - \ln 3\right] - \left[\ln 4 + 2\ln 6 - \ln 1\right]$$

$$= \left[\ln 6 + \ln 8^2 - \ln 3\right] - \left[\ln 4 + \ln 6^2 - 0\right]$$

$$= \left[(\ln 6 + \ln 64) - \ln 3\right] - \left[\ln 4 + \ln 36\right]$$

$$= \left[\ln(6\times 64) - \ln 3\right] - \left[\ln(4\times 36)\right]$$

$$= \left[\ln 384 - \ln 3\right] - \left[\ln 144\right]$$

$$= \left[\ln\left(\frac{384}{3}\right)\right] - \left[\ln 144\right]$$

$$= \ln 128 - \ln 144$$

$$= \ln\left(\frac{128}{144}\right)$$

$$= \ln\left(\frac{8}{9}\right)$$

Hence $\displaystyle\int_4^6 \frac{2x^2-9x-6}{x(x^2-x-6)}\,dx = \ln\frac{m}{n}$ where $m = 8$ and $n = 9$.

<u>Notes</u>

(1) A common error in this question is to assume that $x^2 - x - 6$ is irreducible leading to the incorrect partial fractions $\dfrac{A}{x} + \dfrac{Bx + C}{x^2 - x - 6}$.

It is essential that you check whether a quadratic factor in the denominator is irreducible and factorise the denominator fully before finding the partial fractions.

(2) There are other ways of combining the logarithms to complete the evaluation, eg $\left[\ln 6 + \ln 64 - \ln 3\right] - \left[\ln 4 + \ln 36\right] = \ln 6 + \ln 64 - \ln 3 - \ln 4 - \ln 36$

$$= (\ln 6 + \ln 64) - (\ln 3 + \ln 4 + \ln 36)$$

$$= \ln(6\times 64) - \ln(3\times 4\times 36)$$

$$= \ln 384 - \ln 432$$

$$= \ln\left(\frac{384}{432}\right)$$

$$= \ln\left(\frac{8}{9}\right)$$

16.(a) First factorise the denominator: $x^2 - 4 = (x-2)(x+2)$

The denominator contains distinct linear factors.

$$\frac{4}{x^2-4} = \frac{4}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$$4 = A(x+2) + B(x-2)$$

Let $x = -2 \quad \Rightarrow \quad 4 = A(0) + B(-4) \quad \Rightarrow \quad 4 = -4B \quad \Rightarrow \quad B = -1$

Let $x = 2 \quad \Rightarrow \quad 4 = A(4) + B(0) \quad \Rightarrow \quad 4 = 4A \quad \Rightarrow \quad A = 1$

Hence $\dfrac{4}{x^2-4} = \dfrac{1}{x-2} + \dfrac{-1}{x+2}$ *or* $\dfrac{4}{x^2-4} = \dfrac{1}{x-2} - \dfrac{1}{x+2}$.

(b) To find $\int \frac{x^2}{x^2-4}\,dx$, note that $\frac{x^2}{x^2-4}$ is an **improper algebraic fraction** since the degree of the numerator is equal to the degree of the denominator, so algebraic long division should be used to start the integration process.

$$\begin{array}{r|l} & \quad\quad\quad\;\; 1 \\ \hline x^2-4 & x^2+0x+0 \\ & x^2 \quad\quad\; -4 \\ \hline & \quad\quad\quad\;\; 4 \end{array}$$

Hence $\dfrac{x^2}{x^2-4} = 1 + \dfrac{4}{x^2-4} = 1 + \dfrac{1}{x-2} + \dfrac{1}{x+2}$ [using the partial fractions in (a)]

$$\int \frac{x^2}{x^2-4}\,dx = \int \left(1 + \frac{1}{x-2} + \frac{1}{x+2}\right)dx = x + \ln|x-2| + \ln|x+2| + C$$

<u>Note</u>

The final answer can be written as $x + \ln|(x-2)(x+2)| + C$ or $x + \ln|x^2-4| + C$ but there is no need to do this.

17.(a) First factorise the denominator: $x^2 - 1 = (x-1)(x+1)$

The denominator contains distinct linear factors.

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$x = A(x+1) + B(x-1)$$

Let $x = -1 \quad \Rightarrow \quad -1 = A(0) + B(-2) \quad \Rightarrow \quad -1 = -2B \quad \Rightarrow \quad B = \frac{1}{2}$

Let $x = 1 \quad \Rightarrow \quad 1 = A(2) + B(0) \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$

Hence $\dfrac{x}{x^2-1} = \dfrac{1/2}{x-1} + \dfrac{1/2}{x+1}$.

(b) To find $\int \dfrac{x^3}{x^2-1}\,dx$, note that $\dfrac{x^3}{x^2-1}$ is an **improper algebraic fraction** since the degree of the numerator is greater than the degree of the denominator, so algebraic long division should be used to start the integration process.

$$\begin{array}{r|l} & \quad\quad\quad x \\ \hline x^2-1 & x^3+0x^2+0x+0 \\ & x^3 \quad\quad\quad -x \\ \hline & \quad\quad\quad\quad x \end{array}$$

Hence $\dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1} = x + \dfrac{1/2}{x-1} + \dfrac{1/2}{x+1}$ [using the partial fractions in (a)]

$$\int \frac{x^3}{x^2-1}\,dx = \int \left(x + \frac{1/2}{x-1} + \frac{1/2}{x+1} \right) dx = \frac{x^2}{2} + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

<u>Notes</u>

The partial fractions can be written as $\dfrac{1}{2(x-1)} + \dfrac{1}{2(x+1)}$ but there is no need to do this.

18. The denominator contains a repeated linear factor.

$$\frac{x+4}{(x+1)^2(2x-1)} = \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + B(2x-1)(x+1) + C(2x-1)}{(x+1)^2(2x-1)}$$

$$x+4 = A(x+1)^2 + B(2x-1)(x+1) + C(2x-1)$$

Let $x=-1 \quad \Rightarrow \quad -1+4 = A(0)^2 + B(-3)(0) + C(-3) \quad \Rightarrow \quad 3=-3C \quad \Rightarrow \quad C=-1$

Let $x=\frac{1}{2} \quad \Rightarrow \quad \frac{1}{2}+4 = A\left(\frac{3}{2}\right)^2 + B(0)\left(\frac{3}{2}\right) + C(0) \quad \Rightarrow \quad \frac{9}{2} = \frac{9}{4}A \quad \Rightarrow \quad A=2$

Equating coefficients of $x^2 \quad \Rightarrow \quad 0 = A+2B \quad \Rightarrow \quad 0 = 2+2B \quad \Rightarrow \quad 2B=-2 \quad \Rightarrow \quad B=-1$

Hence $\quad \frac{x+4}{(x+1)^2(2x-1)} = \frac{2}{2x-1} + \frac{-1}{x+1} + \frac{-1}{(x+1)^2}$

or $\quad \frac{x+4}{(x+1)^2(2x-1)} = \frac{2}{2x-1} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$.

$$\int_1^2 \frac{x+4}{(x+1)^2(2x-1)}\,dx = \int_1^2 \left(\frac{2}{2x-1} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx$$

$$= \int_1^2 \left(\frac{2}{2x-1} - \frac{1}{x+1} - (x+1)^{-2}\right)dx$$

$$= \left[2\left(\frac{1}{2}\ln|2x-1|\right) - \ln|x+1| - \frac{(x+1)^{-1}}{(-1)\times 1}\right]_1^2$$

$$= \left[\ln|2x-1| - \ln|x+1| + (x+1)^{-1}\right]_1^2$$

$$= \left[\ln|2x-1| - \ln|x+1| + \frac{1}{x+1}\right]_1^2$$

$$= \left[\ln|3| - \ln|3| + \frac{1}{3}\right] - \left[\ln|1| - \ln|2| + \frac{1}{2}\right]$$

$$= \left[\ln 3 - \ln 3 + \frac{1}{3}\right] - \left[\ln 1 - \ln 2 + \frac{1}{2}\right]$$

$$= \left[2\ln 3 + \frac{1}{3}\right] - \left[0 - \ln 2 + \frac{1}{2}\right]$$

$$= 2\ln 3 + \frac{1}{3} + \ln 2 - \frac{1}{2}$$

$$= \ln 2 - \frac{1}{6}$$

Note

$$\int \frac{2}{2x-1}\,dx = 2\int \frac{1}{2x-1}\,dx = 2\left(\frac{1}{2}\ln|2x-1|\right) + C = \ln|2x-1| + C$$

using the standard integral $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$.

Remember to divide by the coefficient of *x* when integrating.

19.(a) $c(x) = x^3 - x^2 - x - 2$

To find a real root of $c(x)$, use synthetic division and try factors of -2.

$$\begin{array}{r|rrrr} & 1 & -1 & -1 & -2 \\ 2 & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

remainder = 0, so $x = 2$ is a root of $c(x)$ and $c(x) = (x-2)(x^2 + x + 1)$.

Hence $c(x) = l(x)q(x)$ where $l(x) = x - 2$ and $q(x) = x^2 + x + 1$.

(b) For $q(x) = x^2 + x + 1$: $a = 1$, $b = 1$, $c = 1$ $\Rightarrow$ $b^2 - 4ac = 1^2 - 4(1)(1) = -3$

$b^2 - 4ac < 0$, hence $q(x)$ has no real roots and is irreducible (cannot be written as the product of two linear factors with real coefficients).
This means that $c(x)$ cannot be written as the product of three linear factors with real coefficients.

(c) $$\frac{5x+4}{x^3 - x^2 - x - 2} = \frac{A}{l(x)} + \frac{Bx+C}{q(x)} \quad \Rightarrow \quad \frac{5x+4}{x^3 - x^2 - x - 2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$$
$$= \frac{A(x^2+x+1) + (Bx+C)(x-2)}{x^3 - x^2 - x - 2}$$

$5x + 4 = A(x^2 + x + 1) + (Bx + C)(x - 2)$

Let $x = 2$ $\Rightarrow$ $5(2) + 4 = A(2^2 + 2 + 1) + 0$ $\Rightarrow$ $14 = 7A$ $\Rightarrow$ $A = 2$

Equating coefficients of x^2 $\Rightarrow$ $0 = A + B$ $\Rightarrow$ $0 = 2 + B$ $\Rightarrow$ $B = -2$

Equating constants $\Rightarrow$ $4 = A - 2C$ $\Rightarrow$ $4 = 2 - 2C$ $\Rightarrow$ $2C = -2$ $\Rightarrow$ $C = -1$

Hence $A = 2$, $B = -2$, $C = -1$ and $$\frac{5x+4}{x^3 - x^2 - x - 2} = \frac{2}{x-2} + \frac{-2x-1}{x^2+x+1}.$$

$$\int \frac{5x+4}{x^3-x^2-x-2}\,dx = \int\left(\frac{2}{x-2}+\frac{-2x-1}{x^2+x+1}\right)dx$$

$$= \int \frac{2}{x-2}\,dx + \int \frac{-2x-1}{x^2+x+1}\,dx$$

$$= \int \frac{2}{x-2}\,dx + \int \frac{-(2x+1)}{x^2+x+1}\,dx$$

$$= \int \frac{2}{x-2}\,dx - \int \frac{2x+1}{x^2+x+1}\,dx$$

To find $\int \frac{2}{x-2}\,dx$, use the standard integral $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+C$:

$$\int \frac{2}{x-2}\,dx = 2\int \frac{1}{x-2}\,dx = 2\ln|x-2|+C$$

To find $\int \frac{2x+1}{x^2+x+1}\,dx$, note that the numerator is the exact derivative of the denominator, so the integral can be found directly using the result $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$:

$$\int \frac{2x+1}{x^2+x+1}\,dx = \ln|x^2+x+1|+C$$

Hence $\int \frac{5x+4}{x^3-x^2-x-2}\,dx = 2\ln|x-2| - \ln|x^2+x+1|+C$.

<u>Note</u>

Alternatively, to find $\int \frac{2x+1}{x^2+x+1}\,dx$ the substitution $u = x^2+x+1$ could be used.

20. The denominator contains an irreducible quadratic factor x^2+5.
(You can assume that any quadratic factor of the form x^2+a, where $a>0$, is irreducible, or you can show that $b^2-4ac<0$.)

$$\frac{12x^2+20}{x(x^2+5)}=\frac{A}{x}+\frac{Bx+C}{x^2+5}$$
$$=\frac{A(x^2+5)+(Bx+C)x}{x(x^2+5)}$$

$$12x^2+20=A(x^2+5)+(Bx+C)x$$

Let $x=0 \quad\Rightarrow\quad 12(0)^2+20=A(0^2+5)+0 \quad\Rightarrow\quad 20=5A \quad\Rightarrow\quad A=4$

Equating coefficients of $x^2 \quad\Rightarrow\quad 12=A+B \quad\Rightarrow\quad 12=4+B \quad\Rightarrow\quad B=8$

Equating coefficients of $x \quad\Rightarrow\quad 0=C$

[Note that equating constants gives $20=5A \quad\Rightarrow\quad A=4$]

Hence $\dfrac{12x^2+20}{x(x^2+5)}=\dfrac{4}{x}+\dfrac{8x}{x^2+5}$.

(b) $\displaystyle\int\frac{12x^2+20}{x(x^2+5)}\,dx=\int\left(\frac{4}{x}+\frac{8x}{x^2+5}\right)dx=\int\frac{4}{x}\,dx+\int\frac{8x}{x^2+5}\,dx$

$$\int\frac{4}{x}\,dx=4\int\frac{1}{x}dx=4\ln|x|+C$$

To find $\displaystyle\int\frac{8x}{x^2+5}\,dx$, make the numerator the exact derivative of the denominator by adjusting the constant and then integrate using $\displaystyle\int\frac{f'(x)}{f(x)}\,dx=\ln|f(x)|+C$.

Note that $\dfrac{d}{dx}(x^2+5)=2x$, so the numerator must be written as $2x$.

$$\int\frac{8x}{x^2+5}\,dx=4\int\frac{2x}{x^2+5}\,dx=4\ln\left|x^2+5\right|+C$$

Hence $\displaystyle\int\frac{12x^2+20}{x(x^2+5)}\,dx=4\ln|x|+4\ln\left|x^2+5\right|+C$.

$$\int_1^2 \frac{12x^2+20}{x(x^2+5)}\,dx = \Big[4\ln|x| + 4\ln|x^2+5|\Big]_1^2$$

$$= \big[4\ln|2| + 4\ln 9\big] - \big[4\ln|1| + 4\ln|6|\big]$$

$$= \big[4\ln 2 + 4\ln 9\big] - \big[4\ln 1 + 4\ln 6\big]$$

$$= 4\cdot 3944 \text{ (to 4 dp)}$$

<u>Notes</u>

(1) Alternatively, to find $\int \frac{8x}{x^2+5}\,dx$ the substitution $u = x^2+5$ could be used.

(2) The exact value of the definite integral is not required. An approximate value can be given to any reasonable degree of accuracy.

21. The denominator contains an irreducible quadratic factor $1+x^2$.
(You can assume that any quadratic factor of the form x^2+a, where $a>0$, is irreducible, or you can show that $b^2-4ac<0$.)

$$\frac{x^2+3}{x(1+x^2)}=\frac{A}{x}+\frac{Bx+C}{1+x^2}$$
$$=\frac{A(1+x^2)+(Bx+C)x}{x(1+x^2)}$$

$$x^2+3=A(1+x^2)+(Bx+C)x$$

Let $x=0 \quad \Rightarrow \quad 0^2+3=A(1+0^2)+0 \quad \Rightarrow \quad 3=A$

Equating coefficients of $x^2 \quad \Rightarrow \quad 1=A+B \quad \Rightarrow \quad 1=3+B \quad \Rightarrow \quad B=-2$

Equating coefficients of $x \quad \Rightarrow \quad 0=C$ [Note that equating constants gives $3=A$]

Hence $\dfrac{x^2+3}{x(1+x^2)}=\dfrac{3}{x}+\dfrac{-2x}{1+x^2}$ *or* $\dfrac{x^2+3}{x(1+x^2)}=\dfrac{3}{x}-\dfrac{2x}{1+x^2}$.

Before evaluating $\displaystyle\int_{\frac{1}{2}}^{1}\frac{x^2+3}{x(1+x^2)}dx$, first find $\displaystyle\int\frac{x^2+3}{x(1+x^2)}dx$ using the partial fractions.

$$\int\frac{x^2+3}{x(1+x^2)}dx=\int\left(\frac{3}{x}-\frac{2x}{1+x^2}\right)dx=\int\frac{3}{x}dx-\int\frac{2x}{1+x^2}dx$$

$$\int\frac{3}{x}dx=3\int\frac{1}{x}dx=3\ln|x|+C$$

To find $\displaystyle\int\frac{2x}{1+x^2}dx$, note that the numerator is the exact derivative of the denominator, so the integral can be found directly using the result $\displaystyle\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+C$:

$$\int\frac{2x}{1+x^2}dx=\ln|1+x^2|+C$$

Hence $\displaystyle\int\frac{x^2+3}{x(1+x^2)}dx=3\ln|x|-\ln|1+x^2|+C$

$$\int_{\frac{1}{2}}^{1} \frac{x^2+3}{x(1+x^2)}\,dx = \left[3\ln|x| - \ln|1+x^2|\right]_{\frac{1}{2}}^{1}$$

$$= \left[3\ln|1| - \ln|1+1^2|\right] - \left[3\ln\left|\frac{1}{2}\right| - \ln\left|1+\left(\frac{1}{2}\right)^2\right|\right]$$

$$= \left[3\ln 1 - \ln 2\right] - \left[3\ln\left(\frac{1}{2}\right) - \ln\left(\frac{5}{4}\right)\right]$$

$$= 1\cdot 6094 \text{ (to 4 dp)}$$

Notes

(1) Alternatively, to find $\int \frac{2x}{1+x^2}\,dx$ the substitution $u = 1 + x^2$ could be used.

(2) The exact value of the definite integral is not required. An approximate value can be given to any reasonable degree of accuracy.

22. First factorise the denominator fully: $x^3 + x = x(x^2+1)$ and x^2+1 is irreducible

The denominator contains an irreducible quadratic factor.

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$
$$= \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$1 = A(x^2+1)+(Bx+C)x$

Let $x = 0 \quad \Rightarrow \quad 1 = A(0^2+1)+0 \quad \Rightarrow \quad 1 = A$

Equating coefficients of $x^2 \quad \Rightarrow \quad 0 = A+B \quad \Rightarrow \quad 0 = 1+B \quad \Rightarrow \quad B = -1$

Equating coefficients of $x \quad \Rightarrow \quad 0 = C$ [Note that equating constants gives $1 = A$]

Hence $\dfrac{1}{x^3+x} = \dfrac{1}{x} + \dfrac{-x}{x^2+1}$ *or* $\dfrac{1}{x^3+x} = \dfrac{1}{x} - \dfrac{x}{x^2+1}$.

Before finding $I(k) = \int_1^k \frac{1}{x^3+x}\,dx$, first find $\int \frac{1}{x^3+x}\,dx$ using the partial fractions.

$$\int \frac{1}{x^3+x}\,dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \int \frac{1}{x}dx - \int \frac{x}{x^2+1}\,dx$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

To find $\int \frac{x}{x^2+1}\,dx$, make the numerator the exact derivative of the denominator by adjusting the constant and then integrate using $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

Note that $\dfrac{d}{dx}(x^2+1) = 2x$, so the numerator must be written as $2x$.

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| + C$$

Hence $\int \frac{1}{x^3+x}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + C$.

$$I(k)=\int_1^k \frac{1}{x^3+x}\,dx=\left[\ln|x|-\frac{1}{2}\ln\left|x^2+1\right|\right]_1^k$$

$$=\left[\ln|k|-\frac{1}{2}\ln\left|k^2+1\right|\right]-\left[\ln|1|-\frac{1}{2}\ln|2|\right]$$

$$=\left[\ln k-\frac{1}{2}\ln(k^2+1)\right]-\left[\ln 1-\frac{1}{2}\ln 2\right]$$

$$=\left[\ln k-\ln(k^2+1)^{\frac{1}{2}}\right]-\left[0-\ln 2^{\frac{1}{2}}\right]$$

$$=\left[\ln k-\ln\sqrt{k^2+1}\right]-\left[-\ln\sqrt{2}\right]$$

$$=\left[\ln\left(\frac{k}{\sqrt{k^2+1}}\right)\right]-\left[-\ln\sqrt{2}\right]$$

$$=\ln\left(\frac{k}{\sqrt{k^2+1}}\right)+\ln\sqrt{2}$$

$$=\ln\left(\frac{k}{\sqrt{k^2+1}}\times\sqrt{2}\right)$$

$$=\ln\left(\frac{\sqrt{2}k}{\sqrt{k^2+1}}\right)$$

$$e^{I(k)}=e^{\ln\left(\frac{\sqrt{2}k}{\sqrt{k^2+1}}\right)}=\frac{\sqrt{2}k}{\sqrt{k^2+1}}$$

As $k\to\infty$, $k^2+1\approx k^2$, so $\sqrt{k^2+1}\approx\sqrt{k^2}=k$ and $\dfrac{\sqrt{2}k}{\sqrt{k^2+1}}\approx\dfrac{\sqrt{2}k}{k}=\sqrt{2}$.

Hence the limiting value of $e^{I(k)}$ as $k\to\infty$ is $\sqrt{2}$.

Notes

(1) Alternatively, to find $\int\frac{x}{x^2+1}\,dx$ the substitution $u=x^2+1$ could be used.

(2) There are other ways of combining the logarithms to complete the evaluation of $I(k)$,

eg $\left[\ln k-\ln\sqrt{k^2+1}\right]-\left[-\ln\sqrt{2}\right]=\ln k-\ln\sqrt{k^2+1}+\ln\sqrt{2}$

$$=(\ln k+\ln\sqrt{2})-\ln\sqrt{k^2+1}$$

$$=\ln(k\times\sqrt{2})-\ln\sqrt{k^2+1}$$

$$=\ln(\sqrt{2}k)-\ln\sqrt{k^2+1}=\ln\left(\frac{\sqrt{2}k}{\sqrt{k^2+1}}\right)$$

23. To find $\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}\,dx$, note that $\frac{2x^3 - x - 1}{(x-3)(x^2+1)}$ is an **improper algebraic fraction** since the degree of the numerator is equal to the degree of the denominator, so algebraic long division should be used to start the integration process.

First multiply out the brackets in the denominator: $(x-3)(x^2+1) = x^3 - 3x^2 + x - 3$

$$\begin{array}{r|l} & \qquad\qquad\qquad 2 \\ \hline x^3 - 3x^2 + x - 3 & 2x^3 + 0x^2 - x - 1 \\ & 2x^3 - 6x^2 + 2x - 6 \\ \hline & \qquad 6x^2 - 3x + 5 \end{array}$$

Hence $\frac{2x^3 - x - 1}{(x-3)(x^2+1)} = 2 + \frac{6x^2 - 3x + 5}{(x-3)(x^2+1)}$.

Now express $\frac{6x^2 - 3x + 5}{(x-3)(x^2+1)}$ in partial fractions before integrating.

Note that this is a proper algebraic fraction and the denominator contains an irreducible quadratic factor $x^2 + 1$.

$$\frac{6x^2 - 3x + 5}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx + C}{x^2+1}$$
$$= \frac{A(x^2+1) + (Bx+C)(x-3)}{(x-3)(x^2+1)}$$

$$6x^2 - 3x + 5 = A(x^2+1) + (Bx+C)(x-3)$$

Let $x = 3 \quad \Rightarrow \quad 6(3)^2 - 3(3) + 5 = A(3^2+1) + 0 \quad \Rightarrow \quad 50 = 10A \quad \Rightarrow \quad A = 5$

Equating coefficients of $x^2 \quad \Rightarrow \quad 6 = A + B \quad \Rightarrow \quad 6 = 5 + B \quad \Rightarrow \quad B = 1$

Equating constants $\quad \Rightarrow \quad 5 = A - 3C \quad \Rightarrow \quad 5 = 5 - 3C \quad \Rightarrow \quad C = 0$

Hence $\frac{6x^2 - 3x + 5}{(x-3)(x^2+1)} = \frac{5}{x-3} + \frac{x}{x^2+1}$ and $\frac{2x^3 - x - 1}{(x-3)(x^2+1)} = 2 + \frac{5}{x-3} + \frac{x}{x^2+1}$.

$$\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}\,dx = \int \left(2 + \frac{5}{x-3} + \frac{x}{x^2+1} \right) dx$$

$$= \int 2dx + \int \frac{5}{x-3}\,dx + \int \frac{x}{x^2+1}\,dx$$

$$\int 2dx = 2x + C$$

To find $\int \frac{5}{x-3}\,dx$, use the standard integral $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$:

$$\int \frac{5}{x-3}\,dx = 5\int \frac{1}{x-3}\,dx = 5\ln|x-3| + C$$

To find $\int \frac{x}{x^2+1}\,dx$, make the numerator the exact derivative of the denominator by adjusting the constant and then integrate using $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

Note that $\frac{d}{dx}(x^2+1) = 2x$, so the numerator must be written as $2x$.

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| + C$$

Hence $\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}\,dx = 2x + 5\ln|x-3| + \frac{1}{2}\ln|x^2+1| + C$.

Note

It common error in this question is to try to express $\frac{2x^3 - x - 1}{(x-3)(x^2+1)}$ in partial fractions, leading to the incorrect partial fractions $\frac{A}{x-3} + \frac{Bx+C}{x^2+1}$. It is important to note that only a proper algebraic fraction (where the degree of the numerator is less than the degree of the denominator) should be expressed in partial fractions. The correct way to proceed is to use algebraic long division first to give $\frac{2x^3 - x - 1}{(x-3)(x^2+1)} = 2 + \frac{6x^2 - 3x + 5}{(x-3)(x^2+1)}$ and then express the proper algebraic fraction $\frac{6x^2 - 3x + 5}{(x-3)(x^2+1)}$ in partial fractions before integrating.

24. Let $I = \int x \sin 3x dx$.

Use integration by parts with $f(x) = x$ and $g'(x) = \sin 3x$.

$$f(x) = x \qquad \text{and} \qquad g'(x) = \sin 3x$$

$$\Rightarrow \quad f'(x) = 1 \qquad \text{and} \qquad g(x) = -\frac{1}{3}\cos 3x$$

$$\begin{aligned}
I &= f(x)g(x) - \int f'(x)g(x)dx \\
&= x\left(-\frac{1}{3}\cos 3x\right) - \int 1\left(-\frac{1}{3}\cos 3x\right)dx \\
&= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x dx \\
&= -\frac{1}{3}x\cos 3x + \frac{1}{3}\left(\frac{1}{3}\sin 3x\right) + C \\
&= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C
\end{aligned}$$

25. Let $I = \int x \sin x dx$.

Use integration by parts with $f(x) = x$ and $g'(x) = \sin x$.

$$f(x) = x \qquad \text{and} \qquad g'(x) = \sin x$$

$$\Rightarrow \quad f'(x) = 1 \qquad \text{and} \qquad g(x) = -\cos x$$

$$\begin{aligned}
I &= f(x)g(x) - \int f'(x)g(x)dx \\
&= x(-\cos x) - \int 1(-\cos x)dx \\
&= -x\cos x + \int \cos x dx \\
&= -x\cos x + \sin x + C
\end{aligned}$$

Hence $\int_0^{\pi} x \sin x dx = [-x\cos x + \sin x]_0^{\pi}$

$$\begin{aligned}
&= [-\pi\cos\pi + \sin\pi] - [-0\cos 0 + \sin 0] \\
&= [-\pi(-1) + 0] - [0 + 0] \\
&= \pi
\end{aligned}$$

26. Let $I = \int 2x \sin 4x dx$.

Use **integration by parts** with $f(x) = 2x$ and $g'(x) = \sin 4x$.

$$f(x) = 2x \quad \text{and} \quad g'(x) = \sin 4x$$

$$\Rightarrow \quad f'(x) = 2 \quad \text{and} \quad g(x) = -\frac{1}{4}\cos 4x$$

$$\begin{aligned}
I &= f(x)g(x) - \int f'(x)g(x)dx \\
&= 2x\left(-\frac{1}{4}\cos 4x\right) - \int 2\left(-\frac{1}{4}\cos 4x\right)dx \\
&= -\frac{1}{2}x\cos 4x + \frac{1}{2}\int \cos 4x dx \\
&= -\frac{1}{2}x\cos 4x + \frac{1}{2}\left(\frac{1}{4}\sin 4x\right) + C \\
&= -\frac{1}{2}x\cos 4x + \frac{1}{8}\sin 4x + C
\end{aligned}$$

Hence $\displaystyle\int_0^{\frac{\pi}{4}} 2x \sin 4x dx = \left[-\frac{1}{2}x\cos 4x + \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{4}}$

$$\begin{aligned}
&= \left[-\frac{1}{2}\left(\frac{\pi}{4}\right)\cos 4\left(\frac{\pi}{4}\right) + \frac{1}{8}\sin\left(4\left(\frac{\pi}{4}\right)\right)\right] - \left[-\frac{1}{2}(0)\cos(4(0)) + \frac{1}{8}\sin 4(0)\right] \\
&= \left[-\frac{\pi}{8}\cos\pi + \frac{1}{8}\sin\pi\right] - \left[0 + \frac{1}{8}\sin 0\right] \\
&= \left[-\frac{\pi}{8}(-1) + \frac{1}{8}(0)\right] - \left[0 + \frac{1}{8}(0)\right] \\
&= \frac{\pi}{8}
\end{aligned}$$

27. Let $I = \int x \sin 3x dx$.

Use **integration by parts** with $f(x) = x$ and $g'(x) = \sin 3x$.

$$f(x) = x \qquad \text{and} \qquad g'(x) = \sin 3x$$

$$\Rightarrow \qquad f'(x) = 1 \qquad \text{and} \qquad g(x) = -\frac{1}{3}\cos 3x$$

$$\begin{aligned}
I &= f(x)g(x) - \int f'(x)g(x)dx \\
&= x\left(-\frac{1}{3}\cos 3x\right) - \int 1\left(-\frac{1}{3}\cos 3x\right)dx \\
&= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x dx \\
&= -\frac{1}{3}x\cos 3x + \frac{1}{3}\left(\frac{1}{3}\sin 3x\right) + C \\
&= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C
\end{aligned}$$

Hence
$$\begin{aligned}
\int_0^{\frac{\pi}{6}} x \sin 3x dx &= \left[-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{6}} \\
&= \left[-\frac{1}{3}\left(\frac{\pi}{6}\right)\cos 3\left(\frac{\pi}{6}\right) + \frac{1}{9}\sin 3\left(\frac{\pi}{6}\right)\right] - \left[-\frac{1}{3}(0)\cos 3(0) + \frac{1}{9}\sin 3(0)\right] \\
&= \left[-\frac{\pi}{18}\cos\frac{\pi}{2} + \frac{1}{9}\sin\frac{\pi}{2}\right] - \left[0 + \frac{1}{9}\sin 0\right] \\
&= \left[-\frac{\pi}{18}(0) + \frac{1}{9}(1)\right] - \left[0 + \frac{1}{9}(0)\right] \\
&= \frac{1}{9}
\end{aligned}$$

28. Let $I = \int xe^{-2x}dx$.

Use integration by parts with $f(x) = x$ and $g'(x) = e^{-2x}$.

$$f(x) = x \quad \text{and} \quad g'(x) = e^{-2x}$$

$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = -\frac{1}{2}e^{-2x}$$

$$\begin{aligned}
I &= f(x)g(x) - \int f'(x)g(x)dx \\
&= x\left(-\frac{1}{2}e^{-2x}\right) - \int 1\left(-\frac{1}{2}e^{-2x}\right)dx \\
&= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx \\
&= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\left(-\frac{1}{2}e^{-2x}\right) + C \\
&= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C
\end{aligned}$$

Hence
$$\begin{aligned}
\int_0^1 xe^{-2x}dx &= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1 \\
&= \left[-\frac{1}{2}(1)e^{-2(1)} - \frac{1}{4}e^{-2(1)}\right] - \left[-\frac{1}{2}(0)e^{-2(0)} - \frac{1}{4}e^{-2(0)}\right] \\
&= \left[-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right] - \left[0 - \frac{1}{4}e^{0}\right] \\
&= \left[-\frac{3}{4}e^{-2}\right] - \left[-\frac{1}{4}(1)\right] \\
&= -\frac{3}{4}e^{-2} + \frac{1}{4} \\
&= \frac{1}{4} - \frac{3}{4}e^{-2} \\
&= \frac{1}{4}(1 - 3e^{-2})
\end{aligned}$$

29. Let $I = \int x\sqrt{x+1}dx$.

Use integration by parts with $f(x) = x$ and $g'(x) = \sqrt{x+1}$.

Note that $g(x) = \int \sqrt{x+1}dx = \int (x+1)^{\frac{1}{2}} dx = \dfrac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}\times 1} = \dfrac{2(x+1)^{\frac{3}{2}}}{3}$

$$f(x) = x \qquad \text{and} \qquad g'(x) = \sqrt{x+1}$$

$$\Rightarrow \qquad f'(x) = 1 \qquad \text{and} \qquad g(x) = \frac{2(x+1)^{\frac{3}{2}}}{3}$$

$$I = f(x)g(x) - \int f'(x)g(x)dx$$

$$= x\left(\frac{2(x+1)^{\frac{3}{2}}}{3}\right) - \int 1\left(\frac{2(x+1)^{\frac{3}{2}}}{3}\right)dx$$

$$= \frac{2x(x+1)^{\frac{3}{2}}}{3} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}dx$$

$$= \frac{2x(x+1)^{\frac{3}{2}}}{3} - \frac{2}{3}\left(\frac{(x+1)^{\frac{5}{2}}}{\frac{5}{2}\times 1}\right) + C$$

$$= \frac{2x(x+1)^{\frac{3}{2}}}{3} - \frac{2}{3}\left(\frac{2(x+1)^{\frac{5}{2}}}{5}\right) + C$$

$$= \frac{2x\sqrt{(x+1)^3}}{3} - \frac{4\sqrt{(x+1)^5}}{15} + C$$

Hence $\displaystyle\int_0^3 x\sqrt{x+1}dx = \left[\frac{2x\sqrt{(x+1)^3}}{3} - \frac{4\sqrt{(x+1)^5}}{15}\right]_0^3$

$$= \left[\frac{2(3)\sqrt{4^3}}{3} - \frac{4\sqrt{4^5}}{15}\right] - \left[\frac{2(0)\sqrt{1^3}}{3} - \frac{4\sqrt{1^5}}{15}\right]$$

$$= \left[\frac{2(3)(8)}{3} - \frac{4(32)}{15}\right] - \left[0 - \frac{4(1)}{15}\right]$$

$$= \left[\frac{48}{3} - \frac{128}{15}\right] - \left[-\frac{4}{15}\right]$$

$$= \frac{116}{15}$$

30. Let $I = \int x^2 \ln x dx$.

Use integration by parts with $f(x) = \ln x$ and $g'(x) = x^2$.
Note that we must choose $f(x) = \ln x$ as there is no standard integral for $\ln x$ at Advanced Higher.

$$f(x) = \ln x \quad \text{and} \quad g'(x) = x^2$$

$$\Rightarrow \quad f'(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{x^3}{3}$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= \ln x\left(\frac{x^3}{3}\right) - \int \frac{1}{x}\left(\frac{x^3}{3}\right)dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{3}\left(\frac{x^3}{3}\right) + C \\ &= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C \end{aligned}$$

31. Let $I = \int \frac{\ln x}{x^3} dx = \int x^{-3} \ln x dx$.

Use integration by parts with $f(x) = \ln x$ and $g'(x) = x^{-3}$.
Note that we must choose $f(x) = \ln x$ as there is no standard integral for $\ln x$ at Advanced Higher.

$$f(x) = \ln x \quad \text{and} \quad g'(x) = x^{-3}$$

$$\Rightarrow \quad f'(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{x^{-2}}{-2}$$

$$\begin{aligned}
I &= f(x)g(x) - \int f'(x)g(x)dx \\
&= \ln x\left(\frac{x^{-2}}{-2}\right) - \int \frac{1}{x}\left(\frac{x^{-2}}{-2}\right)dx \\
&= \ln x\left(-\frac{1}{2x^2}\right) + \frac{1}{2}\int x^{-3}dx \\
&= -\frac{\ln x}{2x^2} + \frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + C \\
&= -\frac{\ln x}{2x^2} + \frac{x^{-2}}{(-4)} + C \\
&= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C
\end{aligned}$$

32. Let $I = \int \ln x dx$.

Use integration by parts with $f(x) = \ln x$ and $g'(x) = 1$.
Note that we must choose $f(x) = \ln x$ as there is no standard integral for $\ln x$ at Advanced Higher.

$$f(x) = \ln x \quad \text{and} \quad g'(x) = 1$$
$$\Rightarrow \quad f'(x) = \frac{1}{x} \quad \text{and} \quad g(x) = x$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x\ln x - \int \frac{1}{x}(x)dx \\ &= x\ln x - \int 1dx \\ &= x\ln x - x + C \end{aligned}$$

Hence $\int_2^4 \ln x dx = [x\ln x - x]_2^4$

$$\begin{aligned} &= [4\ln 4 - 4] - [2\ln 2 - 2] \\ &= [4\ln 2^2 - 4] - [2\ln 2 - 2] \qquad \text{[write } \ln 4 \text{ as } \ln 2^2\text{]} \\ &= [8\ln 2 - 4] - [2\ln 2 - 2] \\ &= 8\ln 2 - 4 - 2\ln 2 + 2 \\ &= 6\ln 2 - 2 \end{aligned}$$

Note

There are other ways of dealing with the logarithms to complete the evaluation in terms of $\ln 2$, eg $[4\ln 4 - 4] - [2\ln 2 - 2] = [\ln 4^4 - 4] - [\ln 2^2 - 2]$

$$\begin{aligned} &= [\ln 256 - 4] - [\ln 4 - 2] \\ &= \ln 256 - 4 - \ln 4 + 2 \\ &= (\ln 256 - \ln 4) - 2 \\ &= \ln\left(\frac{256}{4}\right) - 2 \\ &= \ln 64 - 2 \\ &= \ln 2^6 - 2 \\ &= 6\ln 2 - 2 \end{aligned}$$

33. Let $I = \int x^2 \sin x dx$.

Use integration by parts with $f(x) = x^2$ and $g'(x) = \sin x$.

$$f(x) = x^2 \quad \text{and} \quad g'(x) = \sin x$$
$$\Rightarrow \quad f'(x) = 2x \quad \text{and} \quad g(x) = -\cos x$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x^2(-\cos x) - \int 2x(-\cos x)dx \\ &= -x^2 \cos x + 2\int x \cos x dx \qquad ...(*) \end{aligned}$$

Integration by parts must be used again to find $\int x \cos x dx$.

Let $J = \int x \cos x dx$.

$$f(x) = x \quad \text{and} \quad g'(x) = \cos x$$
$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = \sin x$$

$$\begin{aligned} J &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x\sin x - \int 1(\sin x)dx \\ &= x\sin x - \int \sin x dx \\ &= x\sin x - (-\cos x) \end{aligned}$$

[there is no need to include a constant of integration here]

$$= x\sin x + \cos x$$

Hence from (*): $I = -x^2 \cos x + 2(x\sin x + \cos x) + C$

$$= -x^2 \cos x + 2x\sin x + 2\cos x + C$$

34. Let $I = \int 8x^2 \sin 4x dx$.

Use integration by parts with $f(x) = 8x^2$ and $g'(x) = \sin 4x$.

$$f(x) = 8x^2 \quad \text{and} \quad g'(x) = \sin 4x$$

$$\Rightarrow \quad f'(x) = 16x \quad \text{and} \quad g(x) = -\frac{1}{4}\cos 4x$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= 8x^2(-\frac{1}{4}\cos 4x) - \int 16x(-\frac{1}{4}\cos 4x)dx \\ &= -2x^2 \cos 4x + 4\int x\cos 4x dx \qquad \text{...(*)} \end{aligned}$$

Integration by parts must be used again to find $\int x\cos 4x dx$.

Let $J = \int x\cos 4x dx$.

$$f(x) = x \quad \text{and} \quad g'(x) = \cos 4x$$

$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = \frac{1}{4}\sin 4x$$

$$\begin{aligned} J &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x\left(\frac{1}{4}\sin 4x\right) - \int 1\left(\frac{1}{4}\sin 4x\right)dx \\ &= \frac{1}{4}x\sin 4x - \frac{1}{4}\int \sin 4x dx \\ &= \frac{1}{4}x\sin 4x - \frac{1}{4}(-\frac{1}{4}\cos 4x) \qquad \text{[there is no need to include a constant of integration here]} \\ &= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x \end{aligned}$$

Hence from (*): $I = -2x^2 \cos 4x + 4\left(\frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x\right) + C$

$$= -2x^2 \cos 4x + x\sin 4x + \frac{1}{4}\cos 4x + C$$

35. Let $I = \int x^2 \cos 3x dx$.

Use integration by parts with $f(x) = x^2$ and $g'(x) = \cos 3x$.

$$f(x) = x^2 \qquad \text{and} \qquad g'(x) = \cos 3x$$

$$\Rightarrow \qquad f'(x) = 2x \qquad \text{and} \qquad g(x) = \frac{1}{3}\sin 3x$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x^2\left(\frac{1}{3}\sin 3x\right) - \int 2x\left(\frac{1}{3}\sin 3x\right)dx \\ &= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x dx \qquad \text{...(*)} \end{aligned}$$

Integration by parts must be used again to find $\int x \sin 3x dx$.

Let $J = \int x \sin 3x dx$.

$$f(x) = x \qquad \text{and} \qquad g'(x) = \sin 3x$$

$$\Rightarrow \qquad f'(x) = 1 \qquad \text{and} \qquad g(x) = -\frac{1}{3}\cos 3x$$

$$\begin{aligned} J &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x\left(-\frac{1}{3}\cos 3x\right) - \int 1\left(-\frac{1}{3}\cos 3x\right)dx \\ &= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x dx \\ &= -\frac{1}{3}x\cos 3x + \frac{1}{3}\left(\frac{1}{3}\sin 3x\right) \qquad \text{[there is no need to include a constant of integration here]} \\ &= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x \end{aligned}$$

Hence from (*):
$$\begin{aligned} I &= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\left(-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right) + C \\ &= \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C \end{aligned}$$

36. Let $I = \int x^2 \sin 5x dx$.

Use integration by parts with $f(x) = x^2$ and $g'(x) = \sin 5x$.

$$f(x) = x^2 \quad \text{and} \quad g'(x) = \sin 5x$$

$$\Rightarrow \quad f'(x) = 2x \quad \text{and} \quad g(x) = -\frac{1}{5}\cos 5x$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x^2\left(-\frac{1}{5}\cos 5x\right) - \int 2x\left(-\frac{1}{5}\cos 5x\right)dx \\ &= -\frac{1}{5}x^2\cos 5x + \frac{2}{5}\int x\cos 5x dx \qquad \text{...(*)} \end{aligned}$$

Integration by parts must be used again to find $\int x\cos 5x dx$.

Let $J = \int x\cos 5x dx$.

$$f(x) = x \quad \text{and} \quad g'(x) = \cos 5x$$

$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = \frac{1}{5}\sin 5x$$

$$\begin{aligned} J &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x\left(\frac{1}{5}\sin 5x\right) - \int 1\left(\frac{1}{5}\sin 5x\right)dx \\ &= \frac{1}{5}x\sin 5x - \frac{1}{5}\int \sin 5x dx \\ &= \frac{1}{5}x\sin 5x - \frac{1}{5}\left(-\frac{1}{5}\cos 5x\right) \qquad \text{[there is no need to include a constant of integration here]} \\ &= \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x \end{aligned}$$

Hence from (*):
$$\begin{aligned} I &= -\frac{1}{5}x^2\cos 5x + \frac{2}{5}\left(\frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x\right) + C \\ &= -\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x + \frac{2}{125}\cos 5x + C \end{aligned}$$

37. Let $I = \int x^2 e^{3x} dx$.

Use **integration by parts** with $f(x) = x^2$ and $g'(x) = e^{3x}$.

$$f(x) = x^2 \quad \text{and} \quad g'(x) = e^{3x}$$

$$\Rightarrow \quad f'(x) = 2x \quad \text{and} \quad g(x) = \frac{1}{3}e^{3x}$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x^2\left(\frac{1}{3}e^{3x}\right) - \int 2x\left(\frac{1}{3}e^{3x}\right)dx \\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}dx \qquad \text{...(*)} \end{aligned}$$

Integration by parts must be used again to find $\int xe^{3x}dx$.

Let $J = \int xe^{4x}dx$.

$$f(x) = x \quad \text{and} \quad g'(x) = e^{3x}$$

$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = \frac{1}{3}e^{3x}$$

$$\begin{aligned} J &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x\left(\frac{1}{3}e^{3x}\right) - \int 1\left(\frac{1}{3}e^{3x}\right)dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{3}\left(\frac{1}{3}e^{3x}\right) \text{ [there is no need to include a constant of integration here]} \\ &= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} \end{aligned}$$

Hence from (*): $$\begin{aligned} I &= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right) + C \\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C \end{aligned}$$

38. Let $I = \int x^2 e^{4x} dx$.

Use **integration by parts** with $f(x) = x^2$ and $g'(x) = e^{4x}$.

$$f(x) = x^2 \quad \text{and} \quad g'(x) = e^{4x}$$

$$\Rightarrow \quad f'(x) = 2x \quad \text{and} \quad g(x) = \frac{1}{4}e^{4x}$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x^2\left(\frac{1}{4}e^{4x}\right) - \int 2x\left(\frac{1}{4}e^{4x}\right)dx \\ &= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}dx \qquad \text{...(*)} \end{aligned}$$

Integration by parts must be used again to find $\int xe^{4x}dx$.

Let $J = \int xe^{4x}dx$.

$$f(x) = x \quad \text{and} \quad g'(x) = e^{4x}$$

$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = \frac{1}{4}e^{4x}$$

$$\begin{aligned} J &= f(x)g(x) - \int f'(x)g(x)dx \\ &= x\left(\frac{1}{4}e^{4x}\right) - \int 1\left(\frac{1}{4}e^{4x}\right)dx \\ &= \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}dx \\ &= \frac{1}{4}xe^{4x} - \frac{1}{4}\left(\frac{1}{4}e^{4x}\right) \qquad \text{[there is no need to include a constant of integration here]} \\ &= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \end{aligned}$$

Hence from (*): $I = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left(\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}\right) + C$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$$

$$\int_0^2 x^2 e^{4x} dx = \left[\frac{1}{4}x^2 e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x}\right]_0^2$$

$$= \left[\frac{1}{4}(2)^2 e^{4(2)} - \frac{1}{8}(2)e^{4(2)} + \frac{1}{32}e^{4(2)}\right] - \left[\frac{1}{4}(0)^2 e^{4(0)} - \frac{1}{8}(0)e^{4(0)} + \frac{1}{32}e^{4(0)}\right]$$

$$= \left[\frac{1}{4}(4)e^8 - \frac{1}{8}(2)e^8 + \frac{1}{32}e^8\right] - \left[0 - 0 + \frac{1}{32}e^0\right]$$

$$= \left[e^8 - \frac{1}{4}e^8 + \frac{1}{32}e^8\right] - \left[\frac{1}{32}(1)\right]$$

$$= \left[\frac{25}{32}e^8\right] - \left[\frac{1}{32}\right]$$

$$= \frac{1}{32}(25e^8 - 1)$$

39. Let $I = \int x^7 (\ln x)^2 dx$.

The integrand is a product of two functions of x which suggests that **integration by parts** could be used.
Use integration by parts with $f(x) = (\ln x)^2$ and $g'(x) = x^7$.

To differentiate $f(x) = (\ln x)^2$, use the **chain rule**:

$$\frac{d}{dx}(\ln x)^2 = 2\ln x \times \frac{1}{x} = \frac{2\ln x}{x}$$

$$\begin{aligned} & f(x) = (\ln x)^2 \quad && \text{and} \quad && g'(x) = x^7 \\ \Rightarrow \quad & f'(x) = \frac{2\ln x}{x} \quad && \text{and} \quad && g(x) = \frac{x^8}{8} \end{aligned}$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= (\ln x)^2\left(\frac{x^8}{8}\right) - \int \frac{2\ln x}{x}\left(\frac{x^8}{8}\right)dx \\ &= \frac{x^8}{8}(\ln x)^2 - \frac{1}{4}\int x^7 \ln x dx \qquad \text{...(*)} \end{aligned}$$

Integration by parts must be used again to find $\int x^7 \ln x dx$.

Let $J = \int x^7 \ln x dx$.

$$\begin{aligned} & f(x) = \ln x \quad && \text{and} \quad && g'(x) = x^7 \\ \Rightarrow \quad & f'(x) = \frac{1}{x} \quad && \text{and} \quad && g(x) = \frac{x^8}{8} \end{aligned}$$

$$\begin{aligned} J &= f(x)g(x) - \int f'(x)g(x)dx \\ &= \ln x\left(\frac{x^8}{8}\right) - \int \frac{1}{x}\left(\frac{x^8}{8}\right)dx \\ &= \frac{x^8}{8}\ln x - \frac{1}{8}\int x^7 dx \\ &= \frac{x^8}{8}\ln x - \frac{1}{8}\left(\frac{x^8}{8}\right) \qquad \text{[there is no need to include a constant of integration here]} \\ &= \frac{x^8}{8}\ln x - \frac{x^8}{64} \end{aligned}$$

Hence from (*): $I = \frac{x^8}{8}(\ln x)^2 - \frac{1}{4}\left(\frac{x^8}{8}\ln x - \frac{x^8}{64}\right) + C$

$$= \frac{x^8}{8}(\ln x)^2 - \frac{x^8}{32}\ln x + \frac{x^8}{256} + C$$

40.(a) $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$

(b) Let $I = \int \dfrac{x}{\sqrt{1-x^2}}\,dx$.

Use the substitution $u = 1 - x^2$ to rewrite the entire integral in terms of *u*.

$$u = 1 - x^2 \quad \Rightarrow \quad \frac{du}{dx} = -2x \quad \Rightarrow \quad du = -2x\,dx \quad \Rightarrow \quad -\frac{1}{2}du = x\,dx$$

$$I = \int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}(x\,dx) = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right) = \int -\frac{1}{2}u^{-\frac{1}{2}}du$$

$$= -\frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= -\frac{1}{2}(2\sqrt{u}) + C$$

$$= -\sqrt{u} + C$$

$$= -\sqrt{1-x^2} + C$$

(c) Let $I = \int \sin^{-1}x \cdot \dfrac{x}{\sqrt{1-x^2}}\,dx$.

Use integration by parts with $f(x) = \sin^{-1}x$ and $g'(x) = \dfrac{x}{\sqrt{1-x^2}}$.

Note that we must choose $f(x) = \sin^{-1}x$ as there is no standard integral for $\sin^{-1}x$ at Advanced Higher.

$$f(x) = \sin^{-1}x \qquad \text{and} \qquad g'(x) = \frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \qquad f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad \text{and} \qquad g(x) = -\sqrt{1-x^2} \qquad \text{[using the result in (b)]}$$

$$I = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$= -\sqrt{1-x^2}\,\sin^{-1}x - \int \frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)dx$$

$$= -\sqrt{1-x^2}\,\sin^{-1}x + \int 1\,dx$$

$$= -\sqrt{1-x^2}\,\sin^{-1}x + x + C$$

41.(a) Let $I = \int \frac{x}{\sqrt{1-x^2}}\,dx$.

Use the substitution $u = 1 - x^2$ to rewrite the entire integral in terms of *u*.

$$u = 1 - x^2 \quad \Rightarrow \quad \frac{du}{dx} = -2x \quad \Rightarrow \quad du = -2x\,dx \quad \Rightarrow \quad -\frac{1}{2}du = x\,dx$$

$$I = \int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}(x\,dx) = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right) = \int -\frac{1}{2}u^{-\frac{1}{2}}du$$

$$= -\frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= -\frac{1}{2}(2\sqrt{u}) + C$$

$$= -\sqrt{u} + C$$

$$= -\sqrt{1-x^2} + C$$

(b) Let $I = \int \sin^{-1} x\,dx$.

Use integration by parts with $f(x) = \sin^{-1} x$ and $g'(x) = 1$.

Note that we must choose $f(x) = \sin^{-1} x$ as there is no standard integral for $\sin^{-1} x$ at Advanced Higher.

$$f(x) = \sin^{-1} x \qquad \text{and} \qquad g'(x) = 1$$

$$\Rightarrow \quad f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad \text{and} \qquad g(x) = x$$

$$I = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$= x\sin^{-1}x - \int \frac{1}{\sqrt{1-x^2}}(x)\,dx$$

$$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}x - \left(-\sqrt{1-x^2}\right) + C \qquad \text{[using the result in (a)]}$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

Hence $\int_0^{\frac{1}{2}} \sin^{-1} x dx = \left[x \sin^{-1} x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$$= \left[\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \sqrt{1-\left(\frac{1}{2}\right)^2}\right] - \left[0 \sin^{-1} 0 + \sqrt{1-0^2}\right]$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{6}\right) + \sqrt{\frac{3}{4}}\right] - \left[0 + \sqrt{1}\right]$$

$$= \left[\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right] - [1]$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

42. Let $I = \int x\tan^{-1}(x^2)dx$.

Use integration by parts with $f(x) = \tan^{-1}(x^2)$ and $g'(x) = x$.
Note that we must choose $f(x) = \tan^{-1} x$ as there is no standard integral for $\tan^{-1} x$ at Advanced Higher.

To differentiate $f(x) = \tan^{-1}(x^2)$, use the **chain rule**:

$$\frac{d}{dx}\tan^{-1}(x^2) = \left(\frac{1}{1+(x^2)^2}\right)\times 2x = \frac{2x}{1+x^4}$$

$$f(x) = \tan^{-1}(x^2) \qquad \text{and} \qquad g'(x) = x$$

$$\Rightarrow \qquad f'(x) = \frac{2x}{1+x^4} \qquad \text{and} \qquad g(x) = \frac{x^2}{2}$$

$$\begin{aligned} I &= f(x)g(x) - \int f'(x)g(x)dx \\ &= \frac{x^2}{2}\tan^{-1}(x^2) - \int\left(\frac{2x}{1+x^4}\right)\frac{x^2}{2}\,dx \\ &= \frac{x^2}{2}\tan^{-1}(x^2) - \int\frac{x^3}{1+x^4}dx \end{aligned}$$

To find $\int\frac{x^3}{1+x^4}\,dx$, make the numerator the exact derivative of the denominator by adjusting the constant and then integrate using $\int\frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

Note that $\frac{d}{dx}(1+x^4) = 4x^3$, so the numerator must be written as $4x^3$.

$$\int\frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int\frac{4x^3}{1+x^4}\,dx = \frac{1}{4}\ln|1+x^4| + C$$

Hence $I = \frac{x^2}{2}\tan^{-1}(x^2) - \frac{1}{4}\ln|1+x^4| + C$.

$$\int_0^1 x\tan^{-1}(x^2)dx = \left[\frac{x^2}{2}\tan^{-1}(x^2) - \frac{1}{4}\ln\left|1+x^4\right|\right]_0^1$$

$$= \left[\frac{1^2}{2}\tan^{-1}(1^2) - \frac{1}{4}\ln\left|1+1^4\right|\right] - \left[\frac{0^2}{2}\tan^{-1}(0^2) - \frac{1}{4}\ln\left|1+0^2\right|\right]$$

$$= \left[\frac{1}{2}\tan^{-1}1 - \frac{1}{4}\ln|2|\right] - \left[0 - \frac{1}{4}\ln|1|\right]$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{4}\ln 2\right] - \left[0 - \frac{1}{4}\ln 1\right]$$

$$= \left[\frac{\pi}{8} - \frac{1}{4}\ln 2\right] - \left[0 - \frac{1}{4}(0)\right]$$

$$= \frac{\pi}{8} - \frac{1}{4}\ln 2$$

43.(a) Let $I = \int xe^{2x} dx$.

The integrand is a product of two functions of x which suggests that **integration by parts** could be used.
Use integration by parts with $f(x) = x$ and $g'(x) = e^{2x}$.

$$f(x) = x \quad \text{and} \quad g'(x) = e^{2x}$$

$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = \frac{1}{2}e^{2x}$$

$$\begin{aligned}
I &= f(x)g(x) - \int f'(x)g(x)dx \\
&= x\left(\frac{1}{2}e^{2x}\right) - \int 1\left(\frac{1}{2}e^{2x}\right)dx \\
&= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx \\
&= \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C \\
&= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C
\end{aligned}$$

Hence $\int_0^1 xe^{2x}dx = \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1$

$$\begin{aligned}
&= \left[\frac{1}{2}(1)e^{2(1)} - \frac{1}{4}e^{2(1)}\right] - \left[\frac{1}{2}(0)e^{2(0)} - \frac{1}{4}e^{2(0)}\right] \\
&= \left[\frac{1}{2}e^{2} - \frac{1}{4}e^{2}\right] - \left[0 - \frac{1}{4}e^{0}\right] \\
&= \left[\frac{1}{4}e^{2}\right] - \left[-\frac{1}{4}(1)\right] \\
&= \frac{1}{4}e^{2} + \frac{1}{4} \\
&= \frac{1}{4}(e^{2} + 1)
\end{aligned}$$

(b) Let $I = \int_0^1 x^2 e^{2x} dx$.

The integrand is a product of two functions of x which suggests that **integration by parts** could be used.
Use integration by parts with $f(x) = x^2$ and $g'(x) = e^{2x}$.

$$f(x) = x^2 \quad \text{and} \quad g'(x) = e^{2x}$$

$$\Rightarrow \quad f'(x) = 2x \quad \text{and} \quad g(x) = \frac{1}{2}e^{2x}$$

$$\begin{aligned}
I &= f(x)g(x) - \int f'(x)g(x)dx \\
&= \left[\frac{1}{2}x^2 e^{2x}\right]_0^1 - \int_0^1 2x\left(\frac{1}{2}e^{2x}\right)dx \\
&= \left[\frac{1}{2}(1)^2 e^{2(1)}\right] - \left[\frac{1}{2}(0)^2 e^{2(0)}\right] - \int_0^1 xe^{2x}dx \\
&= \left[\frac{1}{2}(1)e^2\right] - [0] - \frac{1}{4}(e^2+1) \qquad \text{[using the answer in (a)]} \\
&= \frac{1}{2}e^2 - \frac{1}{4}(e^2+1) \\
&= \frac{1}{2}e^2 - \frac{1}{4}e^2 - \frac{1}{4} \\
&= \frac{1}{4}e^2 - \frac{1}{4} \\
&= \frac{1}{4}(e^2-1)
\end{aligned}$$

(c)
$$\begin{aligned}
\int_0^1 (3x^2+2x)e^{2x}dx &= \int_0^1 (3x^2e^{2x} + 2xe^{2x})dx \\
&= \int_0^1 3x^2e^{2x}dx + \int_0^1 2xe^{2x}dx \\
&= 3\int_0^1 x^2e^{2x}dx + 2\int_0^1 xe^{2x}dx \\
&= 3\left(\frac{1}{4}(e^2-1)\right) + 2\left(\frac{1}{4}(e^2+1)\right) \qquad \text{[using the answers in (a) and (b)]} \\
&= \frac{3}{4}(e^2-1) + \frac{1}{2}(e^2+1) \\
&= \frac{3}{4}e^2 - \frac{3}{4} + \frac{1}{2}e^2 + \frac{1}{2} = \frac{5}{4}e^2 - \frac{1}{4} = \frac{1}{4}(5e^2-1)
\end{aligned}$$

44. Let $I = \int e^x \cos x dx$.

Use integration by parts with $f(x) = e^x$ and $g'(x) = \cos x$.

$$f(x) = e^x \quad \text{and} \quad g'(x) = \cos x$$
$$\Rightarrow \quad f'(x) = e^x \quad \text{and} \quad g(x) = \sin x$$

$$I = f(x)g(x) - \int f'(x)g(x)dx$$
$$= e^x \sin x - \int e^x \sin x dx \qquad \text{...(*)}$$

Integration by parts must be used again to find $\int e^x \sin x dx$.

The choice of $f(x)$ and $g'(x)$ must be consistent with that used already.

Let $J = \int e^x \sin x dx$.

$$f(x) = e^x \quad \text{and} \quad g'(x) = \sin x$$
$$\Rightarrow \quad f'(x) = e^x \quad \text{and} \quad g(x) = -\cos x$$

$$J = f(x)g(x) - \int f'(x)g(x)dx$$
$$= e^x(-\cos x) - \int e^x(-\cos x)dx$$
$$= -e^x \cos x + \int e^x \cos x dx$$

To break this 'infinite loop', note that $\int e^x \cos x dx$ is the original integral, *I*, so $J = -e^x \cos x + I$.

Hence from (*): $I = e^x \sin x - (-e^x \cos x + I)$

$$\Rightarrow \quad I = e^x \sin x + e^x \cos x - I$$
$$\Rightarrow \quad 2I = e^x \sin x + e^x \cos x$$
$$\Rightarrow \quad 2I = e^x(\sin x + \cos x)$$
$$\Rightarrow \quad I = \frac{1}{2}e^x(\sin x + \cos x) + C$$

Hence $\int e^x \cos x dx = \frac{1}{2}e^x(\cos x + \sin x) + C$.

45.(a) Let $I = \int e^x \cos x dx$.

Use integration by parts with $f(x) = e^x$ and $g'(x) = \cos x$.

$$f(x) = e^x \quad \text{and} \quad g'(x) = \cos x$$
$$\Rightarrow \quad f'(x) = e^x \quad \text{and} \quad g(x) = \sin x$$

$$I = f(x)g(x) - \int f'(x)g(x)dx$$
$$= e^x \sin x - \int e^x \sin x dx \qquad ...(*)$$

Integration by parts must be used again to find $\int e^x \sin x dx$.

The choice of $f(x)$ and $g'(x)$ must be consistent with that used already.

Let $J = \int e^x \sin x dx$.

$$f(x) = e^x \quad \text{and} \quad g'(x) = \sin x$$
$$\Rightarrow \quad f'(x) = e^x \quad \text{and} \quad g(x) = -\cos x$$

$$J = f(x)g(x) - \int f'(x)g(x)dx$$
$$= e^x(-\cos x) - \int e^x(-\cos x)dx$$
$$= -e^x \cos x + \int e^x \cos x dx$$

To break this 'infinite loop', note that $\int e^x \cos x dx$ is the original integral, *I*, so

$J = -e^x \cos x + I$.

Hence from (*): $I = e^x \sin x - (-e^x \cos x + I)$

$$\Rightarrow \quad I = e^x \sin x + e^x \cos x - I$$
$$\Rightarrow \quad 2I = e^x \sin x + e^x \cos x$$
$$\Rightarrow \quad 2I = e^x(\sin x + \cos x)$$
$$\Rightarrow \quad I = \frac{1}{2}e^x(\sin x + \cos x) + C$$

Hence $\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) + C$.

(b) **Method 1**

$I_n = \int e^x \cos nx dx$, where $n \neq 0$.

Use integration by parts with $f(x) = e^x$ and $g'(x) = \cos nx$.

$$f(x) = e^x \quad \text{and} \quad g'(x) = \cos nx$$

$$\Rightarrow \quad f'(x) = e^x \quad \text{and} \quad g(x) = \frac{1}{n}\sin nx$$

$$\begin{aligned} I_n &= f(x)g(x) - \int f'(x)g(x)dx \\ &= e^x\left(\frac{1}{n}\sin nx\right) - \int e^x\left(\frac{1}{n}\sin nx\right)dx \\ &= \frac{1}{n}e^x \sin nx - \frac{1}{n}\int e^x \sin nx dx \qquad ...(*) \end{aligned}$$

Integration by parts must be used again to find $\int e^x \sin nx dx$.

The choice of $f(x)$ and $g'(x)$ must be consistent with that used already.

Let $J_n = \int e^x \sin nx dx$.

$$f(x) = e^x \quad \text{and} \quad g'(x) = \sin nx$$

$$\Rightarrow \quad f'(x) = e^x \quad \text{and} \quad g(x) = -\frac{1}{n}\cos nx$$

$$\begin{aligned} J_n &= f(x)g(x) - \int f'(x)g(x)dx \\ &= e^x\left(-\frac{1}{n}\cos nx\right) - \int e^x\left(-\frac{1}{n}\cos nx\right)dx \\ &= -\frac{1}{n}e^x \cos nx + \frac{1}{n}\int e^x \cos nx dx \end{aligned}$$

To break this ‘infinite loop’, note that $\int e^x \cos nx dx$ is the original integral, I_n, so

$$J_n = -\frac{1}{n}e^x \cos nx + \frac{1}{n}I_n.$$

Hence from (*): $$I_n = \frac{1}{n}e^x \sin nx - \frac{1}{n}\left(-\frac{1}{n}e^x \cos nx + \frac{1}{n}I_n\right)$$

$$\Rightarrow \quad I_n = \frac{1}{n}e^x \sin nx + \frac{1}{n^2}e^x \cos nx - \frac{1}{n^2}I_n$$ [$\times n^2$ to clear the fractions]

$$\Rightarrow \quad n^2 I_n = ne^x \sin nx + e^x \cos nx - I_n$$

$$\Rightarrow \quad n^2 I_n + I_n = ne^x \sin nx + e^x \cos nx$$

$$\Rightarrow \quad I_n(n^2+1) = ne^x \sin nx + e^x \cos nx$$

$$\Rightarrow \quad I_n(n^2+1) = e^x(n \sin nx + \cos nx)$$

$$\Rightarrow \quad I_n = \frac{1}{n^2+1}e^x(n \sin nx + \cos nx) + C$$

Method 2

$I_n = \int e^x \cos nx dx$, where $n \neq 0$.

Use integration by parts with $f(x) = \cos nx$ and $g'(x) = e^x$.

	$f(x) = \cos nx$	and	$g'(x) = e^x$
$\Rightarrow$	$f'(x) = -n \sin nx$	and	$g(x) = e^x$

$$I_n = f(x)g(x) - \int f'(x)g(x)dx$$
$$= \cos nx \times e^x - \int -n \sin nx \times e^x dx$$
$$= e^x \cos nx + n\int e^x \sin nxdx \qquad ...(*)$$

Integration by parts must be used again to find $\int e^x \sin nxdx$.

The choice of $f(x)$ and $g'(x)$ must be consistent with that used already.

Let $J_n = \int e^x \sin nxdx$.

	$f(x) = \sin nx$	and	$g'(x) = e^x$
$\Rightarrow$	$f'(x) = n \cos nx$	and	$g(x) = e^x$

$$J_n = f(x)g(x) - \int f'(x)g(x)dx$$
$$= \sin nx \times e^x - \int n \cos nx \times e^x dx$$
$$= e^x \sin nx - n\int e^x \cos nxdx$$

To break this ‘infinite loop’, note that $\int e^x \cos nx dx$ is the original integral, I_n, so $J_n = e^x \sin nx - nI_n$.

Hence from (*):

$$I_n = e^x \cos nx + n(e^x \sin nx - nI_n)$$

$$\Rightarrow \quad I_n = e^x \cos nx + ne^x \sin nx - n^2 I_n$$

$$\Rightarrow \quad I_n + n^2 I_n = e^x \cos nx + ne^x \sin nx$$

$$\Rightarrow \quad I_n(1+n^2) = e^x \cos nx + ne^x \sin nx$$

$$\Rightarrow \quad I_n(1+n^2) = e^x(\cos nx + n\cos nx)$$

$$\Rightarrow \quad I_n = \frac{1}{1+n^2} e^x(\cos nx + n\sin nx) + C$$

$$or \qquad I_n = \frac{1}{n^2+1} e^x(n\sin nx + \cos nx) + C$$

(c) $\int e^x \cos 8x dx = I_8 = \frac{1}{8^2+1} e^x(8\sin 8x + \cos 8x) + C$

$$= \frac{1}{65} e^x(8\sin 8x + \cos 8x) + C$$

Hence $\int_0^{\frac{\pi}{2}} e^x \cos 8x dx = \left[\frac{1}{65} e^x(8\sin 8x + \cos 8x)\right]_0^{\frac{\pi}{2}}$

$$= \left[\frac{1}{65} e^{\frac{\pi}{2}}\left(8\sin 8\left(\frac{\pi}{2}\right) + \cos 8\left(\frac{\pi}{2}\right)\right)\right] - \left[\frac{1}{65} e^0(8\sin 8(0) + \cos 8(0))\right]$$

$$= \left[\frac{1}{65} e^{\frac{\pi}{2}}(8\sin 4\pi + \cos 4\pi)\right] - \left[\frac{1}{65} e^0(8\sin 0 + \cos 0)\right]$$

$$= \left[\frac{1}{65} e^{\frac{\pi}{2}}(8(0)+1)\right] - \left[\frac{1}{65}(1)(8(0)+1)\right]$$

$$= \left[\frac{1}{65} e^{\frac{\pi}{2}}(1)\right] - \left[\frac{1}{65}(1)\right]$$

$$= \frac{1}{65} e^{\frac{\pi}{2}} - \frac{1}{65}$$

$$= \frac{1}{65}\left(e^{\frac{\pi}{2}} - 1\right)$$

46.(a) To evaluate $I_1 = \int_0^1 xe^{-x}dx$, let $I = \int xe^{-x}dx$.

Use integration by parts with $f(x) = x$ and $g'(x) = e^{-x}$.

$$f(x) = x \quad \text{and} \quad g'(x) = e^{-x}$$
$$\Rightarrow \quad f'(x) = 1 \quad \text{and} \quad g(x) = -e^{-x}$$

$$I = f(x)g(x) - \int f'(x)g(x)dx$$
$$= x(-e^{-x}) - \int 1(-e^{-x})dx$$
$$= -xe^{-x} + \int e^{-x}dx$$
$$= -xe^{-x} - e^{-x} + C$$

Hence $I_1 = \left[-xe^{-x} - e^{-x}\right]_0^1 = \left[-1e^{-1} - e^{-1}\right] - \left[-0e^0 - e^0\right]$
$$= \left[-e^{-1} - e^{-1}\right] - \left[0 - 1\right]$$
$$= \left[-2e^{-1}\right] - \left[-1\right]$$
$$= -2e^{-1} + 1$$
$$= 1 - 2e^{-1}$$

(b) $I_n = \int_0^1 x^n e^{-x}dx$ for $n \geq 1$.

Use integration by parts with $f(x) = x^n$ and $g'(x) = e^{-x}$.

$$f(x) = x^n \quad \text{and} \quad g'(x) = e^{-x}$$
$$\Rightarrow \quad f'(x) = nx^{n-1} \quad \text{and} \quad g(x) = -e^{-x}$$

$$I_n = f(x)g(x) - \int f'(x)g(x)dx$$
$$= \left[x^n(-e^{-x})\right]_0^1 - \int_0^1 nx^{n-1}(-e^{-x})dx$$
$$= \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1}e^{-x}dx$$
$$= \left[-1^n e^{-1}\right] - \left[-0^n e^0\right] + nI_{n-1} \qquad \left[\text{since } I_{n-1} = \int_0^1 x^{n-1}e^{-x}dx\right]$$
$$= \left[-1e^{-1}\right] - \left[0\right] + nI_{n-1} \qquad \left[\text{since } 1^n = 1 \text{ and } 0^n = 0\right]$$
$$= nI_{n-1} - e^{-1}$$

Hence $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.

(c) The recurrence relation $I_n = nI_{n-1} - e^{-1}$ with starting value $I_1 = 1 - 2e^{-1}$ can be used to generate the values of I_2, I_3, ...

Using $I_n = nI_{n-1} - e^{-1}$ with $n = 2$ $\Rightarrow$

$$\begin{aligned} I_2 &= 2I_1 - e^{-1} \\ &= 2(1 - 2e^{-1}) - e^{-1} \\ &= 2 - 4e^{-1} - e^{-1} \\ &= 2 - 5e^{-1} \end{aligned}$$

Using $I_n = nI_{n-1} - e^{-1}$ with $n = 3$ $\Rightarrow$

$$\begin{aligned} I_3 &= 3I_2 - e^{-1} \\ &= 3(2 - 5e^{-1}) - e^{-1} \\ &= 6 - 15e^{-1} - e^{-1} \\ &= 6 - 16e^{-1} \end{aligned}$$

Hence $I_3 = 6 - 16e^{-1}$.