ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Maclaurin Series

1.(a) $f(x)=\cos x \quad \Rightarrow \quad f(0)=\cos 0 = 1$

$f'(x)=-\sin x \quad \Rightarrow \quad f'(0)=-\sin 0 = 0$

$f''(x)=-\cos x \quad \Rightarrow \quad f''(0)=-\cos 0 = -1$

$f'''(x)=\sin x \quad \Rightarrow \quad f'''(0)=\sin 0 = 0$

$f^{(4)}(x)=\cos x \quad \Rightarrow \quad f^{(4)}(0)=\cos 0 = 1$

The Maclaurin series for $f(x)$ is given by

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\ldots$$
$$\Rightarrow \quad \cos x = 1+0x+\frac{(-1)}{2}x^2+0x^3+\frac{1}{24}x^4+\ldots$$
$$\Rightarrow \quad \cos x = 1-\frac{1}{2}x^2+\frac{1}{24}x^4+\ldots$$

(b) $g(x)=\frac{1}{2}\cos 2x$

To deduce the Maclaurin series for $\cos 2x$, simply replace x with $2x$ in the Maclaurin series for $\cos x$.

$$\cos 2x = 1-\frac{1}{2}(2x)^2+\frac{1}{24}(2x)^4+\ldots = 1-\frac{1}{2}(4x^2)+\frac{1}{24}(16x^4)+\ldots = 1-2x^2+\frac{2}{3}x^4+\ldots$$

Hence $\quad g(x)=\frac{1}{2}\cos 2x = \frac{1}{2}\left(1-2x^2+\frac{2}{3}x^4+\ldots\right)=\frac{1}{2}-x^2+\frac{1}{3}x^4+\ldots$

(c) To find the Maclaurin series for $g(3x)$, simply replace x with $3x$ in the Maclaurin series for $g(x)$.

$$g(3x)=\frac{1}{2}-(3x)^2+\frac{1}{3}(3x)^4+\ldots = \frac{1}{2}-9x^2+\frac{1}{3}(81x^4)+\ldots = \frac{1}{2}-9x^2+27x^4+\ldots$$

2.(a) Let $f(x) = e^x \quad \Rightarrow \quad f(0) = e^0 = 1$

$f'(x) = e^x \quad \Rightarrow \quad f'(0) = e^0 = 1$

$f''(x) = e^x \quad \Rightarrow \quad f''(0) = e^0 = 1$

$f'''(x) = e^x \quad \Rightarrow \quad f'''(0) = e^0 = 1$

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\Rightarrow \quad e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

(b) **<u>Method 1</u>**

$$(1+e^x)^2 = (1+e^x)(1+e^x) = 1 + 2e^x + e^{2x}$$

To deduce the Maclaurin series for e^{2x}, simply replace x with $2x$ in the Maclaurin series for e^x.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \quad \Rightarrow \quad e^{2x} = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \dots$$

$$= 1 + 2x + \frac{1}{2}(4x^2) + \frac{1}{6}(8x^3) + \dots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$$

Hence $(1+e^x)^2 = 1 + 2e^x + e^{2x}$

$$= 1 + 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right) + \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots\right) + \dots$$

$$= 1 + 2 + 2x + x^2 + \frac{1}{3}x^3 + 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$$

$$= 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \dots$$

Method 2

$$(1+e^x)^2 = \left(1+\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+...\right)\right)^2$$

$$= \left(2+x+\frac{1}{2}x^2+\frac{1}{6}x^3+...\right)^2$$

$$= \left(2+x+\frac{1}{2}x^2+\frac{1}{6}x^3+...\right)\left(2+x+\frac{1}{2}x^2+\frac{1}{6}x^3+...\right)$$

$$= 2(2)+2(x)+2\left(\frac{1}{2}x^2\right)+2\left(\frac{1}{6}x^3\right)+x(2)+x(x)+x\left(\frac{1}{2}x^2\right)+\frac{1}{2}x^2(2)+\frac{1}{2}x^2(x)+\frac{1}{6}x^3(2)+...$$

$$= 4+2x+x^2+\frac{1}{3}x^3+2x+x^2+\frac{1}{2}x^3+x^2+\frac{1}{2}x^3+\frac{1}{3}x^3+...$$

$$= 4+4x+3x^2+\frac{5}{3}x^3+...$$

Method 3

You can find the Maclaurin series for $(1+e^x)^2$ from first principles by finding the derivatives of $f(x)=(1+e^x)^2$.

Let $f(x)=(1+e^x)^2 \quad \Rightarrow \quad f(0)=(1+e^0)^2=(1+1)^2=4$

To differentiate $f(x)$, use the **chain rule**.

$$f'(x)=2(1+e^x)\times e^x = 2e^x(1+e^x)=2e^x+2e^{2x} \quad \Rightarrow \quad f'(0)=2e^0+2e^{2(0)}=4$$

$$f''(x)=2e^x+2(2e^{2x})=2e^x+4e^{2x} \quad \Rightarrow \quad f''(0)=2e^0+4e^{2(0)}=6$$

$$f'''(x)=2e^x+4(2e^{2x})=2e^x+8e^{2x} \quad \Rightarrow \quad f'''(x)=2e^0+8e^{2(0)}=10$$

The Maclaurin series for $f(x)$ is given by

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+...$$

$$\Rightarrow \quad (e^x+1)^2=4+4x+\frac{6}{2}x^2+\frac{10}{6}x^3+...$$

$$\Rightarrow \quad (e^x+1)^2=4+4x+3x^2+\frac{5}{3}x^3+...$$

3. Let $f(x) = \sin 3x \quad \Rightarrow \quad f(0) = \sin 3(0) = \sin 0 = 0$

$f'(x) = 3\cos 3x \quad \Rightarrow \quad f'(0) = 3\cos 3(0) = 3\cos 0 = 3(1) = 3$

$f''(x) = 3(-3\sin 3x) = -9\sin 3x \quad \Rightarrow \quad f''(0) = -9\sin 3(0) = -9\sin 0 = -9(0) = 0$

$f'''(x) = -9(3\cos 3x) = -27\cos 3x \quad \Rightarrow \quad f'''(0) = -27\cos 3(0) = -27\cos 0 = -27(1) = -27$

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$\Rightarrow \quad \sin 3x = 0 + 3x + 0x^2 + \frac{(-27)}{6}x^3 + \ldots$$

$$\Rightarrow \quad \sin 3x = 3x - \frac{9}{2}x^3 + \ldots$$

Let $g(x) = e^{4x} \quad \Rightarrow \quad g(0) = e^{4(0)} = e^0 = 1$

$g'(x) = 4e^{4x} \quad \Rightarrow \quad g'(0) = 4e^{4(0)} = 4(1) = 4$

$g''(x) = 4(4e^{4x}) = 16e^{4x} \quad \Rightarrow \quad g''(0) = 16e^{4(0)} = 16(1) = 16$

$g'''(x) = 16(4e^{4x}) = 64e^{4x} \quad \Rightarrow \quad g'''(0) = 64e^{4(0)} = 64(1) = 64$

The Maclaurin series for $g(x)$ is given by

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \ldots$$

$$\Rightarrow \quad e^{4x} = 1 + 4x + \frac{16}{2}x^2 + \frac{64}{6}x^3 + \ldots$$

$$\Rightarrow \quad e^{4x} = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \ldots$$

Hence $$e^{4x}\sin 3x = \left(1 + 4x + 8x^2 + \frac{32}{3}x^3 + \ldots\right)\left(3x - \frac{9}{2}x^3 + \ldots\right)$$

$$= 1(3x) + 1\left(-\frac{9}{2}x^3\right) + 4x(3x) + 8x^2(3x) + \ldots$$

$$= 3x - \frac{9}{2}x^3 + 12x^2 + 24x^3 + \ldots$$

$$= 3x + 12x^2 + \frac{39}{2}x^3 + \ldots$$

4. Let $f(x)=\cos 3x \quad \Rightarrow \quad f(0)=\cos 3(0)=\cos 0=1$

$$f'(x)=-3\sin 3x \quad \Rightarrow \quad f'(0)=-3\sin 3(0)=-3\sin 0=-3(0)=0$$

$$f''(x)=-3(3\cos 3x)=-9\cos 3x \quad \Rightarrow \quad f''(0)=-9\cos 3(0)=-9\cos 0=-9(1)=-9$$

$$f'''(x)=-9(-3\sin 3x)=27\sin 3x \quad \Rightarrow \quad f'''(0)=27\sin 3(0)=27\sin 0=27(0)=0$$

$$f^{(4)}(x)=27(3\cos 3x)=81\cos 3x \quad \Rightarrow \quad f^{(4)}(0)=81\cos 3(0)=81\cos 0=81(1)=81$$

The Maclaurin series for $f(x)$ is given by

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}+\ldots$$

$$\Rightarrow \quad \cos 3x = 1+0x+\frac{(-9)}{2}x^2+0x^3+\frac{81}{24}x^4+\ldots$$

$$\Rightarrow \quad \cos 3x = 1-\frac{9}{2}x^2+\frac{27}{8}x^4+\ldots$$

Let $g(x)=e^{2x} \quad \Rightarrow \quad g(0)=e^{2(0)}=e^0=1$

$$g'(x)=2e^{2x} \quad \Rightarrow \quad g'(0)=2e^{2(0)}=2(1)=2$$

$$g''(x)=2(2e^{2x})=4e^{2x} \quad \Rightarrow \quad g''(0)=4e^{2(0)}=4(1)=4$$

$$g'''(x)=4(2e^{2x})=8e^{2x} \quad \Rightarrow \quad g'''(0)=8e^{2(0)}=8(1)=8$$

The Maclaurin series for $g(x)$ is given by

$$g(x)=g(0)+g'(0)x+\frac{g''(0)}{2!}x^2+\frac{g'''(0)}{3!}x^3+\ldots$$

$$\Rightarrow \quad e^{2x}=1+2x+\frac{4}{2}x^2+\frac{8}{6}x^3+\ldots$$

$$\Rightarrow \quad e^{2x}=1+2x+2x^2+\frac{4}{3}x^3+\ldots$$

Hence $e^{2x}\cos 3x=\left(1+2x+2x^2+\frac{4}{3}x^3+\ldots\right)\left(1-\frac{9}{2}x^2+\frac{27}{8}x^4+\ldots\right)$

$$=1(1)+1\left(-\frac{9}{2}x^2\right)+2x(1)+2x\left(-\frac{9}{2}x^2\right)+2x^2(1)+\frac{4}{3}x^3(1)+\ldots$$

$$=1-\frac{9}{2}x^2+2x-9x^3+2x^2+\frac{4}{3}x^3+\ldots$$

$$=1+2x-\frac{5}{2}x^2-\frac{23}{3}x^3+\ldots$$

5. Let $f(x)=\sqrt{1+x} \quad \Rightarrow \quad f(0)=\sqrt{1+0}=1$

$$f(x)=(1+x)^{\frac{1}{2}} \quad \Rightarrow \quad f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}\times 1=\frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$\Rightarrow \quad f'(0)=\frac{1}{2}(1+0)^{-\frac{1}{2}}=\frac{1}{2}(1)=\frac{1}{2}$$

$$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}} \quad \Rightarrow \quad f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}\times 1=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$\Rightarrow \quad f''(0)=-\frac{1}{4}(1+0)^{-\frac{3}{2}}=-\frac{1}{4}(1)=-\frac{1}{4}$$

$$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}} \quad \Rightarrow \quad f'''(x)=\frac{3}{8}(1+x)^{-\frac{5}{2}}\times 1=\frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$$\Rightarrow \quad f'''(0)=\frac{3}{8}(1+0)^{-\frac{5}{2}}=\frac{3}{8}(1)=\frac{3}{8}$$

The Maclaurin series for $f(x)$ is given by

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$

$$\Rightarrow \quad \sqrt{1+x}=1+\frac{1}{2}x+\frac{\left(-\frac{1}{4}\right)}{2}x^2+\frac{\frac{3}{8}}{6}x^3+\ldots$$

$$\Rightarrow \quad \sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$$

To deduce the Maclaurin series for $\sqrt{1+x^2}$, simply replace *x* with x^2 in the Maclaurin series for $\sqrt{1+x}$.

$$\sqrt{1+x^2}=1+\frac{1}{2}x^2-\frac{1}{8}(x^2)^2+\frac{1}{16}(x^2)^3+\ldots=1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6+\ldots$$

Hence $\sqrt{(1+x)(1+x^2)}=\sqrt{1+x}\sqrt{1+x^2}$ [since $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$]

$$=\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots\right)\left(1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6+\ldots\right)$$

$$=1(1)+1\left(\frac{1}{2}x^2\right)+\frac{1}{2}x(1)+\frac{1}{2}x\left(\frac{1}{2}x^2\right)-\frac{1}{8}x^2(1)+\frac{1}{16}x^3(1)+\ldots$$

$$=1+\frac{1}{2}x^2+\frac{1}{2}x+\frac{1}{4}x^3-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$$

$$=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x+\ldots$$

Q.(a) (i) $f(x)=e^{3x} \quad \Rightarrow \quad f(0)=e^{3(0)}=e^{0}=1$

$f'(x)=3e^{3x} \quad \Rightarrow \quad f'(0)=3e^{3(0)}=3(1)=3$

$f''(x)=3(3e^{3x})=9e^{3x} \quad \Rightarrow \quad f''(0)=9e^{3(0)}=9(1)=9$

$f'''(x)=9(3e^{3x})=27e^{3x} \quad \Rightarrow \quad f'''(0)=27e^{3(0)}=27(1)=27$

The Maclaurin series for $f(x)$ is given by

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$

$$\Rightarrow \quad e^{3x}=1+3x+\frac{9}{2}x^2+\frac{27}{6}x^3+\ldots$$

$$\Rightarrow \quad e^{3x}=1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\ldots$$

(ii) $g(x)=(x+2)^{-2}=\dfrac{1}{(x+2)^2} \quad \Rightarrow \quad g(0)=\dfrac{1}{2^2}=\dfrac{1}{4}$

$$g(x)=(x+2)^{-2} \quad \Rightarrow \quad g'(x)=-2(x+2)^{-3}\times 1=-2(x+2)^{-3}=\frac{-2}{(x+2)^3}$$

$$\Rightarrow \quad g'(0)=\frac{-2}{2^3}=\frac{-2}{8}=-\frac{1}{4}$$

$$g'(x)=-2(x+2)^{-3} \quad \Rightarrow \quad g''(x)=6(x+2)^{-4}\times 1=6(x+2)^{-4}=\frac{6}{(x+2)^4}$$

$$\Rightarrow \quad g''(0)=\frac{6}{2^4}=\frac{6}{16}=\frac{3}{8}$$

$$g''(x)=6(x+2)^{-4} \quad \Rightarrow \quad g'''(x)=-24(x+2)^{-5}\times 1=-24(x+2)^{-5}=\frac{-24}{(x+2)^5}$$

$$\Rightarrow \quad g'''(0)=\frac{-24}{2^5}=\frac{-24}{32}=-\frac{3}{4}$$

The Maclaurin series for $g(x)$ is given by

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \ldots$$

$$\Rightarrow \quad (x+2)^{-2} = \frac{1}{4} + \left(-\frac{1}{4}\right)x + \frac{3/8}{2}x^2 + \frac{\left(-3/4\right)}{6}x^3 + \ldots$$

$$\Rightarrow \quad (x+2)^{-2} = \frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 - \frac{1}{8}x^3 + \ldots$$

(b) $$h(x) = \frac{xe^{3x}}{(x+2)^2} = xe^{3x}(x+2)^{-2}$$

$$= x\left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \ldots\right)\left(\frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 - \frac{1}{8}x^3 + \ldots\right)$$

$$\left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \ldots\right)\left(\frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 - \frac{1}{8}x^3 + \ldots\right)$$

$$= 1\left(\frac{1}{4}\right) + 1\left(-\frac{1}{4}x\right) + 1\left(\frac{3}{16}x^2\right) + 1\left(-\frac{1}{8}x^3\right) + 3x\left(\frac{1}{4}\right) + 3x\left(-\frac{1}{4}x\right) + 3x\left(\frac{3}{16}x^2\right) + \frac{9}{2}x^2\left(\frac{1}{4}\right)$$

$$+ \frac{9}{2}x^2\left(-\frac{1}{4}x\right) + \frac{9}{2}x^3\left(\frac{1}{4}\right) + \ldots$$

$$= \frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 - \frac{1}{8}x^3 + \frac{3}{4}x - \frac{3}{4}x^2 + \frac{9}{16}x^3 + \frac{9}{8}x^2 - \frac{9}{8}x^3 + \frac{9}{8}x^3 + \ldots$$

$$= \frac{1}{4} + \frac{1}{2}x + \frac{9}{16}x^2 + \frac{7}{16}x^3 + \ldots$$

Hence $$h(x) = x\left(\frac{1}{4} + \frac{1}{2}x + \frac{9}{16}x^2 + \frac{7}{16}x^3 + \ldots\right) = \frac{1}{4}x + \frac{1}{2}x^2 + \frac{9}{16}x^3 + \frac{7}{16}x^4 + \ldots$$

6. Let $f(x) = e^x \quad \Rightarrow \quad f(0) = e^0 = 1$

$$f'(x) = e^x \quad \Rightarrow \quad f'(0) = e^0 = 1$$

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = e^0 = 1$$

$$f'''(x) = e^x \quad \Rightarrow \quad f'''(0) = e^0 = 1$$

$$f^{(4)}(x) = e^x \quad \Rightarrow \quad f^{(4)}(0) = e^0 = 1$$

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

$$\Rightarrow \quad e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

To deduce the Maclaurin series for e^{x^2}, simply replace x with x^2 in the Maclaurin series for e^x.

$$e^{x^2} = 1 + x^2 + \frac{1}{2}(x^2)^2 + \frac{1}{6}(x^2)^3 + \frac{1}{24}(x^2)^4 \ldots = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 ..$$

$$e^{x+x^2} = e^x e^{x^2} = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots\right)\left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \ldots\right)$$

$$= 1(1) + 1(x^2) + 1\left(\frac{1}{2}x^4\right) + x(1) + x(x^2) + \frac{1}{2}x^2(1) + \frac{1}{2}x^2(x^2) + \frac{1}{6}x^3(1) + \frac{1}{24}x^4(1) + \ldots$$

$$= 1 + x^2 + \frac{1}{2}x^4 + x + x^3 + \frac{1}{2}x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{25}{24}x^4 + \ldots$$

Note

The Maclaurin series for e^{x+x^2} can also be found from first principles by finding the derivatives of $f(x) = e^{x+x^2}$, however this approach is not recommended.

7. The easiest way to find the Maclaurin series for $x\ln(2+x)$ is to find the Maclaurin series for $\ln(2+x)$ and then multiply this series by x.

Let $f(x) = \ln(2+x) \quad \Rightarrow \quad f(0) = \ln(2+0) = \ln 2$

$$f'(x) = \left(\frac{1}{2+x}\right) \times 1 = \frac{1}{2+x} \quad \Rightarrow \quad f'(0) = \frac{1}{2+0} = \frac{1}{2}$$

$$f'(x) = (2+x)^{-1} \quad \Rightarrow \quad f''(x) = -(2+x)^{-2} \times 1 = -\frac{1}{(2+x)^2}$$

$$\Rightarrow \quad f''(0) = -\frac{1}{(2+0)^2} = -\frac{1}{4}$$

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

$$\Rightarrow \quad \ln(2+x) = \ln 2 + \frac{1}{2}x + \frac{\left(-\frac{1}{4}\right)}{2}x^2 + \ldots$$

$$\Rightarrow \quad \ln(2+x) = \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$$

Hence $x\ln(2+x) = x\left(\ln 2 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right) = x\ln 2 + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \ldots$

To deduce the Maclaurin series for $x\ln(2-x)$, simply replace x with $-x$ in the Maclaurin series for $x\ln(2+x)$.

$$-x\ln(2+(-x)) = -x\ln 2 + \frac{1}{2}(-x)^2 - \frac{1}{8}(-x)^3 + \ldots$$

$$\Rightarrow \quad -x\ln(2-x) = -x\ln 2 + \frac{1}{2}x^2 - \frac{1}{8}(-x^3) + \ldots \qquad [\text{since } (-x)^2 = x^2 \text{ and } (-x)^3 = -x^3]$$

$$\Rightarrow \quad -x\ln(2-x) = -x\ln 2 + \frac{1}{2}x^2 + \frac{1}{8}x^3 + \ldots \qquad [\times(-1)]$$

$$\Rightarrow \quad x\ln(2-x) = x\ln 2 - \frac{1}{2}x^2 - \frac{1}{8}x^3 + \ldots$$

$$
\begin{aligned}
x\ln(4-x^2) &= x\ln\big((2-x)(2+x)\big) \\
&= x\big(\ln(2-x)+\ln(2+x)\big) \\
&= x\ln(2-x)+x\ln(2+x) \\
&= \left(x\ln 2+\frac{1}{2}x^2-\frac{1}{8}x^3+\ldots\right)+\left(x\ln 2-\frac{1}{2}x^2-\frac{1}{8}x^3+\ldots\right) \\
&= x\ln 2+\frac{1}{2}x^2-\frac{1}{8}x^3+x\ln 2-\frac{1}{2}x^2-\frac{1}{8}x^3+\ldots \\
&= 2x\ln 2-\frac{1}{4}x^3+\ldots
\end{aligned}
$$

Note

The Maclaurin series for $x\ln(2+x)$ can also be found from first principles by finding the derivatives of $f(x)=x\ln(2+x)$.

8. **Method 1**

$f(x) = e^x \sin x$

Find the Maclaurin series for e^x and $\sin x$ and then multiply the two series.

Let $g(x) = e^x \quad \Rightarrow \quad g(0) = e^0 = 1$

$g'(x) = e^x \quad \Rightarrow \quad g'(0) = e^0 = 1$

$g''(x) = e^x \quad \Rightarrow \quad g''(0) = e^0 = 1$

$g'''(x) = e^x \quad \Rightarrow \quad g''(0) = e^0 = 1$

The Maclaurin series for $g(x)$ is given by

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \ldots$$

$$\Rightarrow \quad e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$$

Let $h(x) = \sin x \quad \Rightarrow \quad f(0) = \sin 0 = 0$

$h'(x) = \cos x \quad \Rightarrow \quad h'(0) = \cos 0 = 1$

$h''(x) = -\sin x \quad \Rightarrow \quad h''(0) = -\sin 0 = 0$

$h'''(x) = -\cos x \quad \Rightarrow \quad h'''(0) = -\cos 0 = -1$

The Maclaurin series for $h(x)$ is given by

$$h(x) = h(0) + h'(0)x + \frac{h''(0)}{2!}x^2 + \frac{h'''(0)}{3!}x^3 + \ldots$$

$$\Rightarrow \quad \sin x = 0 + 1x + 0x^2 + \frac{(-1)}{6}x^3 + \ldots$$

$$\Rightarrow \quad \sin x = x - \frac{1}{6}x^3 + \ldots$$

Hence $f(x) = e^x \sin x = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots\right)\left(x - \frac{1}{6}x^3 + \ldots\right)$

$$= 1(x) + 1\left(-\frac{1}{6}x^3\right) + x(x) + \frac{1}{2}x^2(x) + \ldots$$

$$= x - \frac{1}{6}x^3 + x^2 + \frac{1}{2}x^3 + \ldots$$

$$= x + x^2 + \frac{1}{3}x^3 + \ldots$$

Method 2

The Maclaurin series for $f(x) = e^x \sin x$ can be found from first principles by finding the derivatives of $f(x)$.

$f(x) = e^x \sin x \quad \Rightarrow \quad f(0) = e^0 \sin 0 = 1(0) = 0$

To differentiate $f(x) = e^x \sin x$, use the **product rule** as $f(x)$ is the product of two functions of *x*.

$f'(x) = e^x \cos x + \sin x(e^x) = e^x(\sin x + \cos x) \quad \Rightarrow \quad f'(0) = e^0(\sin 0 + \cos 0) = 1(0+1) = 1$

To differentiate $f'(x) = e^x(\sin x + \cos x)$, use the **product rule** again.

$$\begin{aligned} f''(x) &= e^x(\cos x - \sin x) + (\sin x + \cos x)e^x \\ &= e^x(\cos x - \sin x) + e^x(\sin x + \cos x) \\ &= e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \\ &= 2e^x \cos x \qquad \Rightarrow \quad f''(0) = 2e^0 \cos 0 = 2(1)(1) = 2 \end{aligned}$$

To differentiate $f''(x) = 2e^x \cos x$, use the **product rule** again.

$$\begin{aligned} f'''(x) &= 2e^x(-\sin x) + \cos x(2e^x) \\ &= 2e^x \cos x - 2e^x \sin x \\ &= 2e^x(\cos x - \sin x) \qquad \Rightarrow \quad f'''(0) = 2e^0(\cos 0 - \sin 0) = 2(1)(1-0) = 2 \end{aligned}$$

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$\Rightarrow \quad e^x \sin x = 0 + 1x + \frac{2}{2}x^2 + \frac{2}{6}x^3 + \ldots$$

$$\Rightarrow \quad e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$

9. **Method 1**

To find the Maclaurin series for $f(x) = \sin^2 x$, you can square the Maclaurin series for $\sin x$.

Let $g(x) = \sin x \quad \Rightarrow \quad g(0) = \sin 0 = 0$

$$g'(x) = \cos x \quad \Rightarrow \quad g'(0) = \cos 0 = 1$$

$$g''(x) = -\sin x \quad \Rightarrow \quad g''(0) = -\sin 0 = 0$$

$$g'''(x) = -\cos x \quad \Rightarrow \quad g'''(0) = -\cos 0 = -1$$

$$g^{(4)}(x) = \sin x \quad \Rightarrow \quad g^{(4)}(0) = \sin 0 = 0$$

The Maclaurin series for $g(x)$ is given by

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''()}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 + \ldots$$

$$\Rightarrow \quad \sin x = 0 + 1x + 0x^2 + \frac{(-1)}{6}x^3 + 0x^4 + \ldots$$

$$\Rightarrow \quad \sin x = x - \frac{1}{6}x^3 + \ldots$$

$$f(x) = \sin^2 x = \left(x - \frac{1}{6}x^3 + \ldots\right)^2$$

$$= \left(x - \frac{1}{6}x^3 + \ldots\right)\left(x - \frac{1}{6}x^3 + \ldots\right)$$

$$= x(x) + x\left(-\frac{1}{6}x^3\right) - \frac{1}{6}x^3(x) + \ldots$$

$$= x^2 - \frac{1}{6}x^4 - \frac{1}{6}x^4 + \ldots$$

$$= x^2 - \frac{1}{3}x^4 + \ldots$$

The Maclaurin series for $\cos^2 x$ can be deduced from the Maclaurin series for $\sin^2 x$ using the identity $\cos^2 x = 1 - \sin^2 x$.

$$\cos^2 x = 1 - \sin^2 x = 1 - \left(x - \frac{1}{3}x^4 + ...\right) = 1 - x + \frac{1}{3}x^4 + ...$$

Method 2

The Maclaurin series for $f(x) = \sin^2 x$ can be found from first principles by finding the derivatives of $f(x)$.

$$f(x) = \sin^2 x \quad \Rightarrow \quad f(0) = \sin^2 0 = 0^2 = 0$$

$$f(x) = \sin^2 x = (\sin x)^2 \quad \Rightarrow \quad f'(x) = 2\sin x \cos x \quad \Rightarrow \quad f'(0) = 2\sin 0 \cos 0 = 2(0)(1) = 0$$

To differentiate $f'(x) = 2\sin x \cos x$, use the **product rule** as $f'(x)$ is the product of two functions of *x*.

$$\begin{aligned} f''(x) &= 2\sin x(-\sin x) + \cos x(2\cos x) \\ &= 2\cos^2 x - 2\sin^2 x \qquad \Rightarrow \quad f''(0) = 2\cos^2 0 - 2\sin^2 0 = 2(1)^2 - 2(0)^2 = 2 \end{aligned}$$

$$\begin{aligned} f''(x) = 2(\cos x)^2 - 2(\sin x)^2 \quad \Rightarrow \quad f'''(x) &= 4\cos x(-\sin x) - 4\sin x \cos x \\ &= -4\sin x \cos x - 4\sin x \cos x \\ &= -8\sin x \cos x \end{aligned}$$

$$\Rightarrow \quad f'''(0) = -8\sin 0 \cos 0 = -8(0)(1) = 0$$

To differentiate $f'''(x) = -8\sin x \cos x$, use the **product rule**.

$$\begin{aligned} f^{(4)}(x) &= -8\sin x(-\sin x) + \cos x(-8\cos x) \\ &= 8\sin^2 x - 8\cos^2 x \qquad \Rightarrow \quad f^{(4)}(0) = 8\sin^2 0 - 8\cos^2 0 = 8(0)^2 - 8(1)^2 = -8 \end{aligned}$$

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + ...$$

$$\Rightarrow \quad \sin^2 x = 0 + 0x + \frac{2}{2}x^2 + 0x^3 + \frac{(-8)}{24}x^4 + ...$$

$$\Rightarrow \quad \sin^2 x = x^2 - \frac{1}{3}x^4 + ...$$

Note

If you remember the identity $\sin 2x = 2\sin x\cos x$ from Higher, the first derivative of $f(x)$ can be written as $f'(x) = \sin 2x$. This makes it a lot easier to find the other derivatives: $f''(x) = 2\cos 2x$, $f'''(x) = -4\sin 2x$ and $f^{(4)}(x) = -8\cos 2x$.

10.(a) $f(x) = e^{2x} \quad \Rightarrow \quad f(0) = e^{2(0)} = e^0 = 1$

$f'(x) = 2e^{2x} \quad \Rightarrow \quad f'(0) = 2e^{2(0)} = 2(1) = 2$

$f''(x) = 2(2e^{2x}) = 4e^{2x} \quad \Rightarrow \quad f''(0) = 4e^{2(0)} = 4(1) = 4$

$f'''(x) = 4(2e^{2x}) = 8e^{2x} \quad \Rightarrow \quad f'''(0) = 8e^{2(0)} = 8(1) = 8$

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$\Rightarrow \quad e^{2x} = 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \ldots$$

$$\Rightarrow \quad e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$$

(b) (i) $g(x) = \tan x \quad \Rightarrow \quad g'(x) = \sec^2 x$

$g'(x) = (\sec x)^2 \quad \Rightarrow \quad g''(x) = 2\sec x(\sec x\tan x) = 2\sec^2 x\tan x$

To differentiate $g''(x) = 2\sec^2 x\tan x$ use the **product rule** as $g''(x)$ is the product of two functions of x and note that $\frac{d}{dx}(2\sec^2 x) = 2(2\sec^2 x\tan x) = 4\sec^2 x\tan x$.

$$g'''(x) = 2\sec^2 x\sec^2 x + \tan x(4\sec^2 x\tan x)$$
$$= 2\sec^4 x + 4\tan^2 x\sec^2 x$$

(ii) $g(x) = \tan x \quad \Rightarrow \quad g(0) = \tan 0 = 0$

$g'(x) = \sec^2 x \quad \Rightarrow \quad g'(0) = \sec^2 0 = 1^2 = 1 \qquad [\text{since } \sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1]$

$g''(x) = 2\sec^2 x\tan x \quad \Rightarrow \quad g''(0) = 2\sec^2 0\tan 0 = 2(1)^2(0) = 0$

$$g'''(x) = 2\sec^4 x + 4\tan^2 x\sec^2 x \quad \Rightarrow \quad g'''(0) = 2\sec^4 0 + 4\tan^2 0\sec^2 0$$
$$= 2(1)^4 + 4(0)^2(1)$$

$= 2$

The Maclaurin series for $g(x)$ is given by

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + ...$$

$$\Rightarrow \quad \tan x = 0 + 1x + 0x^2 + \frac{2}{6}x^3 + ...$$

$$\Rightarrow \quad \tan x = x + \frac{1}{3}x^3 + ...$$

(c) $e^{2x}\tan x = \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + ...\right)\left(x + \frac{1}{3}x^3 + ...\right)$

$$= 1(x) + 1\left(\frac{1}{3}x^3\right) + 2x(x) + 2x^2(x) + ...$$

$$= x + \frac{1}{3}x^3 + 2x^2 + 2x^3 + ...$$

$$= x + 2x^2 + \frac{7}{3}x^3 + ...$$

(d) To deduce the Maclaurin series for $2e^{2x}\tan x + e^{2x}\sec^2 x$ from the Maclaurin series for $e^{2x}\tan x$, note that $\frac{d}{dx}(e^{2x}\tan x) = e^{2x}\sec^2 x + \tan x(2e^{2x}) = 2e^{2x}\tan x + e^{2x}\sec^2 x$ using the **product rule**.

Start with the Macluarin series for $e^{2x}\tan x$: $\quad e^{2x}\tan x = x + 2x^2 + \frac{7}{3}x^3 + ...$

Now differentiate both sides with respect to x giving

$$2e^{2x}\tan x + e^{2x}\sec^2 x = 1 + 4x + 7x^2 + ...$$

11. The denominator contains a repeated linear factor.

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} = \frac{A}{x-4} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$
$$= \frac{A(x+2)^2 + B(x-4)(x+2) + C(x-4)}{(x+2)^2(x-4)}$$

$$x^2+6x-4 = A(x+2)^2 + B(x-4)(x+2) + C(x-4)$$

Let $x=-2 \quad \Rightarrow \quad (-2)^2+6(-2)-4 = A(0)^2 + B(-6)(0) + C(-6)$

$\Rightarrow \quad -12 = -6C$

$\Rightarrow \quad C = 2$

Let $x = 4 \quad \Rightarrow \quad 4^2+6(4)-4 = A(-6)^2 + B(0)(6) + C(0)$

$\Rightarrow \quad 36 = 36A$

$\Rightarrow \quad A = 1$

Equating coefficients of $x^2 \quad \Rightarrow \quad 1 = A + B \quad \Rightarrow \quad 1 = 1 + B \quad \Rightarrow \quad B = 0$

Hence $\dfrac{x^2+6x-4}{(x+2)^2(x-4)} = \dfrac{1}{x-4} + \dfrac{2}{(x+2)^2}$.

$$f(x) = \frac{x^2+6x-4}{(x+2)^2(x-4)} \quad \Rightarrow \quad f(0) = \frac{-4}{(2)^2(-4)} = \frac{-4}{-16} = \frac{1}{4}$$

To find the derivatives of $f(x)$, differentiate the partial fractions for $f(x)$.

$$f(x) = \frac{1}{x-4} + \frac{2}{(x+2)^2} = (x-4)^{-1} + 2(x+2)^{-2}$$

$$f'(x) = -(x-4)^{-2} \times 1 - 4(x+2)^{-3} \times 1 = -\frac{1}{(x-4)^2} - \frac{4}{(x+2)^3}$$

$$f'(0) = -\frac{1}{(-4)^2} - \frac{4}{2^3} = -\frac{1}{16} - \frac{4}{8} = -\frac{9}{16}$$

$$f'(x) = -(x-4)^{-2} - 4(x+2)^{-3} \quad \Rightarrow \quad f''(x) = 2(x-4)^{-3} \times 1 + 12(x+2)^{-4} \times 1$$

$$= \frac{2}{(x-4)^3} + \frac{12}{(x+2)^4}$$

$$\Rightarrow \quad f''(0) = \frac{2}{(-4)^3} + \frac{12}{2^4} = \frac{2}{-64} + \frac{12}{16} = \frac{23}{32}$$

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

$$\Rightarrow \quad f(x) = \frac{1}{4} + \left(-\frac{9}{16}\right)x + \frac{23/32}{2}x^2 \ldots$$

$$\Rightarrow \quad f(x) = \frac{1}{4} - \frac{9}{16}x + \frac{23}{64}x^2 \ldots$$