ADVANCED HIGHER MATHEMATICS

Solutions to Exam Questions on Matrices

1. $A^2 = AA = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2(2)+1(0) & 2(1)+1(-1) \\ 0(2)+(-1)(0) & 0(1)+(-1)(-1) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix}$

$A^2 - A = \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2I$

Hence $A^2 - A = kI$ where $k = 2$.

2.(a) $AB = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2(4)+(-1)(0) & 2(6)+(-1)(-3) \\ 3(4)+5(0) & 3(6)+5(-3) \end{pmatrix} = \begin{pmatrix} 8 & 15 \\ 12 & 3 \end{pmatrix}$

(b) $4C + D = 4\begin{pmatrix} x & 2 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix} = \begin{pmatrix} 4x & 8 \\ 0 & 4y \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix} = \begin{pmatrix} 4x+2 & 15 \\ 12 & 4y-1 \end{pmatrix}$

(c) $AB = 4C + D \quad \Rightarrow \quad \begin{pmatrix} 8 & 15 \\ 12 & 3 \end{pmatrix} = \begin{pmatrix} 4x+2 & 15 \\ 12 & 4y-1 \end{pmatrix}$

Equating entries $\Rightarrow \quad 4x + 2 = 8 \quad \Rightarrow \quad 4x = 6 \quad \Rightarrow \quad x = \frac{3}{2}$

and $\quad 4y - 1 = 3 \quad \Rightarrow \quad 4y = 4 \quad \Rightarrow \quad y = 1$

Hence $x = \frac{3}{2}$ and $y = 1$.

3.(a) Let $A = \begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}$.

$\det A = 2(3) - x(-1) = 6 + x \quad \Rightarrow \quad A^{-1} = \frac{1}{6+x}\begin{pmatrix} 3 & -x \\ 1 & 2 \end{pmatrix}$

(b) The matrix A is singular when $\det A = 0 \quad \Rightarrow \quad 6 + x = 0 \quad \Rightarrow \quad x = -6$

Note

The inverse of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when $\det A \neq 0$.

4.(a) $A = \begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix}$

$\det A = 5(t+4) - 3(3t) = 5t + 20 - 9t = 20 - 4t$

$$A^{-1} = \frac{1}{20-4t}\begin{pmatrix} 5 & -3t \\ -3 & t+4 \end{pmatrix}$$

(b) The matrix A is singular when $\det A = 0 \Rightarrow 20 - 4t = 0 \Rightarrow 20 = 4t \Rightarrow t = 5$

(c) $A = \begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} t+4 & 3 \\ 3t & 5 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix}$

Equating entries $\Rightarrow t + 4 = 6 \Rightarrow t = 2$ $\quad$ [$or \quad 3t = 6 \Rightarrow t = 2$]

<u>Note</u>

The inverse of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when $\det A \neq 0$.

5.(a) $A^2 = AA = \begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4(4)+x(0) & 4x+x(2) \\ 0(4)+2(0) & 0x+2(2) \end{pmatrix} = \begin{pmatrix} 16 & 6x \\ 0 & 4 \end{pmatrix}$

$3B = 3\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 3 \\ 0 & 3 \end{pmatrix}$

Hence $A^2 - 3B = \begin{pmatrix} 16 & 6x \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 15 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 6x-3 \\ 0 & 1 \end{pmatrix}$

(b) (i) $C = \begin{pmatrix} y & 3 \\ -1 & 2 \end{pmatrix} \Rightarrow \det C = 2y - 3(-1) = 2y + 3 \Rightarrow C^{-1} = \dfrac{1}{2y+3}\begin{pmatrix} 2 & -3 \\ 1 & y \end{pmatrix}$

(ii) The matrix C is singular when $\det C = 0 \Rightarrow 2y + 3 = 0 \Rightarrow 2y = -3$

$\Rightarrow y = -\dfrac{3}{2}$

<u>Note</u>

The inverse of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when $\det A \neq 0$.

6.(a) $A = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix}$

The matrix A is singular when $\det A = 0 \quad \Rightarrow \quad 1(4) - x(x) = 0 \quad \Rightarrow \quad 4 - x^2 = 0$

$$\Rightarrow \quad 4 = x^2$$

$$\Rightarrow \quad x = \pm 2$$

(b) (i) When $x = 2$, $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

$$A^2 = AA = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1(1)+2(2) & 1(2)+2(4) \\ 2(1)+4(2) & 2(2)+4(4) \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} = 5\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 5A$$

Hence $A^2 = pA$ where $p = 5$.

(ii) **<u>Method 1</u>**

An expression for A^4 in terms of A can be found using the fact that $A^2 = 5A$.

$$A^4 = A^2 A^2 = (5A)(5A) = 25A^2 = 25(5A) = 125A$$

Hence $A^4 = qA$ where $q = 125$.

<u>Method 2</u>

An expression for A^4 in terms of A can be found by finding the matrix A^4.

$$A^4 = A^2 A^2 = \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix}\begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} = \begin{pmatrix} 5(5)+10(10) & 5(10)+10(20) \\ 10(5)+20(10) & 10(10)+20(20) \end{pmatrix}$$

$$= \begin{pmatrix} 125 & 250 \\ 250 & 500 \end{pmatrix}$$

$$= 125\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= 125A$$

Hence $A^4 = qA$ where $q = 125$.

7.(a) $A = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix}$

The matrix A is singular when $\det A = 0 \Rightarrow \lambda(\lambda - 3) - 2(2) = 0$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$
$$\Rightarrow (\lambda + 1)(\lambda - 4) = 0$$
$$\Rightarrow \lambda = -1,\ \lambda = 4$$

(b) When $\lambda = 3$, $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$.

$$\det A = 3(0) - 2(2) = -4 \quad \Rightarrow \quad A^{-1} = \frac{1}{-4}\begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & \tfrac{1}{2} \\ \tfrac{1}{2} & -\tfrac{3}{4} \end{pmatrix}$$

<u>Note</u>

The inverse of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when $\det A \neq 0$.

8.(a) (i) $P = \begin{pmatrix} x & 2 \\ -5 & -1 \end{pmatrix}$

$\det P = 2 \quad \Rightarrow \quad x(-1) - 2(-5) = 2 \quad \Rightarrow \quad -x + 10 = 2 \quad \Rightarrow \quad -x = -8 \quad \Rightarrow \quad x = 8$

(ii) We now know that $x = 8$, so $P = \begin{pmatrix} 8 & 2 \\ -5 & -1 \end{pmatrix}$ and $\det P = 2$.

$$P^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -2 \\ 5 & 8 \end{pmatrix} \quad or \quad P^{-1} = \begin{pmatrix} -1/2 & -1 \\ 5/2 & 4 \end{pmatrix}$$

(iii) $Q = \begin{pmatrix} 2 & -3 \\ 4 & y \end{pmatrix} \quad \Rightarrow \quad Q' = \begin{pmatrix} 2 & 4 \\ -3 & y \end{pmatrix}$

$$P^{-1}Q' = \frac{1}{2}\begin{pmatrix} -1 & -2 \\ 5 & 8 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -3 & y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (-1)(2) + (-2)(-3) & (-1)(4) + (-2)y \\ 5(2) + 8(-3) & 5(4) + 8y \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & -4 - 2y \\ -14 & 20 + 8y \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 - y \\ -7 & 10 + 4y \end{pmatrix}$$

(b) $R = \begin{pmatrix} 5 & -2 \\ z & -6 \end{pmatrix}$

The matrix R is singular when $\det R = 0 \quad \Rightarrow \quad 5(-6) - (-2)z = 0 \quad \Rightarrow \quad -30 + 2z = 0$

$\Rightarrow \quad 2z = 30$

$\Rightarrow \quad z = 15$

Note

The inverse of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when $\det A \neq 0$.

9.(a) $A^2 = AA = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 4(4)+p(-2) & 4p+p(1) \\ (-2)(4)+1(-2) & -2p+1(1) \end{pmatrix} = \begin{pmatrix} 16-2p & 5p \\ -10 & 1-2p \end{pmatrix}$

(b) **<u>Method 1</u>**

The matrix A^2 is singular when $\det(A^2) = 0 \Rightarrow (16-2p)(1-2p) - 5p(-10) = 0$

$$\Rightarrow 16 - 34p + 4p^2 + 50p = 0$$
$$\Rightarrow 4p^2 + 16p + 16 = 0$$
$$\Rightarrow 4(p^2 + 4p + 4) = 0$$
$$\Rightarrow 4(p+2)(p+2) = 0$$
$$\Rightarrow p = -2$$

<u>Method 2</u>

The matrix A^2 is singular when $\det(A^2) = 0$.

To find $\det(A^2)$, we can use the fact that $\det(AB) = \det A \times \det B$.

$\det(A^2) = \det(AA) = \det A \times \det A = (\det A)^2$

$A = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} \Rightarrow \det A = 4(1) - p(-2) = 4 + 2p \Rightarrow \det(A^2) = (\det A)^2 = (4+2p)^2$

Hence the matrix A^2 is singular when $(4+2p)^2 = 0 \Rightarrow 4 + 2p = 0 \Rightarrow 2p = -4$

$$\Rightarrow p = -2$$

(c) $A = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 4 & -2 \\ p & 1 \end{pmatrix} \Rightarrow 3A' = 3\begin{pmatrix} 4 & -2 \\ p & 1 \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ 3p & 3 \end{pmatrix}$

$B = 3A' \Rightarrow \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ 3p & 3 \end{pmatrix}$

Equating entries $\Rightarrow x = 12$ and $3p = 1 \Rightarrow p = \frac{1}{3}$

Hence $p = \frac{1}{3}$ and $x = 12$.

10.(a) $A = \begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix} \quad \Rightarrow \quad \det A = 2(-1) - 0\lambda = -2$

(b) $A^2 = AA = \begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix} = \begin{pmatrix} 2(2)+0\lambda & 2(0)+0(-1) \\ \lambda(2)+(-1)\lambda & \lambda(0)+(-1)(-1) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix}$

$$A^2 = pA + qI \quad \Rightarrow \quad \begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix} = p\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 2p & 0 \\ \lambda p & -p \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 2p+q & 0 \\ \lambda p & -p+q \end{pmatrix}$$

Equating entries $\Rightarrow$ $2p + q = 4$ ...(1) and $-p + q = 1$ ...(2)

Solving equations (1) and (2) gives $p = 1$ and $q = 2$.

Hence $A^2 = pA + qI$ where $p = 1$ and $q = 2$.

(c) **<u>Method 1</u>**

A similar expression for A^4 in terms of A and I can be found using the fact that $A^2 = A + 2I$.

$$\begin{aligned} A^4 = A^2A^2 &= (A+2I)(A+2I) \\ &= A(A+2I) + 2I(A+2I) \\ &= A^2 + 2AI + 2IA + 4I^2 \\ &= A^2 + 2A + 2A + 4I \qquad [\text{since } AI = A,\ IA = A \text{ and } I^2 = II = I\,] \\ &= A^2 + 4A + 4I \\ &= (A+2I) + 4A + 4I \qquad [\text{since } A^2 = A + 2I\,] \\ &= 5A + 6I \end{aligned}$$

Hence $A^4 = 5A + 6I$.

Method 2

An expression for A^4 in terms of A and I can be found by finding the matrix A^4.

$$A^4 = A^2A^2 = \begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 4(4)+0\lambda & 4(0)+0(1) \\ \lambda(4)+1\lambda & \lambda(0)+1(1) \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 5\lambda & 1 \end{pmatrix}$$

We wish to find the values of x and y such that $A^4 = xA + yI$.

$$A^4 = xA + yI \quad \Rightarrow \quad \begin{pmatrix} 16 & 0 \\ 5\lambda & 1 \end{pmatrix} = x\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix} + y\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 16 & 0 \\ 5\lambda & 1 \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ \lambda x & -x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 16 & 0 \\ 5\lambda & 1 \end{pmatrix} = \begin{pmatrix} 2x+y & 0 \\ \lambda x & -x+y \end{pmatrix}$$

Equating entries $\Rightarrow$ $2x + y = 16$ ...(1) and $-x + y = 1$ ...(2)

Solving equations (1) and (2) gives $x = 5$ and $y = 6$.

Hence $A^4 = 5A + 6I$.

11.(a) $A=\begin{pmatrix}1 & -2\\ 3 & 0\end{pmatrix} \Rightarrow \det A = 1(0)-(-2)(3)=6$

$$A^{-1}=\frac{1}{6}\begin{pmatrix}0 & 2\\ -3 & 1\end{pmatrix} \quad \text{or} \quad A^{-1}=\begin{pmatrix}0 & 1/3\\ -1/2 & 1/6\end{pmatrix}$$

(b) **Method 1**

$$AB=\begin{pmatrix}-4 & -3\\ 6 & -3\end{pmatrix}$$

To find B, pre-multiply both sides of the equation by A^{-1}.

This gives $A^{-1}(AB)=A^{-1}\begin{pmatrix}-4 & -3\\ 6 & -3\end{pmatrix}$

$$\Rightarrow (A^{-1}A)B=\frac{1}{6}\begin{pmatrix}0 & 2\\ -3 & 1\end{pmatrix}\begin{pmatrix}-4 & -3\\ 6 & -3\end{pmatrix}$$

$$\Rightarrow IB=\frac{1}{6}\begin{pmatrix}0 & 2\\ -3 & 1\end{pmatrix}\begin{pmatrix}-4 & -3\\ 6 & -3\end{pmatrix} \qquad [\text{since } A^{-1}A=I]$$

$$\Rightarrow B=\frac{1}{6}\begin{pmatrix}0 & 2\\ -3 & 1\end{pmatrix}\begin{pmatrix}-4 & -3\\ 6 & -3\end{pmatrix} \qquad [\text{since } IB=B]$$

$$=\frac{1}{6}\begin{pmatrix}0(-4)+2(6) & 0(-3)+2(-3)\\ (-3)(-4)+1(6) & (-3)(-3)+1(-3)\end{pmatrix}$$

$$=\frac{1}{6}\begin{pmatrix}12 & -6\\ 18 & 6\end{pmatrix}$$

$$=\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}$$

Method 2

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$AB = \begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} a-2c & b-2d \\ 3a & 3b \end{pmatrix} = \begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix}$$

Equating entries: $3a = 6 \quad \Rightarrow \quad a = 2$

$3b = -3 \quad \Rightarrow \quad b = -1$

$a - 2c = -4 \quad \Rightarrow \quad 2 - 2c = -4 \quad \Rightarrow \quad -2c = -6 \quad \Rightarrow \quad c = 3$

$b - 2d = -3 \quad \Rightarrow \quad -1 - 2d = -3 \quad \Rightarrow \quad -2d = -2 \quad \Rightarrow \quad d = 1$

Hence $B = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$.

Note

The inverse of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when $\det A \neq 0$.

12.(a) $A = \begin{pmatrix} 1 & 3 & 4 \\ k & 0 & -1 \\ 5 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad \Rightarrow \quad A + B = \begin{pmatrix} 4 & -7 & 6 \\ k-3 & 9 & -1 \\ 5 & 1 & 1 \end{pmatrix}$

(b) Expanding along the top row of matrix A in a '+ – +' pattern gives

$$\det A = 1\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} - 3\begin{vmatrix} k & -1 \\ 5 & 0 \end{vmatrix} + 4\begin{vmatrix} k & 0 \\ 5 & 3 \end{vmatrix}$$
$$= 1\big(0(0) - (-1)(3)\big) - 3\big(k(0) - (-1)(5)\big) + 4\big(k(3) - 0(5)\big)$$
$$= 1(3) - 3(5) + 4(3k)$$
$$= 3 - 15 + 12k$$
$$= -12 + 12k$$
$$= 12k - 12$$

(c) $$BC = \begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 3(3)+(-10)(1)+2(2) & 3(2)+(-10)(1)+2(2) & 3(-6)+(-10)(-2)+2(-1) \\ (-3)(3)+9(1)+0(2) & (-3)(2)+9(1)+0(2) & (-3)(-6)+9(-2)+0(-1) \\ 0(3)+(-2)(1)+1(2) & 0(2)+(-2)(1)+1(2) & 0(-6)+(-2)(-2)+1(-1) \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(d) $$BC = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3I$$

$$BC = 3I \quad \Rightarrow \quad B\left(\frac{1}{3}C\right) = I$$

Hence the matrix $\frac{1}{3}C$ is the inverse of the matrix B, ie $B^{-1} = \frac{1}{3}C$.

[*or* $BC = 3I \quad \Rightarrow \quad \left(\frac{1}{3}B\right)C = I$, hence the matrix $\frac{1}{3}B$ is the inverse of the matrix C,

ie $C^{-1} = \frac{1}{3}B$]

13. $A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$

The matrix A is singular when $\det A = 0$.

Expanding along the top row of matrix A in a '+ – +' pattern gives

$$\det A = p\begin{vmatrix} p & 1 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 3 & p \\ 0 & -1 \end{vmatrix}$$
$$= p\big(p(-1) - 1(-1)\big) - 2\big(3(-1) - 1(0)\big) + 0$$
$$= p(-p+1) - 2(-3)$$
$$= -p^2 + p + 6$$

Hence the matrix A is singular when $-p^2 + p + 6 = 0 \quad [\times(-1)]$

$$\Rightarrow \quad p^2 - p - 6 = 0$$
$$\Rightarrow \quad (p+2)(p-3) = 0$$
$$\Rightarrow \quad p = -2,\ p = 3$$

14. $A = \begin{pmatrix} m & 1 & 1 \\ 0 & m & -2 \\ 1 & 0 & 1 \end{pmatrix}$

The matrix A is singular when $\det A = 0$.

Expanding along the top row of matrix A in a '+ – +' pattern gives

$$\det A = m\begin{vmatrix} m & -2 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & m \\ 1 & 0 \end{vmatrix}$$
$$= m\big(m(1) - (-2)(0)\big) - 1\big(0(1) - (-2)(1)\big) + 1\big(0(0) - m(1)\big)$$
$$= m(m) - 1(2) + 1(-m)$$
$$= m^2 - 2 - m$$
$$= m^2 - m - 2$$

Hence the matrix A is singular when $m^2 - m - 2 = 0$

$$\Rightarrow \quad (m+1)(m-2) = 0$$
$$\Rightarrow \quad m = -1,\ m = 2$$

15. Let $A=\begin{pmatrix}3 & k & 2\\ 3 & -4 & 2\\ k & 0 & 1\end{pmatrix}$.

The matrix A is singular when $\det A = 0$.

Expanding along the top row of matrix A in a '+ – +' pattern gives

$$\det A = 3\begin{vmatrix}-4 & 2\\ 0 & 1\end{vmatrix} - k\begin{vmatrix}3 & 2\\ k & 1\end{vmatrix} + 2\begin{vmatrix}3 & -4\\ k & 0\end{vmatrix}$$
$$= 3\big((-4)(1)-2(0)\big) - k\big(3(1)-2k\big) + 2\big(3(0)-(-4)k\big)$$
$$= 3(-4) - k(3-2k) + 2(4k)$$
$$= -12 - 3k + 2k^2 + 8k$$
$$= 2k^2 + 5k - 12$$

Hence the matrix A is singular when $2k^2 + 5k - 12 = 0$

$\Rightarrow\ (2k-3)(k+4)=0$

$\Rightarrow\ k=\dfrac{3}{2},\ k=-4$

16. Let $A=\begin{pmatrix}1 & 1 & 0\\ 0 & k-2 & -1\\ 1 & 2 & k\end{pmatrix}$.

The matrix A does not have an inverse when $\det A = 0$.

Expanding along the top row of matrix A in a '+ – +' pattern gives

$$\det A = 1\begin{vmatrix}k-2 & -1\\ 2 & k\end{vmatrix} - 1\begin{vmatrix}0 & -1\\ 1 & k\end{vmatrix} + 0\begin{vmatrix}0 & k-2\\ 1 & 2\end{vmatrix}$$
$$= 1\big(k(k-2)-(-1)2\big) - 1\big(0k-(-1)(1)\big) + 0$$
$$= 1(k^2-2k+2) - 1(1)$$
$$= k^2 - 2k + 2 - 1$$
$$= k^2 - 2k + 1$$

Hence the matrix A is singular when $k^2 - 2k + 1 = 0$

$\Rightarrow\ (k-1)(k-1)=0$

$\Rightarrow\ k=1$

17.(a) $C = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad C' = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

$$2C' - D = 2\begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 2 \\ k+3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 & 2 \\ 2 & -2 & 0 \\ 4 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 2 \\ k+3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 1 & 0 \\ -1-k & -2 & -2 \\ 3 & -1 & -3 \end{pmatrix}$$

[since $2-(k+3) = 2-k-3 = -1-k$]

(b) (i) $D = \begin{pmatrix} 1 & 1 & 2 \\ k+3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

Expanding along the top row of matrix D in a '+ – +' pattern gives

$$\det D = 1\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} k+3 & 2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} k+3 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1\big(0(1) - 2(1)\big) - 1\big(1(k+3) - 2(1)\big) + 2\big(1(k+3) - 0(1)\big)$$

$$= 1(-2) - 1(k+1) + 2(k+3)$$

$$= -2 - k - 1 + 2k + 6$$

$$= 3 + k$$

(ii) D^{-1} does not exist when $\det D = 0 \quad \Rightarrow \quad 3 + k = 0 \quad \Rightarrow \quad k = -3$

18.(a) Let $P = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}$.

The matrix P is singular when $\det P = 0$.

Expanding along the top row of matrix P in a '+ – +' pattern gives

$$\det P = 1\begin{vmatrix} 0 & 2 \\ \lambda & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ -1 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -1 & \lambda \end{vmatrix}$$
$$= 1\big(0(6) - 2\lambda\big) - 2\big(3(6) - 2(-1)\big) - 1\big(3\lambda - 0(-1)\big)$$
$$= 1(-2\lambda) - 2(20) - 1(3\lambda)$$
$$= -2\lambda - 40 - 3\lambda$$
$$= -5\lambda - 40$$

Hence the matrix P is singular when $-5\lambda - 40 = 0 \quad \Rightarrow \quad -5\lambda = 40 \quad \Rightarrow \quad \lambda = -8$

(b) $A = \begin{pmatrix} 2 & 2\alpha - \beta & -1 \\ 3\alpha + 2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix} \quad \Rightarrow \quad A' = \begin{pmatrix} 2 & 3\alpha + 2\beta & -1 \\ 2\alpha - \beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$

Equating entries $\Rightarrow$ $3\alpha + 2\beta = -5$ ...(1) and $2\alpha - \beta = -1$ ...(2)

Solving equations (1) and (2) gives $\alpha = -1$ and $\beta = -1$.

19. $$AB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1(1)+1(4)+1(-3) & 1(0)+1(-2)+1(2) & 1(1)+1(-2)+1(1) \\ 1(1)+2(4)+3(-3) & 1(0)+2(-2)+3(2) & 1(1)+2(-2)+3(1) \\ 1(1)+(-1)(4)+(-1)(-3) & 1(0)+(-1)(-2)+(-1)(2) & 1(1)+(-1)(-2)+(-1)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2I$$

Hence $AB = kI$ where $k = 2$.

(i) To find the inverse matrix A^{-1} from the equation $AB = 2I$, rearrange the equation into the form $A(...) = I$.

$$AB = 2I \quad \Rightarrow \quad A\left(\frac{1}{2}B\right) = I \quad \Rightarrow \quad A^{-1} = \frac{1}{2}B = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 2 & -1 & -1 \\ -3/2 & 1 & 1/2 \end{pmatrix}$$

(ii) $A^2B = (AA)B = A(AB)$

$= A(2I)$ [since $AB = 2I$]

$= 2AI$

$= 2A$ [since $AI = A$]

$$= 2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & -2 & -2 \end{pmatrix}$$

20. $A^2 = 4A - 3I$

$$\begin{aligned} A^4 = A^2A^2 &= (4A-3I)(4A-3I) \\ &= 4A(4A-3I) - 3I(4A-3I) \\ &= 16A^2 - 12AI - 12IA + 9I^2 \\ &= 16A^2 - 12A - 12A + 9I \qquad [\text{since } AI = A,\ IA = A \text{ and } I^2 = II = I\ ] \\ &= 16A^2 - 24A + 9I \\ &= 16(4A-3I) - 24A + 9I \qquad [\text{since } A^2 = 4A - 3I\ ] \\ &= 64A - 48I - 24A + 9I \\ &= 40A - 39I \end{aligned}$$

Hence $A^4 = pA + qI$ where $p = 40$ and $q = -39$.

21. $A^2 = AA = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 0(0)+4(1)+2(-1) & 0(4)+4(0)+2(-2) & 0(2)+4(1)+2(-3) \\ 1(0)+0(1)+1(-1) & 1(4)+0(0)+1(-2) & 1(2)+0(1)+1(-3) \\ (-1)(0)+(-2)(1)+(-3)(-1) & (-1)(4)+(-2)(0)+(-3)(-2) & (-1)(2)+(-2)(1)+(-3)(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{pmatrix}$$

$$A^2 + A = \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2I$$

Hence $A^2 + A = kI$ where $k = 2$.

To find the inverse matrix A^{-1} from the equation $A^2 + A = 2I$, rearrange the equation into the form $A(...) = I$.

$$A^2 + A = 2I \quad \Rightarrow \quad A(A+I) = 2I \quad \Rightarrow \quad A\left(\frac{1}{2}(A+I)\right) = I \quad \Rightarrow \quad A^{-1} = \frac{1}{2}(A+I)$$

$$\Rightarrow \quad A^{-1} = \frac{1}{2}A + \frac{1}{2}I$$

Hence $A^{-1} = pA + qI$ where $p = \frac{1}{2}$ and $q = \frac{1}{2}$.

22.(a) $AB = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1(x+2)+0(-4)+(-1)(2) & 1(x-2)+0(4)+(-1)(-2) & 1(x+3)+0(2)+(-1)(3) \\ 0(x+2)+1(-4)+(-1)(2) & 0(x-2)+1(4)+(-1)(-2) & 0(x+3)+1(2)+(-1)(3) \\ 0(x+2)+1(-4)+2(2) & 0(x-2)+1(4)+2(-2) & 0(x+3)+1(2)+2(3) \end{pmatrix}$$

$$= \begin{pmatrix} x+2-2 & x-2+2 & x+3-3 \\ -4-2 & 4+2 & 2-3 \\ -4+4 & 4-4 & 2+6 \end{pmatrix}$$

$$= \begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix}$$

(b) $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad \Rightarrow \quad \det A = 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$

$$= 1\big(1(2)-(-1)(1)\big) - 0 - 1\big(0(1)-1(0)\big)$$
$$= 1(3) - 1(0)$$
$$= 3$$

$AB = \begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix} \quad \Rightarrow \quad \det(AB) = x\begin{vmatrix} 6 & -1 \\ 0 & 8 \end{vmatrix} - x\begin{vmatrix} -6 & -1 \\ 0 & 8 \end{vmatrix} + x\begin{vmatrix} -6 & 6 \\ 0 & 0 \end{vmatrix}$

$$= x\big(6(8)-(-1)(0)\big) - x\big((-6)(8)-(-1)(0)\big) + x\big((-6)(0)-6(0)\big)$$
$$= x(48) - x(-48) + x(0)$$
$$= 48x + 48x$$
$$= 96x$$

There are two ways of finding $\det B$.

Method 1

Using the fact that $\det(AB) = \det A \times \det B \quad \Rightarrow \quad 96x = 3 \times \det B$

$$\Rightarrow \quad \det B = \frac{96x}{3} = 32x$$

Method 2

Find $\det B$ directly from the matrix $B = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}$.

$$\det B = (x+2)\begin{vmatrix} 4 & 2 \\ -2 & 3 \end{vmatrix} - (x-2)\begin{vmatrix} -4 & 2 \\ 2 & 3 \end{vmatrix} + (x+3)\begin{vmatrix} -4 & 4 \\ 2 & -2 \end{vmatrix}$$

$$= (x+2)\big(4(3) - 2(-2)\big) - (x-2)\big((-4)(3) - 2(2)\big) + (x+3)\big((-4)(-2) - 4(2)\big)$$

$$= (x+2)(16) - (x-2)(-16) + (x+3)(0)$$

$$= 16(x+2) + 16(x-2) + 0$$

$$= 16x + 32 + 16x - 32$$

$$= 32x$$

23. **Method 1**

$A + A^{-1} = I \quad \Rightarrow \quad A = I - A^{-1}$

$A^2 = AA = A(I - A^{-1}) = AI - AA^{-1} = A - I$ [since $AI = A$ and $AA^{-1} = I$]

$A^3 = AA^2 = A(A - I)$ [since $A^2 = A - I$]

$= A^2 - AI$

$= A^2 - A$ [since $AI = A$]

$= (A - I) - A$ [since $A^2 = A - I$]

$= -I$

Hence $A^3 = kI$ where $k = -1$.

Method 2

Starting with the equation $A + A^{-1} = I$, pre-multiply both sides of the equation by *A*.

$A + A^{-1} = I \quad \Rightarrow \quad A(A + A^{-1}) = AI$

$\Rightarrow \quad A^2 + AA^{-1} = A$ [since $AI = A$]

$\Rightarrow \quad A^2 + I = A$ [since $AA^{-1} = I$]

Now pre-multiply both sides of the above equation by *A*.

$A^2 + I = A \quad \Rightarrow \quad A(A^2 + I) = AA$

$\Rightarrow \quad A^3 + AI = A^2$

$\Rightarrow \quad A^3 + A = A^2$ [since $AI = A$]

$\Rightarrow \quad A^3 = A^2 - A$

$\Rightarrow \quad A^3 = (A - I) - A$ [since $A^2 + I = A$ and so $A^2 = A - I$]

$\Rightarrow \quad A^3 = -I$

Hence $A^3 = kI$ where $k = -1$.

24.(a) $M_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \dfrac{\pi}{2}$

$$= \begin{pmatrix} \cos\dfrac{\pi}{2} & -\sin\dfrac{\pi}{2} \\ \sin\dfrac{\pi}{2} & \cos\dfrac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(b) Under reflection in the x-axis: $P(x, y) \to P'(x, -y)$

This can be represented by the matrix equation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

Hence $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) $M_2 M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

Under the transformation represented by $M_2 M_1$: $P(x, y) \to P'(-y, -x)$

Hence the transformation represented by $M_2 M_1$ is equivalent to reflection in the line with equation $y = -x$.

25.(a) Under reflection in the y-axis: $P(x, y) \to P'(-x, y)$

This can be represented by the matrix equation $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$.

Hence $M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) $M_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \dfrac{\pi}{2}$

$$= \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(c) The matrix associated with the transformation in (b) followed by the transformation in (a) is $M_1 M_2$.

$$M_3 = M_1 M_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

Under the transformation associated with M_3: $P(x, y) \to P'(y, x)$

Hence the single transformation associated with M_3 is reflection in the line with equation $y = x$.

26. Under reflection in the x-axis: $P(x, y) \to P'(x, -y)$

This can be represented by the matrix equation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

Hence $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{where } \theta = 30°$$

$$= \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

The matrix associated with the transformation A followed by B is BA.

$$BA = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}x + \frac{1}{2}y \\ \frac{1}{2}x - \frac{\sqrt{3}}{2}y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}x + y}{2} \\ \frac{x - \sqrt{3}y}{2} \end{pmatrix}$$

Under the transformation A followed by B: $P(x, y) \to P'\left(\frac{\sqrt{3}x + y}{2}, \frac{x - \sqrt{3}y}{2}\right)$

Hence the image of a point (x, y) under the transformation A followed by B is $\left(\frac{kx + y}{2}, \frac{x - ky}{2}\right)$, where $k = \sqrt{3}$.

27.(a) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \frac{\pi}{3}$

$$= \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(b) Under reflection in the x-axis: $P(x, y) \to P'(x, -y)$

This can be represented by the matrix equation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

Hence $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) The matrix associated with the transformation in (a) followed by the transformation in (b) is BA.

$$P = BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(d) The matrix associated with an anticlockwise rotation through an angle of θ about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Note that in this matrix the top left and bottom right entries are equal.

In the matrix $P = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, the top left and bottom right entries are not equal, hence the matrix P is not associated with rotation about the origin.

28. Let A be the matrix associated with an enlargement, scale factor 2.

Under an enlargement, scale factor 2: $P(x, y) \rightarrow P'(2x, 2y)$

This can be represented by the matrix equation $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$.

Hence $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Let B be the matrix associated with a **clockwise** rotation of $60°$ about the origin.

Note that a **clockwise** rotation of $60°$ about the origin is equivalent to an **anticlockwise** rotation of $300°$ about the origin.

Hence $B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta = 300°$

$$= \begin{pmatrix} \cos 300° & -\sin 300° \\ \sin 300° & \cos 300° \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

The matrix associated with an enlargement, scale factor 2, followed by a clockwise rotation of $60°$ about the origin is BA.

$$M = BA = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$