

Solutions to Exam Questions on Partial Fractions

1. The denominator contains distinct linear factors.

$$\begin{aligned}\frac{3x+32}{(x+4)(6-x)} &= \frac{A}{x+4} + \frac{B}{6-x} \\ &= \frac{A(6-x) + B(x+4)}{(x+4)(6-x)}\end{aligned}$$

$$3x + 32 = A(6 - x) + B(x + 4)$$

$$\text{Let } x = 6 \Rightarrow 3(6) + 32 = A(0) + B(10) \Rightarrow 50 = 10B \Rightarrow B = 5$$

$$\text{Let } x = -4 \Rightarrow 3(-4) + 32 = A(10) + B(0) \Rightarrow 20 = 10A \Rightarrow A = 2$$

$$\text{Hence } \frac{3x+32}{(x+4)(6-x)} = \frac{2}{x+4} + \frac{5}{6-x}.$$

2. First factorise the denominator: $x^2 - 4x = x(x - 4)$

The denominator contains distinct linear factors.

$$\begin{aligned}\frac{5x-4}{x^2-4x} &= \frac{5x-4}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4} \\ &= \frac{A(x-4) + Bx}{x(x-4)}\end{aligned}$$

$$5x - 4 = A(x - 4) + Bx$$

$$\text{Let } x = 4 \Rightarrow 5(4) - 4 = A(0) + B(4) \Rightarrow 16 = 4B \Rightarrow B = 4$$

$$\text{Let } x = 0 \Rightarrow 5(0) - 4 = A(-4) + B(0) \Rightarrow -4 = -4A \Rightarrow A = 1$$

$$\text{Hence } \frac{5x-4}{x^2-4x} = \frac{1}{x} + \frac{4}{x-4}.$$

3. First factorise the denominator: $x^2 - 2x - 15 = (x + 3)(x - 5)$

The denominator contains distinct linear factors.

$$\begin{aligned}\frac{3x-7}{x^2-2x-15} &= \frac{3x-7}{(x+3)(x-5)} = \frac{A}{x+3} + \frac{B}{x-5} \\ &= \frac{A(x-5) + B(x+3)}{(x+3)(x-5)}\end{aligned}$$

$$3x - 7 = A(x - 5) + B(x + 3)$$

$$\text{Let } x = 5 \Rightarrow 3(5) - 7 = A(0) + B(8) \Rightarrow 8 = 8B \Rightarrow B = 1$$

$$\text{Let } x = -3 \Rightarrow 3(-3) - 7 = A(-8) + B(0) \Rightarrow -16 = -8A \Rightarrow A = 2$$

$$\text{Hence } \frac{3x-7}{x^2-2x-15} = \frac{2}{x+3} + \frac{1}{x-5}.$$

4. First factorise the denominator: $x^2 + x - 2 = (x + 2)(x - 1)$

The denominator contains distinct linear factors.

$$\begin{aligned}\frac{11-2x}{x^2+x-2} &= \frac{11-2x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \\ &= \frac{A(x-1) + B(x+2)}{(x+2)(x-1)}\end{aligned}$$

$$11 - 2x = A(x - 1) + B(x + 2)$$

$$\text{Let } x = 1 \Rightarrow 11 - 2(1) = A(0) + B(3) \Rightarrow 9 = 3B \Rightarrow B = 3$$

$$\text{Let } x = -2 \Rightarrow 11 - 2(-2) = A(-3) + B(0) \Rightarrow 15 = -3A \Rightarrow A = -5$$

$$\text{Hence } \frac{11-2x}{x^2+x-2} = \frac{-5}{x+2} + \frac{3}{x-1} \text{ or } \frac{11-2x}{x^2+x-2} = \frac{3}{x-1} - \frac{5}{x+2}.$$

5. First factorise the denominator: $x^2 - 4 = (x - 2)(x + 2)$

The denominator contains distinct linear factors.

$$\begin{aligned}\frac{4}{x^2 - 4} &= \frac{4}{(x - 2)(x + 2)} = \frac{A}{x - 2} + \frac{B}{x + 2} \\ &= \frac{A(x + 2) + B(x - 2)}{(x - 2)(x + 2)}\end{aligned}$$

$$4 = A(x + 2) + B(x - 2)$$

$$\text{Let } x = -2 \Rightarrow 4 = A(0) + B(-4) \Rightarrow 4 = -4B \Rightarrow B = -1$$

$$\text{Let } x = 2 \Rightarrow 4 = A(4) + B(0) \Rightarrow 4 = 4A \Rightarrow A = 1$$

$$\text{Hence } \frac{4}{x^2 - 4} = \frac{1}{x - 2} + \frac{-1}{x + 2} \text{ or } \frac{4}{x^2 - 4} = \frac{1}{x - 2} - \frac{1}{x + 2}.$$

6. First factorise the denominator: $x^2 - 1 = (x - 1)(x + 1)$

The denominator contains distinct linear factors.

$$\begin{aligned}\frac{x}{x^2 - 1} &= \frac{x}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} \\ &= \frac{A(x + 1) + B(x - 1)}{(x - 1)(x + 1)}\end{aligned}$$

$$x = A(x + 1) + B(x - 1)$$

$$\text{Let } x = -1 \Rightarrow -1 = A(0) + B(-2) \Rightarrow -1 = -2B \Rightarrow B = \frac{1}{2}$$

$$\text{Let } x = 1 \Rightarrow 1 = A(2) + B(0) \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$\text{Hence } \frac{x}{x^2 - 1} = \frac{1/2}{x - 1} + \frac{1/2}{x + 1} \text{ or } \frac{x}{x^2 - 1} = \frac{1}{2(x - 1)} + \frac{1}{2(x + 1)}.$$

7. First factorise the denominator: $1 - y^2 = (1 - y)(1 + y)$

The denominator contains distinct linear factors.

$$\begin{aligned}\frac{1}{1 - y^2} &= \frac{1}{(1 - y)(1 + y)} = \frac{A}{1 - y} + \frac{B}{1 + y} \\ &= \frac{A(1 + y) + B(1 - y)}{(1 - y)(1 + y)}\end{aligned}$$

$$1 = A(1 + y) + B(1 - y)$$

$$\text{Let } y = -1 \Rightarrow 1 = A(0) + B(2) \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$$

$$\text{Let } y = 1 \Rightarrow 1 = A(2) + B(0) \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$\text{Hence } \frac{1}{1 - y^2} = \frac{1/2}{1 - y} + \frac{1/2}{1 + y} \quad \text{or} \quad \frac{1}{1 - y^2} = \frac{1}{2(1 - y)} + \frac{1}{2(1 + y)}.$$

8. First factorise the denominator: $x^2 - x - 6 = (x + 2)(x - 3)$

The denominator contains distinct linear factors.

$$\begin{aligned}\frac{1}{x^2 - x - 6} &= \frac{1}{(x + 2)(x - 3)} = \frac{A}{x + 2} + \frac{B}{x - 3} \\ &= \frac{A(x - 3) + B(x + 2)}{(x + 2)(x - 3)}\end{aligned}$$

$$1 = A(x - 3) + B(x + 2)$$

$$\text{Let } x = 3 \Rightarrow 1 = A(0) + B(5) \Rightarrow 1 = 5B \Rightarrow B = \frac{1}{5}$$

$$\text{Let } x = -2 \Rightarrow 1 = A(-5) + B(0) \Rightarrow 1 = -5A \Rightarrow A = -\frac{1}{5}$$

$$\text{Hence } \frac{1}{x^2 - x - 6} = \frac{-1/5}{x + 2} + \frac{1/5}{x - 3} \quad \text{or} \quad \frac{1}{x^2 - x - 6} = \frac{1}{5(x - 3)} - \frac{1}{5(x + 2)}.$$

9. The denominator contains distinct linear factors.

$$\begin{aligned}\frac{3x+5}{(x+1)(x+2)(x+3)} &= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \\ &= \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)}{(x+1)(x+2)(x+3)}\end{aligned}$$

$$3x+5 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

$$\text{Let } x = -2 \Rightarrow 3(-2) + 5 = A(0)(1) + B(-1)(1) + C(-1)(0) \Rightarrow -1 = -B \Rightarrow B = 1$$

$$\text{Let } x = -3 \Rightarrow 3(-3) + 5 = A(-1)(0) + B(-2)(0) + C(-2)(-1) \Rightarrow -4 = 2C \Rightarrow C = -2$$

$$\text{Let } x = -1 \Rightarrow 3(-1) + 5 = A(1)(2) + B(0)(2) + C(0)(1) \Rightarrow 2 = 2A \Rightarrow A = 1$$

$$\text{Hence } \frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{-2}{x+3}$$

$$\text{or } \frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x+3}.$$

10. The denominator contains distinct linear factors.

$$\begin{aligned}\frac{8}{x(x+2)(x+4)} &= \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4} \\ &= \frac{A(x+2)(x+4) + Bx(x+4) + Cx(x+2)}{x(x+2)(x+4)}\end{aligned}$$

$$8 = A(x+2)(x+4) + Bx(x+4) + Cx(x+2)$$

$$\text{Let } x = -2 \Rightarrow 8 = A(0)(2) + B(-2)(2) + C(-2)(0) \Rightarrow 8 = -4B \Rightarrow B = -2$$

$$\text{Let } x = -4 \Rightarrow 8 = A(-2)(0) + B(-4)(0) + C(-4)(-2) \Rightarrow 8 = 8C \Rightarrow C = 1$$

$$\text{Let } x = 0 \Rightarrow 8 = A(2)(4) + B(0)(4) + C(0)(2) \Rightarrow 8 = 8A \Rightarrow A = 1$$

$$\text{Hence } \frac{8}{x(x+2)(x+4)} = \frac{1}{x} + \frac{-2}{x+2} + \frac{1}{x+4} \quad \text{or} \quad \frac{8}{x(x+2)(x+4)} = \frac{1}{x} - \frac{2}{x+2} + \frac{1}{x+4}.$$

11. The denominator contains distinct linear factors.

$$\begin{aligned}\frac{13+6x+5x^2}{(1+x)(2-x)(3+x)} &= \frac{A}{1+x} + \frac{B}{2-x} + \frac{C}{3+x} \\ &= \frac{A(2-x)(3+x) + B(1+x)(3+x) + C(1+x)(2-x)}{(1+x)(2-x)(3+x)}\end{aligned}$$

$$13+6x+5x^2 = A(2-x)(3+x) + B(1+x)(3+x) + C(1+x)(2-x)$$

$$\text{Let } x = 2 \Rightarrow 13+6(2)+5(2)^2 = A(0)(5) + B(3)(5) + C(3)(0) \Rightarrow 45 = 15B \Rightarrow B = 3$$

$$\begin{aligned}\text{Let } x = -3 \Rightarrow 13+6(-3)+5(-3)^2 &= A(5)(0) + B(-2)(0) + C(-2)(5) \Rightarrow 40 = -10C \\ &\Rightarrow C = -4\end{aligned}$$

$$\begin{aligned}\text{Let } x = -1 \Rightarrow 13+6(-1)+5(-1)^2 &= A(3)(2) + B(0)(2) + C(0)(3) \Rightarrow 12 = 6A \\ &\Rightarrow A = 2\end{aligned}$$

$$\text{Hence } \frac{13+6x+5x^2}{(1+x)(2-x)(3+x)} = \frac{2}{1+x} + \frac{3}{2-x} + \frac{-4}{3+x}$$

$$\text{or } \frac{13+6x+5x^2}{(1+x)(2-x)(3+x)} = \frac{2}{1+x} + \frac{3}{2-x} - \frac{4}{3+x}.$$

12. First check whether $x^2 - x - 6$ is irreducible:

$$a = 1, b = -1, c = -6 \Rightarrow b^2 - 4ac = (-1)^2 - 4(1)(-6) = 25$$

$b^2 - 4ac > 0$, so $x^2 - x - 6$ has two real and distinct roots and can be written as the product of two distinct linear factors, ie $x^2 - x - 6 = (x + 2)(x - 3)$.

This means that the denominator can be written as $x(x + 2)(x - 3)$ and contains distinct linear factors.

$$\begin{aligned} \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} &= \frac{2x^2 - 9x - 6}{x(x + 2)(x - 3)} = \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x - 3} \\ &= \frac{A(x + 2)(x - 3) + Bx(x - 3) + Cx(x + 2)}{x(x + 2)(x - 3)} \end{aligned}$$

$$2x^2 - 9x - 6 = A(x + 2)(x - 3) + Bx(x - 3) + Cx(x + 2)$$

$$\begin{aligned} \text{Let } x = -2 &\Rightarrow 2(-2)^2 - 9(-2) - 6 = A(0)(-5) + B(-2)(-5) + C(-2)(0) \\ &\Rightarrow 20 = 10B \\ &\Rightarrow B = 2 \end{aligned}$$

$$\begin{aligned} \text{Let } x = 3 &\Rightarrow 2(3)^2 - 9(3) - 6 = A(5)(0) + B(3)(0) + C(3)(5) \\ &\Rightarrow -15 = 15C \\ &\Rightarrow C = -1 \end{aligned}$$

$$\begin{aligned} \text{Let } x = 0 &\Rightarrow 2(0)^2 - 9(0) - 6 = A(2)(-3) + B(0)(-3) + C(0)(2) \\ &\Rightarrow -6 = -6A \\ &\Rightarrow A = 1 \end{aligned}$$

$$\text{Hence } \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \frac{1}{x} + \frac{2}{x + 2} + \frac{-1}{x - 3} \quad \text{or} \quad \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \frac{1}{x} + \frac{2}{x + 2} - \frac{1}{x - 3}.$$

Note

A common error in this question is to assume that $x^2 - x - 6$ is irreducible leading to the incorrect partial fractions $\frac{A}{x} + \frac{Bx + C}{x^2 - x - 6}$.

It is essential that you check whether a quadratic factor in the denominator is irreducible and factorise the denominator fully before finding the partial fractions.

13. The denominator contains a repeated linear factor.

$$\begin{aligned}\frac{x^2 - 6x + 20}{(x+1)(x-2)^2} &= \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \\ &= \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2}\end{aligned}$$

$$x^2 - 6x + 20 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

$$\text{Let } x = 2 \Rightarrow 2^2 - 6(2) + 20 = A(0)^2 + B(3)(0) + C(3) \Rightarrow 12 = 3C \Rightarrow C = 4$$

$$\begin{aligned}\text{Let } x = -1 &\Rightarrow (-1)^2 - 6(-1) + 20 = A(-3)^2 + B(0)(-3) + C(0) \\ &\Rightarrow 27 = 9A \\ &\Rightarrow A = 3\end{aligned}$$

$$\text{Equating coefficients of } x^2 \Rightarrow 1 = A + B \Rightarrow 1 = 3 + B \Rightarrow B = -2$$

$$\text{Hence } \frac{x^2 - 6x + 20}{(x+1)(x-2)^2} = \frac{3}{x+1} + \frac{-2}{x-2} + \frac{4}{(x-2)^2}$$

$$\text{or } \frac{x^2 - 6x + 20}{(x+1)(x-2)^2} = \frac{3}{x+1} - \frac{2}{x-2} + \frac{4}{(x-2)^2}.$$

14. The denominator contains a repeated linear factor.

$$\begin{aligned}\frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)} &= \frac{A}{x - 4} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2} \\ &= \frac{A(x + 2)^2 + B(x - 4)(x + 2) + C(x - 4)}{(x + 2)^2(x - 4)}\end{aligned}$$

$$x^2 + 6x - 4 = A(x + 2)^2 + B(x - 4)(x + 2) + C(x - 4)$$

$$\begin{aligned}\text{Let } x = -2 &\Rightarrow (-2)^2 + 6(-2) - 4 = A(0)^2 + B(-6)(0) + C(-6) \\ &\Rightarrow -12 = -6C \\ &\Rightarrow C = 2\end{aligned}$$

$$\begin{aligned}\text{Let } x = 4 &\Rightarrow 4^2 + 6(4) - 4 = A(-6)^2 + B(0)(6) + C(0) \\ &\Rightarrow 36 = 36A \\ &\Rightarrow A = 1\end{aligned}$$

$$\text{Equating coefficients of } x^2 \Rightarrow 1 = A + B \Rightarrow 1 = 1 + B \Rightarrow B = 0$$

$$\text{Hence } \frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)} = \frac{1}{x - 4} + \frac{2}{(x + 2)^2}.$$

15. The denominator contains a repeated linear factor.

$$\begin{aligned}\frac{x+4}{(x+1)^2(2x-1)} &= \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\ &= \frac{A(x+1)^2 + B(2x-1)(x+1) + C(2x-1)}{(x+1)^2(2x-1)}\end{aligned}$$

$$x+4 = A(x+1)^2 + B(2x-1)(x+1) + C(2x-1)$$

$$\text{Let } x = -1 \Rightarrow -1+4 = A(0)^2 + B(-3)(0) + C(-3) \Rightarrow 3 = -3C \Rightarrow C = -1$$

$$\text{Let } x = \frac{1}{2} \Rightarrow \frac{1}{2} + 4 = A\left(\frac{3}{2}\right)^2 + B(0)\left(\frac{3}{2}\right) + C(0) \Rightarrow \frac{9}{2} = \frac{9}{4}A \Rightarrow A = 2$$

$$\text{Equating coefficients of } x^2 \Rightarrow 0 = A + 2B \Rightarrow 0 = 2 + 2B \Rightarrow 2B = -2 \Rightarrow B = -1$$

$$\text{Hence } \frac{x+4}{(x+1)^2(2x-1)} = \frac{2}{2x-1} + \frac{-1}{x+1} + \frac{-1}{(x+1)^2}$$

$$\text{or } \frac{x+4}{(x+1)^2(2x-1)} = \frac{2}{2x-1} - \frac{1}{x+1} - \frac{1}{(x+1)^2}.$$

- 16.** The denominator contains an irreducible quadratic factor $x^2 + 5$.
(You can assume that any quadratic factor of the form $x^2 + a$, where $a > 0$, is irreducible, or you can show that $b^2 - 4ac < 0$.)

$$\begin{aligned}\frac{3x^2 + 4x + 17}{(x-3)(x^2 + 5)} &= \frac{A}{x-3} + \frac{Bx + C}{x^2 + 5} \\ &= \frac{A(x^2 + 5) + (Bx + C)(x-3)}{(x-3)(x^2 + 5)}\end{aligned}$$

$$3x^2 + 4x + 17 = A(x^2 + 5) + (Bx + C)(x-3)$$

$$\text{Let } x = 3 \Rightarrow 3(3)^2 + 4(3) + 17 = A(3^2 + 5) + 0 \Rightarrow 56 = 14A \Rightarrow A = 4$$

$$\text{Equating coefficients of } x^2 \Rightarrow 3 = A + B \Rightarrow 3 = 4 + B \Rightarrow B = -1$$

$$\begin{aligned}\text{Equating constants} \Rightarrow 17 = 5A - 3C &\Rightarrow 17 = 5(4) - 3C \Rightarrow 17 = 20 - 3C \\ &\Rightarrow 3C = 3 \\ &\Rightarrow C = 1\end{aligned}$$

$$\text{Hence } \frac{3x^2 + 4x + 17}{(x-3)(x^2 + 5)} = \frac{4}{x-3} + \frac{-x+1}{x^2 + 5} \quad \text{or} \quad \frac{3x^2 + 4x + 17}{(x-3)(x^2 + 5)} = \frac{4}{x-3} + \frac{1-x}{x^2 + 5}.$$

17. The denominator contains an irreducible quadratic factor $x^2 + 5$.
(You can assume that any quadratic factor of the form $x^2 + a$, where $a > 0$, is irreducible, or you can show that $b^2 - 4ac < 0$.)

$$\begin{aligned}\frac{12x^2 + 20}{x(x^2 + 5)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 5} \\ &= \frac{A(x^2 + 5) + (Bx + C)x}{x(x^2 + 5)}\end{aligned}$$

$$12x^2 + 20 = A(x^2 + 5) + (Bx + C)x$$

$$\text{Let } x = 0 \Rightarrow 12(0)^2 + 20 = A(0^2 + 5) + 0 \Rightarrow 20 = 5A \Rightarrow A = 4$$

$$\text{Equating coefficients of } x^2 \Rightarrow 12 = A + B \Rightarrow 12 = 4 + B \Rightarrow B = 8$$

$$\text{Equating coefficients of } x \Rightarrow 0 = C$$

$$[\text{Note that equating constants gives } 20 = 5A \Rightarrow A = 4]$$

$$\text{Hence } \frac{12x^2 + 20}{x(x^2 + 5)} = \frac{4}{x} + \frac{8x}{x^2 + 5}.$$

- 18.** The denominator contains an irreducible quadratic factor $1 + x^2$.
(You can assume that any quadratic factor of the form $x^2 + a$, where $a > 0$, is irreducible, or you can show that $b^2 - 4ac < 0$.)

$$\begin{aligned}\frac{x^2 + 3}{x(1 + x^2)} &= \frac{A}{x} + \frac{Bx + C}{1 + x^2} \\ &= \frac{A(1 + x^2) + (Bx + C)x}{x(1 + x^2)}\end{aligned}$$

$$x^2 + 3 = A(1 + x^2) + (Bx + C)x$$

$$\text{Let } x = 0 \Rightarrow 0^2 + 3 = A(1 + 0^2) + 0 \Rightarrow 3 = A$$

$$\text{Equating coefficients of } x^2 \Rightarrow 1 = A + B \Rightarrow 1 = 3 + B \Rightarrow B = -2$$

$$\text{Equating coefficients of } x \Rightarrow 0 = C$$

[Note that equating constants gives $3 = A$]

$$\text{Hence } \frac{x^2 + 3}{x(1 + x^2)} = \frac{3}{x} + \frac{-2x}{1 + x^2} \quad \text{or} \quad \frac{x^2 + 3}{x(1 + x^2)} = \frac{3}{x} - \frac{2x}{1 + x^2}.$$

19. First factorise the denominator fully: $x^3 + x = x(x^2 + 1)$ and $x^2 + 1$ is irreducible

The denominator contains an irreducible quadratic factor.

$$\begin{aligned}\frac{1}{x^3 + x} &= \frac{1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \\ &= \frac{A(x^2 + 1) + (Bx + C)x}{x(x^2 + 1)}\end{aligned}$$

$$1 = A(x^2 + 1) + (Bx + C)x$$

$$\text{Let } x = 0 \Rightarrow 1 = A(0^2 + 1) + 0 \Rightarrow 1 = A$$

$$\text{Equating coefficients of } x^2 \Rightarrow 0 = A + B \Rightarrow 0 = 1 + B \Rightarrow B = -1$$

$$\text{Equating coefficients of } x \Rightarrow 0 = C$$

[Note that equating constants gives $1 = A$]

$$\text{Hence } \frac{1}{x^3 + x} = \frac{1}{x} + \frac{-x}{x^2 + 1} \text{ or } \frac{1}{x^3 + x} = \frac{1}{x} - \frac{x}{x^2 + 1}.$$

20.(a) $c(x) = x^3 - x^2 - x - 2$

To find a real root of $c(x)$, use synthetic division and try factors of -2 .

$$2 \begin{array}{r|rrrr} & 1 & -1 & -1 & -2 \\ & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

remainder = 0, so $x = 2$ is a root of $c(x)$ and $c(x) = (x - 2)(x^2 + x + 1)$.

Hence $c(x) = l(x)q(x)$ where $l(x) = x - 2$ and $q(x) = x^2 + x + 1$.

(b) For $q(x) = x^2 + x + 1$: $a = 1, b = 1, c = 1 \Rightarrow b^2 - 4ac = 1^2 - 4(1)(1) = -3$

$b^2 - 4ac < 0$, hence $q(x)$ has no real roots and is irreducible (cannot be written as the product of two linear factors with real coefficients).

This means that $c(x)$ cannot be written as the product of three linear factors with real coefficients.

(c)
$$\frac{5x+4}{x^3-x^2-x-2} = \frac{A}{l(x)} + \frac{Bx+C}{q(x)} \Rightarrow \frac{5x+4}{x^3-x^2-x-2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$$

$$= \frac{A(x^2+x+1) + (Bx+C)(x-2)}{x^3-x^2-x-2}$$

$$5x + 4 = A(x^2 + x + 1) + (Bx + C)(x - 2)$$

Let $x = 2 \Rightarrow 5(2) + 4 = A(2^2 + 2 + 1) + 0 \Rightarrow 14 = 7A \Rightarrow A = 2$

Equating coefficients of $x^2 \Rightarrow 0 = A + B \Rightarrow 0 = 2 + B \Rightarrow B = -2$

Equating constants $\Rightarrow 4 = A - 2C \Rightarrow 4 = 2 - 2C \Rightarrow 2C = -2 \Rightarrow C = -1$

Hence $A = 2, B = -2$ and $C = -1$.

Now check equivalence of the denominators:

$$\begin{aligned}(x+1)(x+3)(x-2) &= (x+1)(x^2+x-6) \\ &= x(x^2+x-6)+1(x^2+x-6) \\ &= x^3+x^2-6x+x^2+x-6 \\ &= x^3+2x^2-5x-6\end{aligned}$$

$$\text{Hence } 3 + \frac{2x^2+15x+7}{x^3+2x^2-5x-6} = \frac{3x^3+8x^2-11}{(x+1)(x+3)(x-2)}.$$

$$(b) \frac{3x^3+8x^2-11}{(x+1)(x+3)(x-2)} = 3 + \frac{2x^2+15x+7}{x^3+2x^2-5x-6} = 3 + \frac{2x^2+15x+7}{(x+1)(x+3)(x-2)}$$

Note that $\frac{2x^2+15x+7}{(x+1)(x+3)(x-2)}$ is a proper algebraic fraction and can be expressed in partial fractions.

The denominator contains distinct linear factors.

$$\begin{aligned}\frac{2x^2+15x+7}{(x+1)(x+3)(x-2)} &= \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{x-2} \\ &= \frac{A(x+3)(x-2) + B(x+1)(x-2) + C(x+1)(x+3)}{(x+1)(x+3)(x-2)}\end{aligned}$$

$$2x^2+15x+7 = A(x+3)(x-2) + B(x+1)(x-2) + C(x+1)(x+3)$$

$$\begin{aligned}\text{Let } x = -3 &\Rightarrow 2(-3)^2 + 15(-3) + 7 = A(0)(-5) + B(-2)(-5) + C(-2)(0) \\ &\Rightarrow -20 = 10B \\ &\Rightarrow B = -2\end{aligned}$$

$$\begin{aligned}\text{Let } x = 2 &\Rightarrow 2(2)^2 + 15(2) + 7 = A(5)(0) + B(3)(0) + C(3)(5) \\ &\Rightarrow 45 = 15C \\ &\Rightarrow C = 3\end{aligned}$$

$$\begin{aligned}\text{Let } x = -1 &\Rightarrow 2(-1)^2 + 15(-1) + 7 = A(2)(-3) + B(0)(-3) + C(0)(2) \\ &\Rightarrow -6 = -6A \\ &\Rightarrow A = 1\end{aligned}$$

$$\text{Hence } \frac{3x^3 + 8x^2 - 11}{(x+1)(x+3)(x-2)} = 3 + \frac{1}{x+1} + \frac{-2}{x+3} + \frac{3}{x-2}$$

$$\text{or } \frac{3x^3 + 8x^2 - 11}{(x+1)(x+3)(x-2)} = 3 + \frac{1}{x+1} - \frac{2}{x+3} + \frac{3}{x-2}.$$

Note

You should only attempt to express a proper algebraic fraction (where the degree of the numerator is less than the degree of the denominator) as the sum of partial fractions.

For an improper algebraic fraction, algebraic long division should be used first before finding partial fractions for the proper fraction remaining.