ADVANCED HIGHER MATHEMATICS

Exam Questions on Vectors

1. Given $\boldsymbol{u} = -2\boldsymbol{i} + 5\boldsymbol{k}$, $\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{w} = -\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$, calculate $\boldsymbol{u}\,.\,(\boldsymbol{v} \times \boldsymbol{w})$.

2. Three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are given by

$$\boldsymbol{u} = 5\boldsymbol{i} + 13\boldsymbol{j}, \quad \boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}, \quad \boldsymbol{w} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}.$$

Calculate $\boldsymbol{u}\,.\,(\boldsymbol{v} \times \boldsymbol{w})$.

Interpret your answer geometrically.

3. Find the point of intersection of the line

$$\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2}$$

and the plane with equation $2x + y - z = 4$.

4. Obtain the equation of the plane passing through the points $P(-2, 1, -1)$, $Q(1, 2, 3)$ and $R(3, 0, 1)$.

5. (a) Find the equation of the plane π_1 which contains the points $A(1, 1, 0)$, $B(3, 1, -1)$ and $C(2, 0, -3)$.

(b) Given that π_2 is the plane whose equation is $x + 2y + z = 3$, calculate the size of the acute angle between the planes π_1 and π_2.

6. (a) Find the equation of the plane π_1 containing the points A(1, 0, 3), B(0, 2, −1) and C(1, 1, 0).

(b) Calculate the size of the acute angle between π_1 and the plane π_2 with equation $x + y - z = 0$.

(c) Find the point of intersection of plane π_2 and the line

$$\frac{x-11}{4} = \frac{y-15}{5} = \frac{z-12}{2}.$$

7. (a) Find the equation of the plane π_1, through the points $A(0, -1, 3)$, $B(1, 0, 3)$ and $C(0, 0, 5)$.

(b) π_2 is the plane through A with normal in the direction $-\mathbf{j} + \mathbf{k}$.
Find the equation of the plane π_2.

(c) Determine the acute angle between planes π_1 and π_2.

8. Two lines L_1 and L_2 are given by the equations:

$$L_1: \quad \mathbf{x} = 4 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = -7\lambda$$

$$L_2: \quad \frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3}$$

(a) Show that the lines L_1 and L_2 intersect and find the point of intersection.

(b) Calculate the obtuse angle between the lines L_1 and L_2.

9. Let L_1 and L_2 be the lines

$$L_1: \quad x = 8 - 2t, \quad y = -4 + 2t, \quad z = 3 + t$$

$$L_2: \quad \frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2}.$$

(a)(i) Show that L_1 and L_2 intersect and find their point of intersection.

(ii) Verify that the acute angle between them is $\cos^{-1}\left(\frac{4}{9}\right)$.

(b)(i) Obtain the equation of the plane Π that is perpendicular to L_2 and passes through the point $(1, -4, 2)$.

(ii) Find the coordinates of the point of intersection of the plane Π and the line L_1.

10. The lines L_1 and L_2 are given by the following equations.

$$L_1: \quad \frac{x+6}{3} = \frac{y-1}{-1} = \frac{z-2}{2}$$

$$L_2: \quad \frac{x+5}{4} = \frac{y+4}{1} = \frac{z}{4}$$

(a) Show that the lines L_1 and L_2 intersect and state the coordinates of the point of intersection.

(b) Find the equation of the plane π containing L_1 and L_2.

(c) A third line, L_3, is given by the equation $\frac{x-1}{2} = \frac{y+7}{4} = \frac{z-3}{-1}$.

Calculate the acute angle between L_3 and the plane π.
Give your answer in degrees correct to 2 decimal places.

11. A line, L_1, passes through the point P(2, 4, 1) and is parallel to

$$\mathbf{d_1} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

and a second line, L_2, passes through Q(–5, 2, 5) and is parallel to

$$\mathbf{d_2} = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

(a) Write down parametric equations for L_1 and L_2.

(b) Show that the lines L_1 and L_2 intersect and find the point of intersection.

(c) Determine the equation of the plane containing L_1 and L_2.

12. The lines L_1 and L_2 are given by the equations

$$\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} \quad \text{and} \quad \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2},$$

respectively.

Find:

(a) the value of k for which L_1 and L_2 intersect and the point of intersection;

(b) the acute angle between L_1 and L_2.

13. The equations of two planes are $x - 4y + 2z = 1$ and $x - y - z = -5$.

By letting $z = t$, or otherwise, obtain parametric equations of the line of intersection of the two planes.

Show that this line lies in the plane with equation $x + 2y - 4z = -11$.

14. (a) Show that the line of intersection, L, of the planes $x + y - z = 6$ and $2x - 3y + 2z = 2$ has parametric equations

$$x = \lambda$$
$$y = 4\lambda - 14$$
$$z = 5\lambda - 20.$$

(b) Find the acute angle between line L and the plane $-5x + 2y - 4z = 1$.

15. (a) Find the equation of the plane passing through the point $P(1, 1, 0)$ which is perpendicular to the line L given by

$$\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1}.$$

(b) Find the coordinates of the point Q where the plane and L intersect.

(c) Hence, or otherwise, obtain the shortest distance from P to L and explain why this is the shortest distance.

16. (a) Find the equation of the plane π_1 through the points $A(1, 1, 1)$, $B(2, -1, 1)$ and $C(0, 3, 3)$.

(b) The plane π_2 has equation $x + 3y - z = 2$.
By letting $x = t$, or otherwise, find parametric equations for the line of intersection of the planes π_1 and π_2.

(c) Find the size of the acute angle between the planes π_1 and π_2.

17. (a) A beam of light passes through the points B(7, 8, 1) and T(–3, –22, 6).
Obtain parametric equations for the line representing the beam of light.

(b) A sheet of metal is represented by a plane containing the points P(2, 1, 9), Q(1, 2, 7) and R(–3, 7, 1).
Find the Cartesian equation of the plane.

(c) The beam of light passes through a hole in the metal at point H.
Find the coordinates of H.

18. Lines L_1 and L_2 are given by the parametric equations

$$L_1:\ x = 2 + s,\ y = -s,\ z = 2 - s \quad \text{and} \quad L_2:\ x = -1 - 2t,\ y = t,\ z = 2 + 3t.$$

(a) Show that L_1 and L_2 do not intersect.

(b) The line L_3 passes through the point $P(1, 1, 3)$ and its direction is perpendicular to the directions of both L_1 and L_2. Obtain parametric equations for L_3.

(c) Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1.

(d) PQ is the shortest distance between the lines L_1 and L_2. Calculate PQ.

19. Planes π_1, π_2 and π_3 have equations:

$$\pi_1: \quad x - 2y + z = -4$$
$$\pi_2: \quad 3x - 5y - 2z = 1$$
$$\pi_3: \quad -7x + 11y + az = -11$$

where $a \in \mathbf{R}$.

(a) Use Gaussian elimination to find the value of a such that the intersection of the planes π_1, π_2 and π_3 is a line.

(b) By letting $z = t$, or otherwise, find parametric equations for the line of intersection of the planes when a takes this value.

(c) The plane π_4 has equation $-9x + 15y + 6z = 20$.

(i) Find the acute angle between π_1 and π_4.

(ii) Describe the relationship between π_2 and π_4.
Justify your answer.