Solutions to Exam Questions on Vectors

1. $\boldsymbol{u} = -2\boldsymbol{i} + 5\boldsymbol{k}$, $\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{w} = -\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$

$$\boldsymbol{v}\times\boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= \boldsymbol{i}\big(2(4) - (-1)(1)\big) - \boldsymbol{j}\big(3(4) - (-1)(-1)\big) + \boldsymbol{k}\big(3(1) - 2(-1)\big)$$

$$= \boldsymbol{i}(9) - \boldsymbol{j}(11) + \boldsymbol{k}(5)$$

$$= 9\boldsymbol{i} - 11\boldsymbol{j} + 5\boldsymbol{k}$$

$$\boldsymbol{u}\,\boldsymbol{.}\,(\boldsymbol{v}\times\boldsymbol{w}) = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} \boldsymbol{.} \begin{pmatrix} 9 \\ -11 \\ 5 \end{pmatrix} = (-2)(9) + 0(-11) + 5(5) = 7$$

2. $\boldsymbol{u} = 5\boldsymbol{i} + 13\boldsymbol{j}$, $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{w} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$

$$\boldsymbol{v}\times\boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= \boldsymbol{i}\big(1(-1) - 3(4)\big) - \boldsymbol{j}\big(2(-1) - 3(1)\big) + \boldsymbol{k}\big(2(4) - 1(1)\big)$$

$$= \boldsymbol{i}(-13) - \boldsymbol{j}(-5) + \boldsymbol{k}(8)$$

$$= -13\boldsymbol{i} + 5\boldsymbol{j} + 8\boldsymbol{k}$$

$$\boldsymbol{u}\,\boldsymbol{.}\,(\boldsymbol{v}\times\boldsymbol{w}) = \begin{pmatrix} 5 \\ 13 \\ 0 \end{pmatrix} \boldsymbol{.} \begin{pmatrix} -13 \\ 5 \\ 8 \end{pmatrix} = 5(-13) + 13(5) + 0(8) = 0$$

$\boldsymbol{u}\,\boldsymbol{.}\,(\boldsymbol{v}\times\boldsymbol{w}) = 0$, so the vector $\boldsymbol{u}$ is perpendicular to the vector $\boldsymbol{v}\times\boldsymbol{w}$.

3. First find parametric equations for the line.

$$\frac{x-3}{4}=\frac{y-2}{-1}=\frac{z+1}{2}=t \quad \Rightarrow \quad x=4t+3, \quad y=-t+2, \quad z=2t-1$$

To find the point of intersection of the line and the plane, substitute the parametric equations for the line into the equation of the plane.

$$\begin{aligned}2x+y-z=4 \quad &\Rightarrow \quad 2(4t+3)+(-t+2)-(2t-1)=4\\ &\Rightarrow \quad 8t+6-t+2-2t+1=4\\ &\Rightarrow \quad 5t+9=4\\ &\Rightarrow \quad 5t=-5\\ &\Rightarrow \quad t=-1\end{aligned}$$

Then $x=4t+3=4(-1)+3=-1$

$y=-t+2=-(-1)+2=3$

$z=2t-1=2(-1)-1=-3$

Hence the point of intersection of the line and the plane is (–1, 3, –3).

4. First find two vectors which lie in the plane, eg $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ}=\boldsymbol{q}-\boldsymbol{p}=\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}-2\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\1\\4\end{pmatrix} \quad \text{and} \quad \overrightarrow{PR}=\boldsymbol{r}-\boldsymbol{p}=\begin{pmatrix}3\\0\\1\end{pmatrix}-\begin{pmatrix}-2\\1\\-1\end{pmatrix}=\begin{pmatrix}5\\-1\\2\end{pmatrix}$$

A normal vector for the plane is then given by

$$\begin{aligned}\boldsymbol{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\3 & 1 & 4\\5 & -1 & 2\end{vmatrix}&=\boldsymbol{i}\begin{vmatrix}1 & 4\\-1 & 2\end{vmatrix}-\boldsymbol{j}\begin{vmatrix}3 & 4\\5 & 2\end{vmatrix}+\boldsymbol{k}\begin{vmatrix}3 & 1\\5 & -1\end{vmatrix}\\ &=\boldsymbol{i}\big(1(2)-4(-1)\big)-\boldsymbol{j}\big(3(2)-4(5)\big)+\boldsymbol{k}\big(3(-1)-1(5)\big)\\ &=\boldsymbol{i}(6)-\boldsymbol{j}(-14)+\boldsymbol{k}(-8)\\ &=6\boldsymbol{i}+14\boldsymbol{j}-8\boldsymbol{k}\end{aligned}$$

Equation of plane: $6x+14y-8z=\lambda$ for some constant λ

To evaluate λ, substitute the coordinates of a point in the plane, eg $R(3, 0, 1)$.

$$6x+14y-8z=\lambda \quad \Rightarrow \quad 6(3)+14(0)-8(1)=\lambda \quad \Rightarrow \quad \lambda=10$$

The equation of the plane is $6x+14y-8z=10$ *or* $3x+7y-4z=5$.

5.(a) First find two vectors which lie in the plane, eg $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

A normal vector for the plane is then given by

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & -1 \\ 1 & -1 & -3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 0 & -1 \\ -1 & -3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= \boldsymbol{i}\big(0(-3) - (-1)(-1)\big) - \boldsymbol{j}\big(2(-3) - (-1)(1)\big) + \boldsymbol{k}\big(2(-1) - 0(1)\big)$$

$$= \boldsymbol{i}(-1) - \boldsymbol{j}(-5) + \boldsymbol{k}(-2)$$

$$= -\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$$

Equation of plane: $-x + 5y - 2z = \lambda$ for some constant λ

To evaluate λ, substitute the coordinates of a point in the plane, eg $A(1, 1, 0)$.

$$-x + 5y - 2z = \lambda \quad \Rightarrow \quad -1 + 5(1) - 2(0) = \lambda \quad \Rightarrow \quad \lambda = 4$$

The equation of the plane is $-x + 5y - 2z = 4$.

(b) plane π_1: $-x+5y-2z=4$

plane π_2: $x+2y+z=3$

The angle, θ, between the planes π_1 and π_2 is the angle between the two normal vectors for the planes.

Hence $\cos\theta = \dfrac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$ where $\boldsymbol{n}_1 = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

$$\boldsymbol{n}_1 \bullet \boldsymbol{n}_2 = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = (-1)(1) + 5(2) + (-2)(1) = 7$$

$$|\boldsymbol{n}_1| = \sqrt{(-1)^2 + 5^2 + (-2)^2} = \sqrt{1+25+4} = \sqrt{30} \quad \text{and} \quad |\boldsymbol{n}_2| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}$$

Hence $\cos\theta = \dfrac{7}{\sqrt{30}\sqrt{6}} = 0\cdot5217... \quad \Rightarrow \quad \theta = \cos^{-1} 0\cdot5217... = 58\cdot55°$ (to 2 dp)

The acute angle between the planes π_1 and π_2 is $58\cdot55°$.

6.(a) First find two vectors which lie in the plane, eg $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

A normal vector for the plane is then given by

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & -4 \\ 0 & 1 & -3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 2 & -4 \\ 1 & -3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -1 & -4 \\ 0 & -3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= \boldsymbol{i}\big(2(-3) - (-4)(1)\big) - \boldsymbol{j}\big((-1)(-3) - (-4)(0)\big) + \boldsymbol{k}\big((-1)(1) - 2(0)\big)$$

$$= \boldsymbol{i}(-2) - \boldsymbol{j}(3) + \boldsymbol{k}(-1)$$

$$= -2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$$

Equation of plane: $-2x - 3y - z = \lambda$ for some constant λ

To evaluate λ, substitute the coordinates of a point in the plane, eg A(1, 0, 3).

$$-2x - 3y - z = \lambda \quad \Rightarrow \quad -2(1) - 3(0) - 3 = \lambda \quad \Rightarrow \quad \lambda = -5$$

The equation of the plane is $-2x - 3y - z = -5$ *or* $2x + 3y + z = 5$.

(b) plane π_1: $2x+3y+z=5$

plane π_2: $x+y-z=0$

The angle, θ, between the planes π_1 and π_2 is the angle between the two normal vectors for the planes.

Hence $\cos\theta = \dfrac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$ where $\boldsymbol{n}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$$\boldsymbol{n}_1 \bullet \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2(1)+3(1)+1(-1) = 4$$

$$|\boldsymbol{n}_1| = \sqrt{2^2+3^2+1^2} = \sqrt{4+9+1} = \sqrt{14} \quad \text{and} \quad |\boldsymbol{n}_2| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

Hence $\cos\theta = \dfrac{4}{\sqrt{14}\sqrt{3}} = 0\cdot6172\ldots \Rightarrow \theta = \cos^{-1} 0\cdot6172\ldots = 51\cdot9°$ (to 1 dp)

The acute angle between the planes π_1 and π_2 is $51\cdot9°$.

(c) First find parametric equations for the line.

$$\frac{x-11}{4} = \frac{y-15}{5} = \frac{z-12}{2} \Rightarrow x = 4t+11, \quad y = 5t+15, \quad z = 2t+12$$

To find the point of intersection of the line and plane π_2, substitute the parametric equations for the line into the equation of the plane.

$$\begin{aligned} x+y-z=0 \quad &\Rightarrow \quad 4t+11+(5t+15)-(2t+12)=0 \\ &\Rightarrow \quad 4t+11+5t+15-2t-12=0 \\ &\Rightarrow \quad 7t+14=0 \\ &\Rightarrow \quad 7t=-14 \\ &\Rightarrow \quad t=-2 \end{aligned}$$

Then $x = 4t+11 = 4(-2)+11 = 3$

$y = 5t+15 = 5(-2)+15 = 5$

$z = 2t+12 = 2(-2)+12 = 8$

Hence the point of intersection of the line and plane π_2 is (3, 5, 8).

7.(a) First find two vectors which lie in the plane, eg $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

A normal vector for the plane is then given by

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= \boldsymbol{i}\big(1(2) - 0(1)\big) - \boldsymbol{j}\big(1(2) - 0(0)\big) + \boldsymbol{k}\big(1(1) - 1(0)\big)$$

$$= \boldsymbol{i}(2) - \boldsymbol{j}(2) + \boldsymbol{k}(1)$$

$$= 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

Equation of plane: $2x - 2y + z = \lambda$ for some constant λ

To evaluate λ, substitute the coordinates of a point in the plane, eg $C(0, 0, 5)$.

$$2x - 2y + z = \lambda \quad \Rightarrow \quad 2(0) - 2(0) + 5 = \lambda \quad \Rightarrow \quad \lambda = 5$$

The equation of the plane is $2x - 2y + z = 5$.

(b) Plane π_2 has normal vector $-\boldsymbol{j} + \boldsymbol{k}$.

Equation of plane π_2: $-y + z = \mu$ for some constant μ

To evaluate μ, substitute the coordinates of $A(0, -1, 3)$.

$$-y + z = \mu \quad \Rightarrow \quad -(-1) + 3 = \mu \quad \Rightarrow \quad \mu = 4$$

The equation of the plane π_2 is $-y + z = 4$.

(c) plane π_1: $2x - 2y + z = 5$

plane π_2: $-y + z = 4$

The angle, θ, between planes π_1 and π_2 is the angle between the two normal vectors for the planes.

Hence $\cos\theta = \dfrac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$ where $\boldsymbol{n}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

$$\boldsymbol{n}_1 \bullet \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 2(0) + (-2)(-1) + 1(1) = 3$$

$$|\boldsymbol{n}_1| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$|\boldsymbol{n}_2| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{0+1+1} = \sqrt{2}$$

Hence $\cos\theta = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 45°$

The acute angle between planes π_1 and π_2 is $45°$.

8.(a) To find the point of intersection of the lines L_1 and L_2, first find parametric equations for each line.

L_1: $x = 4 + 3\lambda$, $y = 2 + 4\lambda$, $z = -7\lambda$

L_2: $\dfrac{x-3}{-2} = \dfrac{y-8}{1} = \dfrac{z+1}{3} = t \quad \Rightarrow \quad x = -2t + 3, \quad y = t + 8, \quad z = 3t - 1$

Equating $x \Rightarrow 4 + 3\lambda = -2t + 3 \Rightarrow 3\lambda + 2t = -1$ ...(1)

Equating $y \Rightarrow 2 + 4\lambda = t + 8 \Rightarrow 4\lambda - t = 6$ ...(2)

Equating $z \Rightarrow -7\lambda = 3t - 1 \Rightarrow -7\lambda - 3t = -1$
$\Rightarrow 7\lambda + 3t = 1.$ ...(3)

Solving equations (1) and (2) gives $\lambda = 1$ and $t = -2$.

To verify that the lines L_1 and L_2 intersect, we must check that the values of λ and t also satisfy equation (3).

Check in (3): $7\lambda + 3t = 7(1) + 3(-2) = 1$ √

The values of λ and t also satisfy equation (3), hence the lines L_1 and L_2 intersect.

$x = 4 + 3\lambda = 4 + 3(1) = 7$ *or* $x = -2t + 3 = -2(-2) + 3 = 7$
$y = 2 + 4\lambda = 2 + 4(1) = 6$ *or* $y = t + 8 = -2 + 8 = 6$
$z = -7\lambda = -7(1) = -7$ *or* $z = 3t - 1 = 3(-2) - 1 = -7$

The point of intersection is (7, 6, –7).

(b) The angle, θ, between the lines L_1 and L_2 is the angle between the two direction vectors for the lines.

Hence $\cos\theta = \dfrac{\boldsymbol{d}_1 \bullet \boldsymbol{d}_2}{|\boldsymbol{d}_1||\boldsymbol{d}_2|}$.

$$L_1: \quad x = 4 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = -7\lambda \quad \Rightarrow \quad \boldsymbol{d}_1 = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}$$

$$L_2: \quad \frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3} \quad \Rightarrow \quad \boldsymbol{d}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$$\boldsymbol{d}_1 \bullet \boldsymbol{d}_2 = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = 3(-2) + 4(1) + (-7)(3) = -23$$

$$|\boldsymbol{d}_1| = \sqrt{3^2 + 4^2 + (-7)^2} = \sqrt{9 + 16 + 49} = \sqrt{74}$$

$$|\boldsymbol{d}_2| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

Hence $\cos\theta = \dfrac{-23}{\sqrt{74}\sqrt{14}} = -0 \cdot 7145... \quad \Rightarrow \quad \theta = \cos^{-1}(-0 \cdot 7145...) = 135 \cdot 6^\circ$ (to 1 dp)

The obtuse angle between the lines L_1 and L_2 is $135 \cdot 6^\circ$.

9.(a) (i) To find the point of intersection of the lines L_1 and L_2, first find parametric equations for each line.

L_1: $x = 8 - 2t$, $y = -4 + 2t$, $z = 3 + t$

L_2: $\dfrac{x}{-2} = \dfrac{y+2}{-1} = \dfrac{z-9}{2} = \lambda \Rightarrow x = -2\lambda$, $y = -\lambda - 2$, $z = 2\lambda + 9$

Equating $x \Rightarrow 8 - 2t = -2\lambda \Rightarrow -2t + 2\lambda = -8$ ...(1)

Equating $y \Rightarrow -4 + 2t = -\lambda - 2 \Rightarrow 2t + \lambda = 2$ ...(2)

Equating $z \Rightarrow 3 + t = 2\lambda + 9 \Rightarrow t - 2\lambda = 6$ ...(3)

Solving equations (1) and (2) gives $t = 2$ and $\lambda = -2$.

To verify that the lines L_1 and L_2 intersect, we must check that the values of t and λ also satisfy equation (3).

Check in (3): $t - 2\lambda = 2 - 2(-2) = 6$ √

The values of t and λ also satisfy equation (3), hence the lines L_1 and L_2 intersect.

$x = 8 - 2t = 8 - 2(2) = 4$ *or* $x = -2\lambda = -2(-2) = 4$
$y = -4 + 2t = -4 + 2(2) = 0$ *or* $y = -\lambda - 2 = -(-2) - 2 = 0$
$z = 3 + t = 3 + 2 = 5$ *or* $z = 2\lambda + 9 = 2(-2) = 9 = 5$

The point of intersection is (4, 0, 5).

(ii) The angle, θ, between the lines L_1 and L_2 is the angle between the two direction vectors for the lines.

Hence $\cos\theta = \dfrac{\boldsymbol{d}_1 \bullet \boldsymbol{d}_2}{|\boldsymbol{d}_1||\boldsymbol{d}_2|}$.

$$L_1: \quad x = 8 - 2t, \quad y = -4 + 2t, \quad z = 3 + t \quad \Rightarrow \quad \boldsymbol{d}_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$L_2: \quad \frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2} \quad \Rightarrow \quad \boldsymbol{d}_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

$$\boldsymbol{d}_1 \bullet \boldsymbol{d}_2 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = (-2)(-2) + 2(-1) + 1(2) = 4$$

$$|\boldsymbol{d}_1| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$|\boldsymbol{d}_2| = \sqrt{(-2)^2 + (-1)^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

Hence $\cos\theta = \dfrac{4}{3(3)} = \dfrac{4}{9} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\dfrac{4}{9}\right)$

The acute angle between the lines L_1 and L_2 is $\cos^{-1}\left(\dfrac{4}{9}\right)$.

(b) (i) The plane Π is perpendicular to the line L_2, so a direction vector for the line L_2 will also be a normal vector for the plane Π.

$$L_2: \quad \frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2} \quad \Rightarrow \quad \boldsymbol{d} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{n} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = -2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

Equation of plane Π: $\quad -2x - y + 2z = \lambda \quad$ for some constant λ

To evaluate λ, substitute the coordinates of the point (1, –4, 2) which is in the plane Π.

$$-2x - y + 2z = \lambda \quad \Rightarrow \quad -2(1) - (-4) + 2(2) = \lambda \quad \Rightarrow \quad \lambda = 6$$

The equation of the plane Π is $\quad -2x - y + 2z = 6$.

(ii) $L_1: \quad x = 8 - 2t, \quad y = -4 + 2t, \quad z = 3 + t$

To find the point of intersection of the line L_1 and the plane Π, substitute the parametric equations for L_1 into the equation of the plane Π.

$$\begin{aligned} -2x - y + 2z = 6 \quad &\Rightarrow \quad -2(8-2t) - (-4+2t) + 2(3+t) = 6 \\ &\Rightarrow \quad -16 + 4t + 4 - 2t + 6 + 2t = 6 \\ &\Rightarrow \quad 4t - 6 = 6 \\ &\Rightarrow \quad 4t = 12 \\ &\Rightarrow \quad t = 3 \end{aligned}$$

Then $\quad x = 8 - 2t = 8 - 2(3) = 2$

$\quad y = -4 + 2t = -4 + 2(3) = 2$

$\quad z = 3 + t = 3 + 3 = 6$

Hence the point of intersection of the line L_1 and the plane Π is (2, 2, 6).

10.(a) To find the point of intersection of the lines L_1 and L_2, first find parametric equations for each line.

$$L_1: \quad \frac{x+6}{3} = \frac{y-1}{-1} = \frac{z-2}{2} = t \quad \Rightarrow \quad x = 3t-6, \quad y = -t+1, \quad z = 2t+2$$

$$L_2: \quad \frac{x+5}{4} = \frac{y+4}{1} = \frac{z}{4} = \lambda \quad \Rightarrow \quad x = 4\lambda-5, \quad y = \lambda-4, \quad z = 4\lambda$$

Equating $x \quad \Rightarrow \quad 3t-6 = 4\lambda-5 \quad \Rightarrow \quad 3t-4\lambda = 1 \quad ...(1)$

Equating $y \quad \Rightarrow \quad -t+1 = \lambda-4 \quad \Rightarrow \quad -t-\lambda = -5 \quad \Rightarrow \quad t+\lambda = 5 \quad ...(2)$

Equating $z \quad \Rightarrow \quad 2t+2 = 4\lambda \quad \Rightarrow \quad 2t-4\lambda = -2 \quad ...(3)$

Solving equations (1) and (2) gives $t = 3$ and $\lambda = 2$.

To verify that the lines L_1 and L_2 intersect, we must check that the values of t and λ also satisfy equation (3).

Check in (3): $2t-4\lambda = 2(3)-4(2) = -2 \quad \surd$

The values of t and λ also satisfy equation (3), hence the lines L_1 and L_2 intersect.

$x = 3t-6 = 3(3)-6 = 3 \qquad$ *or* $\qquad x = 4\lambda-5 = 4(2)-5 = 3$
$y = -t+1 = -3+1 = -2 \qquad$ *or* $\qquad y = \lambda-4 = 2-4 = -2$
$z = 2t+2 = 2(3)+2 = 8 \qquad$ *or* $\qquad z = 4\lambda = 4(2) = 8$

The point of intersection is (3, –2, 8).

(b) First find a direction vector for each line.

$$L_1: \quad \frac{x+6}{3} = \frac{y-1}{-1} = \frac{z-2}{2} = t \quad \Rightarrow \quad \boldsymbol{d}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$L_2: \quad \frac{x+5}{4} = \frac{y+4}{1} = \frac{z}{4} = \lambda \quad \Rightarrow \quad \boldsymbol{d}_2 = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}$$

The direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are parallel to the plane containing the lines L_1 and L_2.

Hence a normal vector for the plane π is given by

$$\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -1 & 2 \\ 4 & 1 & 4 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= \boldsymbol{i}\big((-1)(4) - 2(1)\big) - \boldsymbol{j}\big(3(4) - 2(4)\big) + \boldsymbol{k}\big(3(1) - (-1)(4)\big)$$

$$= \boldsymbol{i}(-6) - \boldsymbol{j}(4) + \boldsymbol{k}(7)$$

$$= -6\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}$$

Equation of plane: $-6x - 4y + 7z = \mu$ for some constant μ

To evaluate μ, substitute the coordinates of a point in the plane.
The point of intersection of the lines L_1 and L_2 is in the plane containing the lines L_1 and L_2, so we can use the point (3, –2, 8) to find μ.

$$-6x - 4y + 7z = \mu \quad \Rightarrow \quad -6(3) - 4(-2) + 7(8) = \mu \quad \Rightarrow \quad \mu = 46$$

The equation of the plane is $-6x - 4y + 7z = 46$.

(c) The angle, θ, between a line and a plane can be calculated using the formula $\sin\theta = \dfrac{\boldsymbol{d} \bullet \boldsymbol{n}}{|\boldsymbol{d}||\boldsymbol{n}|}$,

where $\boldsymbol{d}$ is a direction vector for the line and $\boldsymbol{n}$ is a normal vector for the plane.

line L_3: $\dfrac{x-1}{2} = \dfrac{y+7}{4} = \dfrac{z-3}{-1} \quad \Rightarrow \quad \boldsymbol{d} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$

plane π: $\boldsymbol{n} = -6\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k} = \begin{pmatrix} -6 \\ -4 \\ 7 \end{pmatrix}$

$$\boldsymbol{d} \bullet \boldsymbol{n} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -6 \\ -4 \\ 7 \end{pmatrix} = 2(-6) + 4(-4) + (-1)(7) = -35$$

$$|\boldsymbol{d}| = \sqrt{2^2 + 4^2 + (-1)^2} = \sqrt{4 + 16 + 1} = \sqrt{21}$$

$$|\boldsymbol{n}| = \sqrt{(-6)^2 + (-4)^2 + 7^2} = \sqrt{36 + 16 + 49} = \sqrt{101}$$

Hence $\sin\theta = \dfrac{-35}{\sqrt{21}\sqrt{101}} = -0 \cdot 7599... \quad \Rightarrow \quad \theta = \sin^{-1}(-0 \cdot 7599..) = -49 \cdot 46°$ (to 2 dp)

The acute angle between line L_3 and the plane π is $49 \cdot 46°$.

11.(a) Line L_1 passes through the point P(2, 4, 1) and is parallel to the vector $\boldsymbol{d}_1 = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

Equation of line L_1 in symmetric form: $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z-1}{-1} = t$

Parametric equations for line L_1: $x = t + 2$, $y = 2t + 4$, $z = -t + 1$

Line, L_2 passes through the point Q(–5, 2, 5) and is parallel to the vector $\boldsymbol{d}_2 = -4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$.

Equation of line L_2 in symmetric form: $\frac{x+5}{-4} = \frac{y-2}{4} = \frac{z-5}{1} = \lambda$

Parametric equations for line L_2: $x = -4\lambda - 5$, $y = 4\lambda + 2$, $z = \lambda + 5$

(b) L_1: $x = t + 2$, $y = 2t + 4$, $z = -t + 1$

L_2: $x = -4\lambda - 5$, $y = 4\lambda + 2$, $z = \lambda + 5$

Equating x $\Rightarrow$ $t + 2 = -4\lambda - 5$ $\Rightarrow$ $t + 4\lambda = -7$ ...(1)

Equating y $\Rightarrow$ $2t + 4 = 4\lambda + 2$ $\Rightarrow$ $2t - 4\lambda = -2$ ...(2)

Equating z $\Rightarrow$ $-t + 1 = \lambda + 5$ $\Rightarrow$ $-t - \lambda = 4$ ...(3)

Solving equations (1) and (2) gives $t = -3$ and $\lambda = -1$.

To verify that the lines L_1 and L_2 intersect, we must check that the values of t and λ also satisfy equation (3).

Check in (3): $-t - \lambda = -(-3) - (-1) = 3 + 1 = 4$ $\surd$

The values of t and λ also satisfy equation (3), hence the lines L_1 and L_2 intersect.

$x = t + 2 = -3 + 2 = -1$ *or* $x = -4\lambda - 5 = -4(-1) - 5 = -1$

$y = 2t + 4 = 2(-3) + 4 = -2$ *or* $y = 4\lambda + 2 = 4(-1) + 2 = -2$

$z = -t + 1 = -(-3) + 1 = 4$ *or* $z = \lambda + 5 = -1 + 5 = 4$

The point of intersection is (–1, –2, 4).

(c) $\boldsymbol{d}_1 = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{d}_2 = -4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$

The direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are parallel to the plane containing the lines L_1 and L_2.

Hence a normal vector for the plane is given by

$$\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -1 \\ -4 & 4 & 1 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 2 \\ -4 & 4 \end{vmatrix}$$

$$= \boldsymbol{i}\big(2(1) - (-1)(4)\big) - \boldsymbol{j}\big(1(1) - (-1)(-4)\big) + \boldsymbol{k}\big(1(4) - 2(-4)\big)$$
$$= \boldsymbol{i}(6) - \boldsymbol{j}(-3) + \boldsymbol{k}(12)$$
$$= 6\boldsymbol{i} + 3\boldsymbol{j} + 12\boldsymbol{k}$$

Equation of plane: $6x + 3y + 12z = \mu$ for some constant μ

To evaluate μ, substitute the coordinates of a point in the plane.
The point of intersection of the lines L_1 and L_2 is in the plane containing the lines L_1 and L_2, so we can use the point (–1, –2, 4) to find μ.

$$6x + 3y + 12z = \mu \quad \Rightarrow \quad 6(-1) + 3(-2) + 12(4) = \mu \quad \Rightarrow \quad \mu = 36$$

The equation of the plane is $6x + 3y + 12z = 36$ *or* $2x + y + 4z = 12$

<u>Note</u>

Parametric equations for the lines L_1 and L_2 can also be written down directly using the coordinates of the points and components of the direction vectors.

Line L_1 passes through the point P(2, 4, 1) and is parallel to the vector $\boldsymbol{d}_1 = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, so parametric equations for line L_1 are $x = 2 + t$, $y = 4 + 2t$, $z = 1 - t$.

Line L_2 passes through Q(–5, 2, 5) and is parallel to the vector $\boldsymbol{d}_2 = -4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$, so parametric equations for line L_2 are $x = -5 - 4\lambda$, $y = 2 + 4\lambda$, $z = 5 + \lambda$.

12.(a) To find when the lines L_1 and L_2 intersect, first find parametric equations for each line.

$$L_1: \quad \frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} = t \quad \Rightarrow \quad x = kt+1, \quad y = -t, \quad z = t-3$$

$$L_2: \quad \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2} = \lambda \quad \Rightarrow \quad x = \lambda+4, \quad y = \lambda-3, \quad z = 2\lambda-3$$

Equating x $\Rightarrow$ $kt+1 = \lambda+4$ $\Rightarrow$ $kt - \lambda = 3$ ...(1)

Equating y $\Rightarrow$ $-t = \lambda - 3$ $\Rightarrow$ $-t - \lambda = -3$ $\Rightarrow$ $t + \lambda = 3$ ...(2)

Equating z $\Rightarrow$ $t - 3 = 2\lambda - 3$ $\Rightarrow$ $t - 2\lambda = 0$ ...(3)

Solving equations (2) and (3) gives $t = 2$ and $\lambda = 1$.

When the lines L_1 and L_2 intersect, the values of t and λ will also satisfy equation (1).

Substitute $t = 2$ and $\lambda = 1$ into equation (1): $kt - \lambda = 3$ $\Rightarrow$ $k(2) - 1 = 3$

$$\Rightarrow \quad 2k = 4$$

$$\Rightarrow \quad k = 2$$

Hence the lines L_1 and L_2 intersect when $k = 2$.

$x = kt+1 = 2t+1 = 2(2)+1 = 5$ *or* $x = \lambda + 4 = 1+4 = 5$

$y = -t = -2$ *or* $y = \lambda - 3 = 1 - 3 = -2$

$z = t - 3 = 2 - 3 = -1$ *or* $z = 2\lambda - 2 = 2(1) - 3 = -1$

The point of intersection is (5, −2, −1).

(b) The angle, θ, between the lines L_1 and L_2 is the angle between the two direction vectors for the lines.

Hence $\cos\theta = \dfrac{\boldsymbol{d}_1 \bullet \boldsymbol{d}_2}{|\boldsymbol{d}_1||\boldsymbol{d}_2|}$.

$$L_1: \quad \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1} \quad \Rightarrow \quad \boldsymbol{d}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$L_2: \quad \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2} \quad \Rightarrow \quad \boldsymbol{d}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\boldsymbol{d}_1 \bullet \boldsymbol{d}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2(1) + (-1)(1) + 1(2) = 3$$

$$|\boldsymbol{d}_1| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$|\boldsymbol{d}_2| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$$

Hence $\cos\theta = \dfrac{3}{\sqrt{6}\sqrt{6}} = \dfrac{3}{6} = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\dfrac{1}{2}\right) = 60°$

The acute angle between the lines L_1 and L_2 is $60°$.

13. Let $z = t$.

plane π_1: $x - 4y + 2z = 1 \Rightarrow x - 4y + 2t = 1 \Rightarrow x - 4y = 1 - 2t$ ...(1)

plane π_2: $x - y - z = -5 \Rightarrow x - y - t = -5 \Rightarrow x - y = -5 + t$ ...(2)

To find parametric equations for the line of intersection of the two planes, solve equations (1) and (2) simultaneously to find expressions for *x* and *y* in terms of *t*.

(1) − (2) eliminates *x* $\Rightarrow (x - 4y) - (x - y) = (1 - 2t) - (-5 + t)$

$\Rightarrow x - 4y - x + y = 1 - 2t + 5 - t$

$\Rightarrow -3y = 6 - 3t$

$\Rightarrow y = -2 + t$ *or* $y = t - 2$

To find *x*, substitute $y = t - 2$ into equation (2):

$x - y = -5 + t \Rightarrow x - (t - 2) = -5 + t$

$\Rightarrow x - t + 2 = -5 + t$

$\Rightarrow x = -5 + t + t - 2$

$\Rightarrow x = 2t - 7$

Hence parametric equations for the line of intersection of the two planes are
$x = 2t - 7$, $y = t - 2$, $z = t$.

To show that the line of intersection lies in the plane with equation $x + 2y - 4z = -11$, verify that the parametric equations for *x*, *y* and *z* satisfy the equation of the plane.

$$x + 2y - 4z = 2t - 7 + 2(t - 2) - 4t$$
$$= 2t - 7 + 2t - 4 - 4t$$
$$= -11$$

Hence the line of intersection lies in the plane with equation $x + 2y - 4z = -11$.

14.(a) To find parametric equations for the line of intersection of the two planes, let $x = \lambda$.

plane π_1: $x + y - z = 6$ $\Rightarrow$ $\lambda + y - z = 6$ $\Rightarrow$ $y - z = 6 - \lambda$ ...(1)

plane π_2: $2x - 3y + 2z = 2$ $\Rightarrow$ $2\lambda - 3y + 2z = 2$ $\Rightarrow$ $-3y + 2z = 2 - 2\lambda$ ...(2)

Now solve equations (1) and (2) simultaneously to find expressions for *x* and *y* in terms of λ .

3 × (1) + (2) eliminates *y* $\Rightarrow$ $3(y - z) + (-3y + 2z) = 3(6 - \lambda) + (2 - 2\lambda)$

$\Rightarrow$ $3y - 3z - 3y + 2z = 18 - 3\lambda + 2 - 2\lambda$

$\Rightarrow$ $-z = 20 - 5\lambda$

$\Rightarrow$ $z = -20 + 5\lambda$

$\Rightarrow$ $z = 5\lambda - 20$

To find z, substitute $z = 5\lambda - 20$ into equation (1):

$y - z = 6 - \lambda$ $\Rightarrow$ $y - (5\lambda - 20) = 6 - \lambda$

$\Rightarrow$ $y - 5\lambda + 20 = 6 - \lambda$

$\Rightarrow$ $y = 6 - \lambda + 5\lambda - 20$

$\Rightarrow$ $y = -14 + 4\lambda$

$\Rightarrow$ $y = 4\lambda - 14$

Hence parametric equations for the line of intersection of the two planes are
$x = \lambda$, $y = 4\lambda - 14$, $z = 5\lambda - 20$.

(b) The angle, θ, between a line and a plane can be calculated using the formula $\sin\theta = \dfrac{\boldsymbol{d} \bullet \boldsymbol{n}}{|\boldsymbol{d}||\boldsymbol{n}|}$,

where $\boldsymbol{d}$ is a direction vector for the line and $\boldsymbol{n}$ is a normal vector for the plane.

line L: $x = \lambda$, $y = 4\lambda - 14$, $z = 5\lambda - 20$ $\Rightarrow$ $\boldsymbol{d} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$

plane: $-5x + 2y - 4z = 1$ $\Rightarrow$ $\boldsymbol{n} = \begin{pmatrix} -5 \\ 2 \\ -4 \end{pmatrix}$

$$\boldsymbol{d} \bullet \boldsymbol{n} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -5 \\ 2 \\ -4 \end{pmatrix} = 1(-5) + 4(2) + 5(-4) = -17$$

$$|\boldsymbol{d}| = \sqrt{1^2 + 4^2 + 5^2} = \sqrt{1 + 16 + 25} = \sqrt{42}$$

$$|\boldsymbol{n}| = \sqrt{(-5)^2 + 2^2 + (-4)^2} = \sqrt{25 + 4 + 16} = \sqrt{45}$$

Hence $\sin\theta = \dfrac{-17}{\sqrt{42}\sqrt{45}} = -0 \cdot 3910...$ $\Rightarrow$ $\theta = \sin^{-1}(-0 \cdot 3910...) = -23 \cdot 02°$ (to 2 dp)

The acute angle between line L and the plane is $23 \cdot 02°$.

15.(a) The plane is perpendicular to the line L, so a direction vector for the line L will also be a normal vector for the plane.

line L: $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1} \quad \Rightarrow \quad \boldsymbol{d} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

Equation of plane: $2x + y - z = \lambda$ for some constant λ

To evaluate λ, substitute the coordinates of the point P(1, 1, 0) which is in the plane.

$$2x + y - z = \lambda \quad \Rightarrow \quad 2(1) + 1 - 0 = \lambda \quad \Rightarrow \quad \lambda = 3$$

The equation of the plane is $2x + y - z = 3$.

(b) First find parametric equations for the line.

$$\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1} = t \quad \Rightarrow \quad x = 2t - 1, \quad y = t + 2, \quad z = -t$$

To find the point of intersection of the line L and the plane, substitute the parametric equations for the line L into the equation of the plane.

$$\begin{aligned} 2x + y - z = 3 \quad &\Rightarrow \quad 2(2t-1) + (t+2) - (-t) = 3 \\ &\Rightarrow \quad 4t - 2 + t + 2 + t = 3 \\ &\Rightarrow \quad 6t = 3 \\ &\Rightarrow \quad t = \frac{1}{2} \end{aligned}$$

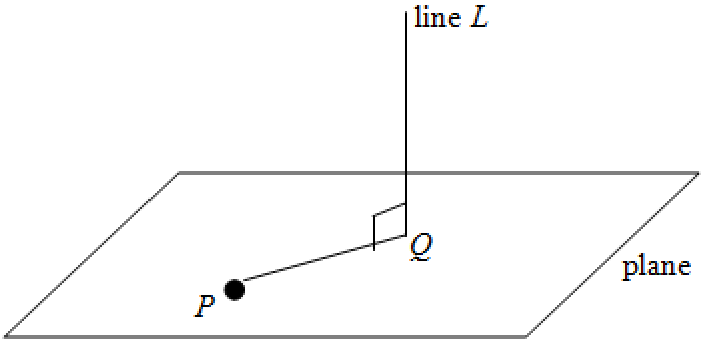
Then $x = 2t - 1 = 2\left(\frac{1}{2}\right) - 1 = 0$

$y = t + 2 = \frac{1}{2} + 2 = \frac{5}{2}$

$z = -t = -\frac{1}{2}$

Hence the point of intersection of the line and the plane is $Q\left(0, \frac{5}{2}, -\frac{1}{2}\right)$.

(c) The shortest distance from the point P to the line L is the distance PQ, since PQ is perpendicular to the line L (see diagram below).



The distance PQ will be the magnitude of the vector $\overrightarrow{PQ}$.

$$P(1,\ 1,\ 0) \quad \text{and} \quad Q\left(0, \frac{5}{2}, -\frac{1}{2}\right) \quad \Rightarrow \quad \overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \begin{pmatrix} 0 \\ 5/2 \\ -1/2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3/2 \\ -1/2 \end{pmatrix}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(-1)^2 + \left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1 + \frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{14}{4}} = \frac{\sqrt{14}}{2}$$

Hence the shortest distance from P to L is $PQ = \dfrac{\sqrt{14}}{2}$ and this is the shortest distance since PQ is perpendicular to L.

16.(a) First find two vectors which lie in the plane, eg $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

A normal vector for the plane is then given by

$$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 0 \\ -1 & 2 & 2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -2 & 0 \\ 2 & 2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix}$$

$$= \boldsymbol{i}\big((-2)(2) - 0(2)\big) - \boldsymbol{j}\big(1(2) - 0(-1)\big) + \boldsymbol{k}\big(1(2) - (-2)(-1)\big)$$

$$= \boldsymbol{i}(-4) - \boldsymbol{j}(2) + \boldsymbol{k}(0)$$

$$= -4\boldsymbol{i} - 2\boldsymbol{j}$$

Equation of plane: $-4x - 2y = \lambda$ for some constant λ

To evaluate λ, substitute the coordinates of a point in the plane, eg $A(1, 1, 1)$.

$$-4x - 2y = \lambda \quad \Rightarrow \quad -4(1) - 2(1) = \lambda \quad \Rightarrow \quad \lambda = -6$$

The equation of the plane is $-4x - 2y = -6$ *or* $4x + 2y = 6$ *or* $2x + y = 3$.

(b) Let $x = t$.

plane π_1: $2x + y = 3 \quad \Rightarrow \quad 2t + y = 3 \quad \Rightarrow \quad y = 3 - 2t$

Hence the parametric equation for y is $y = 3 - 2t$.

plane π_2: $x + 3y - z = 2 \quad \Rightarrow \quad t + 3(3 - 2t) - z = 2$

$$\Rightarrow \quad t + 9 - 6t - z = 2$$
$$\Rightarrow \quad -5t + 9 - z = 2$$
$$\Rightarrow \quad -z = 2 + 5t - 9$$
$$\Rightarrow \quad -z = -7 + 5t$$
$$\Rightarrow \quad z = 7 - 5t$$

Hence parametric equations for the line of intersection of the two planes are
$x = t$, $y = 3 - 2t$, $z = 7 - 5t$.

(c) plane π_1: $2x + y = 3$

plane π_2: $x + 3y - z = 2$

The angle, θ, between planes π_1 and π_2 is the angle between the two normal vectors for the planes.

Hence $\cos\theta = \dfrac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$ where $\boldsymbol{n}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

$$\boldsymbol{n}_1 \bullet \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = 2(1) + 1(3) + 0(-1) = 5$$

$$|\boldsymbol{n}_1| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{4 + 1 + 0} = \sqrt{5}$$

$$|\boldsymbol{n}_2| = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{1 + 9 + 1} = \sqrt{11}$$

Hence $\cos\theta = \dfrac{5}{\sqrt{5}\sqrt{11}} = 0 \cdot 6741...$ $\Rightarrow$ $\theta = \cos^{-1} 0 \cdot 6741... = 47 \cdot 6°$ (to 1 dp)

The acute angle between planes π_1 and π_2 is $47 \cdot 6°$.

17.(a) The vector $\overrightarrow{BT}$ is a direction vector for the line BT.

$$\overrightarrow{BT} = \boldsymbol{t} - \boldsymbol{b} = \begin{pmatrix} -3 \\ -22 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ 8 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ -30 \\ 5 \end{pmatrix}$$

Note that $\overrightarrow{BT} = 5\begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix}$, so we can use the vector $\boldsymbol{d} = \begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix}$ as a direction vector.

Using the point B(7, 8, 1), the equation of the line in symmetric form is

$$\frac{x-7}{-2} = \frac{y-8}{-6} = \frac{z-1}{1} = t \quad \Rightarrow \quad x = -2t + 7, \quad y = -6t + 8, \quad z = t + 1$$

Hence parametric equations for the line BT are $x = -2t + 7$, $y = -6t + 8$, $z = t + 1$.

(b) First find two vectors which lie in the plane, eg $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{PR} = \boldsymbol{r} - \boldsymbol{p} = \begin{pmatrix} -3 \\ 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ -8 \end{pmatrix}$$

A normal vector for the plane is then given by

$$\boldsymbol{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & -2 \\ -5 & 6 & -8 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 1 & -2 \\ 6 & -8 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -1 & -2 \\ -5 & -8 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -1 & 1 \\ -5 & 6 \end{vmatrix}$$

$$= \boldsymbol{i}\big(1(-8) - (-2)(6)\big) - \boldsymbol{j}\big((-1)(-8) - (-2)(-5)\big) + \boldsymbol{k}\big((-1)(6) - 1(-5)\big)$$
$$= \boldsymbol{i}(4) - \boldsymbol{j}(-2) + \boldsymbol{k}(-1)$$
$$= 4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

Equation of plane: $4x + 2y - z = \lambda$ for some constant λ

To evaluate λ, substitute the coordinates of a point in the plane, eg P(2, 1, 9).

$$4x + 2y - z = \lambda \quad \Rightarrow \quad 4(2) + 2(1) - 9 = \lambda \quad \Rightarrow \quad \lambda = 1$$

The equation of the plane is $4x + 2y - z = 1$.

(c) H is the point of intersection of the line in (a) and the plane in (b).

To find the point of intersection of the line and the plane, substitute the parametric equations for the line into the equation of the plane.

$$
\begin{aligned}
4x + 2y - z = 1 \quad &\Rightarrow \quad 4(-2t + 7) + 2(-6t + 8) - (t + 1) = 1 \\
&\Rightarrow \quad -8t + 28 - 12t + 16 - t - 1 = 1 \\
&\Rightarrow \quad -21t + 43 = 1 \\
&\Rightarrow \quad -21t = -42 \\
&\Rightarrow \quad t = 2
\end{aligned}
$$

Then $x = -2t + 7 = -2(2) + 7 = 3$

$y = -6t + 8 = -6(2) + 8 = -4$

$z = t + 1 = 2 + 1 = 3$

Hence the point of intersection of the line and the plane is H(3, –4, 3).

Notes

(1) Any multiple of the vector $\overrightarrow{BT} = \begin{pmatrix} -10 \\ -30 \\ 5 \end{pmatrix}$ can be used as a direction vector for the line BT.

(2) Parametric equations for the line BT can be written down directly using the coordinates of a point on the line and components of a direction vector.

The line BT passes through the point B(7, 8, 1) and is parallel to the vector $\boldsymbol{d} = -2\boldsymbol{i} - 6\boldsymbol{j} + \boldsymbol{k}$, so parametric equations for line BT are

$x = 7 - 2t, \quad y = 8 - 6t, \quad z = 1 + t$.

18.(a) $L_1: \quad x = 2 + s, \quad y = -s, \quad z = 2 - s$

$L_2: \quad x = -1 - 2t, \quad y = t, \quad z = 2 + 3t$

Equating $x \quad \Rightarrow \quad 2 + s = -1 - 2t \quad \Rightarrow \quad s + 2t = -3$ ...(1)

Equating $y \quad \Rightarrow \quad -s = t \quad \Rightarrow \quad -s - t = 0 \quad \Rightarrow \quad s + t = 0$ ...(2)

Equating $z \quad \Rightarrow \quad 2 - s = 2 + 3t \quad \Rightarrow \quad -s - 3t = 0 \quad \Rightarrow \quad s + 3t = 0$ ...(3)

Solving equations (1) and (2) gives $s = 3$ and $t = -3$.

If the lines L_1 and L_2 intersect, the values of s and t will also satisfy equation (3).

Check in (3): $\quad s + 3t = 3 + 3(-3) = -6$

$s + 3t \neq 0$, hence the values of s and t do not satisfy equation (3) and the lines L_1 and L_2 do not intersect.

(b) The direction of line L_3 is perpendicular to the directions of both L_1 and L_2, hence a direction vector for line L_3 is given by $\boldsymbol{d}_3 = \boldsymbol{d}_1 \times \boldsymbol{d}_2$, where $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ and $\boldsymbol{d}_3$ are direction vectors for the lines L_1, L_2 and L_3 respectively.

$$L_1: \quad x = 2 + s, \quad y = -s, \quad z = 2 - s \quad \Rightarrow \quad \boldsymbol{d}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$L_2: \quad x = -1 - 2t, \quad y = t, \quad z = 2 + 3t \quad \Rightarrow \quad \boldsymbol{d}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$$\boldsymbol{d}_3 = \boldsymbol{d}_1 \times \boldsymbol{d}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix}$$

$$= \boldsymbol{i}\big((-1)(3) - (-1)(1)\big) - \boldsymbol{j}\big(1(3) - (-1)(-2)\big) + \boldsymbol{k}\big(1(1) - (-1)(-2)\big)$$

$$= \boldsymbol{i}(-2) - \boldsymbol{j}(1) + \boldsymbol{k}(-1)$$

$$= -2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$

Line L_3 passes through the point $P(1, 1, 3)$ and is parallel to the vector $\boldsymbol{d}_3 = -2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ **.**

Equation of line L_3 in symmetric form: $\dfrac{x-1}{-2} = \dfrac{y-1}{-1} = \dfrac{z-3}{-1} = \lambda$

Parametric equations for line L_3: $x = -2\lambda + 1$, $y = -\lambda + 1$, $z = -\lambda + 3$

(c) L_3: $x = -2\lambda + 1$, $y = -\lambda + 1$, $z = -\lambda + 3$

L_2: $x = -1 - 2t$, $y = t$, $z = 2 + 3t$

Equating $x \Rightarrow -2\lambda + 1 = -1 - 2t \Rightarrow -2\lambda + 2t = -2$ ...(1)

Equating $y \Rightarrow -\lambda + 1 = t \Rightarrow -\lambda - t = -1 \Rightarrow \lambda + t = 1$ ..(2)

Equating $z \Rightarrow -\lambda + 3 = 2 + 3t \Rightarrow -\lambda - 3t = -1 \Rightarrow \lambda + 3t = 1$ ...(3)

Solving equations (1) and (2) gives $\lambda = 1$ and $t = 0$.

To verify that the lines L_3 and L_2 intersect, we must check that the values of λ and t also satisfy equation (3).

Check in (3): $\lambda + 3t = 1 + 3(0) = 1$ √

The values of λ and t also satisfy equation (3), hence the lines L_3 and L_2 intersect.

$x = -2\lambda + 1 = -2(1) + 1 = -1$ *or* $x = -1 - 2t = -1 - 2(0) = -1$
$y = -\lambda + 1 = -1 + 1 = 0$ *or* $y = t = 0$
$z = -\lambda + 3 = -1 + 3 = 2$ *or* $z = 2 + 3t = 2 + 3(0) = 2$

The point of intersection is $Q(-1, 0, 2)$.

To verify that the point P lies on line L_1, we must verify that the coordinates of *P* satisfy the equation of line L_1.

$L_1: \quad x = 2 + s, \quad y = -s, \quad z = 2 - s$

At the point P(1, 1, 3) where $x = 1,\ y = 1$ and $z = 3$:

$x = 2 + s \quad \Rightarrow \quad 1 = 2 + s \quad \Rightarrow \quad s = -1$

$y = -s \quad \Rightarrow \quad 1 = -s \quad \Rightarrow \quad s = -1$

$z = 2 - s \quad \Rightarrow \quad 3 = 2 - s \quad \Rightarrow \quad 1 = -s \quad \Rightarrow \quad s = -1$

Hence $s = -1$ gives $x = 1,\ y = 1$ and $z = 3$, thus giving the point *P*(1, 1, 3).
So the point *P* lies on line L_1.

(d) The distance *PQ* will be the magnitude of the vector $\overrightarrow{PQ}$.

$$P(1, 1, 3) \quad \text{and} \quad Q(-1, 0, 2) \quad \Rightarrow \quad \overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(-2)^2 + (-1)^2 + (-1)^2} = \sqrt{4+1+1} = \sqrt{6}$$

Hence $\quad PQ = \sqrt{6}$

Notes

(1) Note that $\boldsymbol{d}_3 = -2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} = -(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$, so the vector $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ could instead be used as a direction vector for line L_3. Any multiple of $\boldsymbol{d}_3$ can be used as a direction vector for line L_3.

(2) Parametric equations for the line L_3 can be written down directly using the coordinates of the point *P* and components of a direction vector.

The line L_3 passes through the point *P*(1, 1, 3) and is parallel to the vector $\boldsymbol{d}_3 = -2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$, so parametric equations for line L_3 are
$x = 1 - 2\lambda, \quad y = 1 - \lambda, \quad z = 3 - \lambda$.

19.(a) The intersection of the planes π_1, π_2 and π_3 is a line when the system of equations contains a redundant equation.

$$\begin{aligned} x - 2y + z &= -4 \\ 3x - 5y - 2z &= 1 \\ -7x + 11y + az &= -11 \end{aligned}$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 3 & -5 & -2 & 1 \\ -7 & 11 & a & -11 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 7R_1 \end{array}$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -5 & 13 \\ 0 & -3 & a+7 & -39 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -5 & 13 \\ 0 & 0 & a-8 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Redundancy will occur when all the entries in R_3 are zero.

Hence redundancy will occur when $a - 8 = 0 \quad \Rightarrow \quad a = 8$

The intersection of the planes π_1, π_2 and π_3 is a line when $a = 8$.

(b) Let $z = t$.

$R_2 \quad \Rightarrow \quad y - 5z = 13 \quad \Rightarrow \quad y - 5t = 13 \quad \Rightarrow \quad y = 13 + 5t$

$$\begin{aligned} R_1 \quad \Rightarrow \quad x - 2y + z = -4 \quad &\Rightarrow \quad x - 2(13 + 5t) + t = -4 \\ &\Rightarrow \quad x - 26 - 10t + t = -4 \\ &\Rightarrow \quad x - 26 - 9t = -4 \\ &\Rightarrow \quad x = -4 + 26 + 9t \\ &\Rightarrow \quad x = 22 + 9t \end{aligned}$$

Hence parametric equations for the line of intersection of the planes are
$x = 22 + 9t$, $y = 13 + 5t$, $z = t$.

(c) (i) plane π_1: $x - 2y + z = -4$

plane π_4: $-9x + 15y + 6z = 20$

The angle, θ, between the planes π_1 and π_4 is the angle between the two normal vectors for the planes.

Hence $\cos\theta = \dfrac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_4}{|\boldsymbol{n}_1||\boldsymbol{n}_4|}$ where $\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{n}_4 = \begin{pmatrix} -9 \\ 15 \\ 6 \end{pmatrix}$.

$$\boldsymbol{n}_1 \bullet \boldsymbol{n}_4 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -9 \\ 15 \\ 6 \end{pmatrix} = 1(-9) + (-2)(15) + 1(6) = -33$$

$$|\boldsymbol{n}_4| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$|\boldsymbol{n}_4| = \sqrt{(-9)^2 + 15^2 + 6^2} = \sqrt{81 + 225 + 36} = \sqrt{342}$$

Hence $\cos\theta = \dfrac{-33}{\sqrt{6}\sqrt{342}} = -0 \cdot 7284\ldots \quad \Rightarrow \quad \theta = \cos^{-1}(-0 \cdot 7284\ldots)$

$= 136 \cdot 8°$ (to 1 dp)

The acute angle between the planes π_1 and π_4 is $180° - 136 \cdot 8° = 43 \cdot 2°$.

(ii) Let $\boldsymbol{n}_2$ and $\boldsymbol{n}_4$ be normal vectors for the planes π_2 and π_4 respectively.

plane π_2: $3x - 5y - 2z = 1 \quad \Rightarrow \quad \boldsymbol{n}_2 = \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$

plane π_4: $-9x + 15y + 6z = 20 \quad \Rightarrow \quad \boldsymbol{n}_4 = \begin{pmatrix} -9 \\ 15 \\ 6 \end{pmatrix}$

$$\boldsymbol{n}_4 = \begin{pmatrix} -9 \\ 15 \\ 6 \end{pmatrix} = -3\begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix} = -3\boldsymbol{n}_2$$

$\boldsymbol{n}_4 = -3\boldsymbol{n}_2$, hence the normal vectors $\boldsymbol{n}_2$ and $\boldsymbol{n}_4$ are parallel meaning that the planes π_2 and π_4 are parallel.

Note

If the system of equations has a unique solution for x, y and z, then the intersection of the planes π_1, π_2 and π_3 will be a point.

If the system of equations is inconsistent with no solution, then the planes π_1, π_2 and π_3 will not intersect.