Partial Fractions

Partial fractions involves splitting a rational function into more manageable fractions. This process is essential in many areas of mathematics including integration and curve sketching.

We know how to write the sum or difference of algebraic fractions as one fraction with a common denominator.

e.g. $\frac{5}{x+2}+\frac{3}{x+1}=\frac{5(x+1)+3(x+2)}{(x+2)(x+1)}=\frac{8x+11}{(x+2)(x+1)}$

The opposite process of expressing $\frac{8x+11}{(x+2)(x+1)}$ as $\frac{5}{x+2}+\frac{3}{x+1}$ is called writing a rational function in partial fractions.

A **proper rational function** $\frac{P(x)}{Q(x)}$ is one in which the degree of the numerator is less than the degree of the denominator e.g. $\frac{x+3}{x^2+3x-4}$, $\frac{3x^2-11x+5}{x^3-3x^2+2x}$.

An **improper rational function** $\frac{P(x)}{Q(x)}$, is one in which the degree of the numerator is greater than or equal to the degree of the denominator e.g. $\frac{x^2-x+6}{x^2+x-2}$, $\frac{x^4+1}{x^2+2x}$.

Improper rational fractions must be simplified first by algebraic long division before proceeding with partial fractions.

Proper Rational Functions

Distinct linear factors in denominator

Express in partial fractions:

① $\frac{x+7}{(x-2)(x+1)}$

Let $\frac{x+7}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$

Multiply through by $(x-2)(x+1)$

$x+7=A(x+1)+B(x-2)$

Pick values of x to find A and B.

② $\frac{2x}{(x-1)(x+1)}$

Let $\frac{2x}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$

Multiply through by $(x-1)(x+1)$

$2x=A(x+1)+B(x-1)$

Pick values of x to find A and B.

Let $x=2$ Let $x=-1$

$9=3A$ $6=-3B$

$A=3$ $B=-2$

$$\therefore \frac{x+7}{(x-2)(x+1)}=\frac{3}{x-2}-\frac{2}{x+1}$$

③ $\dfrac{4-6x}{x^2+4x}$

Factorise the denominator first.

$$\frac{4-6x}{x(x+4)}$$

Let $\dfrac{4-6x}{x(x+4)}=\dfrac{A}{x}+\dfrac{B}{x+4}$

Multiply through by $x(x+4)$

$4-6x=A(x+4)+Bx$

Let $x=0$ Let $x=-4$

$4=4A$ $28=-4B$

$A=1$ $B=-7$

$$\therefore \frac{4-6x}{x(x+4)}=\frac{1}{x}-\frac{7}{x+4}$$

⑤ $\dfrac{6x-1}{4x^2-1}$

Factorise the denominator first.

$$\frac{6x-1}{(2x-1)(2x+1)}$$

Let $\dfrac{6x-1}{(2x-1)(2x+1)}=\dfrac{A}{2x-1}+\dfrac{B}{2x+1}$

Multiply through by $(2x-1)(2x+1)$

$6x-1=A(2x+1)+B(2x-1)$

Let $x=\dfrac{1}{2}$ Let $x=-\dfrac{1}{2}$

$2=2A$ $-4=-2B$

$A=1$ $B=2$

$$\therefore \frac{6x-1}{4x^2-1}=\frac{1}{2x-1}+\frac{2}{2x+1}$$

Let $x=1$ Let $x=-1$

$2=2A$ $-2=-2B$

$A=1$ $B=1$

$$\therefore \frac{2x}{(x-1)(x+1)}=\frac{1}{x-1}+\frac{1}{x+1}$$

④ $\dfrac{4x-10}{x^2-3x}$

Factorise the denominator first.

$$\frac{4x-10}{x(x-3)}$$

Let $\dfrac{4x-10}{x(x-3)}=\dfrac{A}{x}+\dfrac{B}{x-3}$

Multiply through by $x(x-3)$

$4x-10=A(x-3)+Bx$

Let $x=0$ Let $x=3$

$-10=3A$ $2=3B$

$A=\dfrac{10}{3}$ $B=\dfrac{2}{3}$

$$\therefore \frac{4x-10}{x(x-3)}=\frac{10/3}{x}+\frac{2/3}{x-3}$$

$$\frac{4x-10}{x(x-3)}=\frac{10}{3x}+\frac{2}{3(x-3)}$$

⑥ $\dfrac{6x^2-10x+2}{x^3-3x^2+2x}$

Factorise the denominator first.

$$\frac{6x^2-10x+2}{x(x-2)(x-1)}$$

Let $\dfrac{6x^2-10x+2}{x(x-2)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{x-1}$

Multiply through by $x(x-2)(x-1)$

$$6x^2-10x+2=A(x-2)(x-1)+Bx(x-1)+Cx(x-2)$$

Let $x=0$	Let $x=2$	Let $x=1$
$2=2A$	$6=2B$	$-2=-C$
$A=1$	$B=3$	$C=2$

$$\therefore \frac{6x^2-10x+2}{x(x-2)(x-1)}=\frac{1}{x}+\frac{3}{x-2}+\frac{2}{x-1}$$

Questions

Express in partial fractions:

(a) $\dfrac{10x}{(x-2)(x+3)}$ (b) $\dfrac{7x+5}{(x-1)(x+3)}$ (c) $\dfrac{-x-7}{(x-3)(x+1)}$ (d) $\dfrac{4x+1}{2x^2+3x+1}$ (e) $\dfrac{6+3x-x^2}{x^3-4x^2+3x}$

Repeated linear factor in denominator

When we have a repeated linear factor in the denominator we use a different form of RHS to allow us to achieve partial fractions.

Express in partial fractions:

① $\dfrac{x+3}{(x-2)^2}$

Let $\dfrac{x+3}{(x-2)^2}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}$

Multiply through by $(x-2)^2$.

$$x+3=A(x-2)+B$$

Let $x=2$	Let $x=3$
$B=5$	$6=A+B$
	$A=1$

$$\therefore \frac{x+3}{(x-2)^2}=\frac{1}{x-2}+\frac{5}{(x-2)^2}$$

② $\dfrac{3x^2-11x+5}{(x-2)(x-1)^2}$

Let $\dfrac{3x^2-11x+5}{(x-2)(x-1)^2}=\dfrac{A}{x-2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$

Multiply through by $(x-2)(x-1)^2$

$3x^2-11x+5=A(x-1)^2+B(x-2)(x-1)+C(x-2)$

Let $x=2$ Let $x=1$ Let $x=3$

$12-22+5=A$ $3-11+5=-C$ $27-33+5=4A+2B+C$

$A=-5$ $C=3$ $-1=-20+2B+3$

$B=8$

$\therefore \dfrac{3x^2-11x+5}{(x-2)(x-1)^2}=\dfrac{8}{x-1}-\dfrac{5}{x-2}+\dfrac{3}{(x-1)^2}$

③ $\dfrac{7x^2+1}{x^3-x^2}$

Factorise the denominator first.

$\dfrac{7x^2+1}{x^2(x-1)}$

Let $\dfrac{7x^2+1}{x^2(x-1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{(x-1)}$

Multiply through by $x^2(x-1)$.

$7x^2+1=Ax(x-1)+B(x-1)+Cx^2$

Let $x=0$ Let $x=1$ Let $x=2$

$1=-B$ $C=8$ $29=2A+B+4C$

$B = -1$ $29=2A-1+32$

$A=-1$

$\therefore \dfrac{7x^2+1}{x^2(x-1)}=\dfrac{8}{x-1}-\dfrac{1}{x}-\dfrac{1}{x^2}$

④ $\dfrac{16}{(x^2-2x-3)(x-3)}$

Factorise the denominator first.

$\dfrac{16}{(x-3)(x+1)(x-3)}=\dfrac{16}{(x+1)(x-3)^2}$

Let $\dfrac{16}{(x+1)(x-3)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$

Multiply through by $(x+1)(x-3)^2$.

$$16 = A(x-3)^2 + B(x+1)(x-3) + C(x+1)$$

Let $x=-1$

$$16 = 16A$$
$$A = 1$$

Let $x=3$

$$16 = 4C$$
$$C = 4$$

Let $x=4$

$$16 = A + 5B + 5C$$
$$16 = 1 + 5B + 20$$
$$B = -1$$

$$\therefore \frac{16}{(x^2-2x-3)(x-3)} = \frac{1}{x+1} - \frac{1}{x-3} + \frac{4}{(x-3)^2}$$

Questions

Express in partial fractions:

(a) $\frac{6x^2+x-7}{(x+2)^2(x-3)}$ (b) $\frac{4x^2-8x+15}{(x-2)^2(x+1)}$ (c) $\frac{x^2-x-1}{x^2(x-1)}$

(d) $\frac{1}{x^2-3x^3}$ (e) $\frac{x^2-9x-2}{(x-1)(x^2-1)}$ (f) $\frac{2x^2+7x+3}{x^3+2x^2+x}$

Irreducible quadratic factor in denominator

If the denominator contains a quadratic which cannot be factorised (i.e. is irreducible), we have another form of partial fractions to use.

Express in partial fractions:

① $\frac{3}{(x+1)(x^2-x+1)}$

Let $\frac{3}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

Multiply through by $(x+1)(x^2-x+1)$

$$3 = A(x^2-x+1) + (Bx+C)(x+1)$$

Let $x=-1$

$$3 = 3A$$
$$A = 1$$

Let $x=0$

$$3 = A + C$$
$$C = 2$$

Let $x=1$

$$3 = A + 2B + 2C$$
$$3 = 1 + 2B + 4$$
$$B = -1$$

$$\therefore \frac{3}{(x+1)(x^2-x+1)} = \frac{1}{x+1} + \frac{2-x}{x^2-x+1}$$

② $\dfrac{3x^2-4}{x\left(x^2+1\right)}$

Let $\dfrac{3x^2-4}{x\left(x^2+1\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$

Multiply through by $x\left(x^2+1\right)$

$3x^2-4=A\left(x^2+1\right)+\left(Bx+C\right)x$

Let $x=0$	Equate x^2 terms	Let $x=1$
$A=-4$	$3=A+B$	$-1=2A+B+C$
	$B=7$	$-1=-8+7+C$
		$C=0$

$\therefore \dfrac{3x^2-4}{x\left(x^2+1\right)}=\dfrac{-4}{x}+\dfrac{7x}{x^2+1}$

<u>Questions</u>

Express in partial fractions:

(a) $\dfrac{16x^2}{(x-3)\left(2x^2-x+1\right)}$ (b) $\dfrac{2x^2+3x+1}{(x-1)\left(x^2+x+1\right)}$ (c) $\dfrac{3x^2+2x+1}{\left(x^2+2x+2\right)(x+1)}$

(d) $\dfrac{x^2+3x+3}{(2x+1)\left(x^2+x+2\right)}$ (e) $\dfrac{4x^2-4x+1}{\left(x^2-x+1\right)(x-2)}$ (f) $\dfrac{3x-2}{x^3+2x}$

<u>Improper Rational Functions</u>

If we have an improper rational function (i.e. if in $\dfrac{P(x)}{Q(x)}$, $P(x)$ has a degree $\geq Q(x)$) we must use algebraic long division first to express the function as a constant or linear function + a proper rational function before finding partial fractions.

We will practise some algebraic long division first.

Examples

① Express $\dfrac{x^2-9x-10}{x+1}$ as a proper rational function.

$$\begin{array}{r|l} & x-10 \\ x+1 & x^2-9x-10 \\ & \underline{x^2+x} \\ & -10x-10 \\ & \underline{-10x-10} \\ & 0 \end{array}$$

So $\dfrac{x^2-9x-10}{x+1}=x-10$

This result could also have been achieved by factorising the numerator then simplifying.

② Express $\dfrac{x^3+3x^2+4x-5}{x^2+1}$ as a proper rational function.

$$\begin{array}{r|l} & x+3 \\ x^2+1 & x^3+3x^2+4x-5 \\ & \underline{x^3 \quad\quad +x} \\ & 3x^2+3x-5 \\ & \underline{3x^2 \quad\quad +3} \\ & 3x-8 \end{array}$$

So $\dfrac{x^3+3x^2+4x-5}{x^2+1}=x+3+\dfrac{3x-8}{x^2+1}$

③ Simplify $\dfrac{x^2+3}{x^2-4}$ by algebraic long division.

$$\begin{array}{r|l} & 1 \\ x^2-4 & x^2+3 \\ & \underline{x^2-4} \\ & 7 \end{array}$$

So $\dfrac{x^2+3}{x^2-4}=1+\dfrac{7}{x^2-4}$

④ Simplify $\dfrac{x^3+1}{x+2}$ by algebraic long division.

$$
\begin{array}{r}
x^2-2x+4 \\
x+2\,\big)\overline{x^3 \qquad\qquad +1} \\
\underline{x^3+2x^2} \qquad\quad \\
-2x^2 \qquad +1 \\
\underline{-2x^2-4x} \quad \\
4x+1 \\
\underline{4x+8} \\
-7
\end{array}
$$

So $\dfrac{x^3+1}{x+2}=x^2-2x+4-\dfrac{7}{x+2}$

<u>Now applying with partial fractions</u>:

Express each of these functions as a polynomial function and partial fractions.

① $\dfrac{x^2-x+6}{x^2+x-2}$ First perform algebraic long division.

$$
\begin{array}{r}
1 \\
x^2+x-2\,\big)\overline{x^2-x+6} \\
\underline{x^2+x-2} \\
-2x+8
\end{array}
$$

So $\dfrac{x^2-x+6}{x^2+x-2}=1+\dfrac{8-2x}{x^2+x-2}$

Now express $\dfrac{8-2x}{x^2+x-2}$ in partial fractions.

$$\frac{8-2x}{x^2+x-2}=\frac{8-2x}{(x+2)(x-1)}$$

Let $\dfrac{8-2x}{(x+2)(x-1)}=\dfrac{A}{x+2}+\dfrac{B}{x-1}$

Multiply through by $(x+2)(x-1)$.

$8-2x=A(x-1)+B(x+2)$

Let $x=-2$ Let $x=1$

$12=-3A$ $6=3B$

$A=-4$ $B=2$

$\therefore \dfrac{x^2-x+6}{x^2+x-2}=1+\dfrac{4}{x+2}+\dfrac{2}{x-1}$

② $\dfrac{x^3-5x^2+11x-12}{x^2-5x+6}$ First perform algebraic long division.

$$\begin{array}{r} x \\ x^2-5x+6\overline{)\,x^3-5x^2+11x-12} \\ \underline{x^3-5x^2+6x} \\ 5x-12 \end{array}$$

So $\dfrac{x^3-5x^2+11x-12}{x^2-5x+6}=x+\dfrac{5x-12}{x^2-5x+6}$ Now express $\dfrac{5x-12}{x^2-5x+6}$ in partial fractions.

$$\frac{5x-12}{x^2-5x+6}=\frac{5x-12}{(x-3)(x-2)}=\frac{A}{x-3}+\frac{B}{x-2}$$

Multiply through by $(x-3)(x-2)$.

$$5x-12=A(x-2)+B(x-3)$$

Let $x=3$ Let $x=2$

$A=3$ $-2=-B$

$B=2$

$$\therefore \frac{x^3-5x^2+11x-12}{x^2-5x+6}=x+\frac{3}{x-3}+\frac{2}{x-2}$$

③ $\dfrac{x^4+1}{x^3+2x}$ First perform algebraic long division.

$$\begin{array}{r} x \\ x^3+2x\overline{)\,x^4 +1} \\ \underline{x^4+2x^2} \\ -2x^2 +1 \end{array}$$

So $\dfrac{x^4+1}{x^3+2x}=x+\dfrac{1-2x^2}{x^3+2x}$ Now express $\dfrac{1-2x^2}{x^3+2x}$ in partial fractions.

$$\frac{1-2x^2}{x^3+2x}=\frac{1-2x^2}{x(x^2+2)}=\frac{A}{x}+\frac{Bx+C}{x^2+2}$$

Multiply through by $x(x^2+2)$

$$1-2x^2=A(x^2+2)+(Bx+C)x$$

Let $x=0$ Equate x^2 terms Let $x=1$

$1=2A$ $-2=A+B$ $-1=3A+B+C$

$A=\dfrac{1}{2}$ $B=\dfrac{-5}{2}$ $C=0$

$$\therefore \frac{x^4+1}{x^3+2x}=x+\frac{1}{2x}-\frac{5x}{2(x^2+2)}$$

④ $\dfrac{(x+1)(x+3)(x-2)}{(x-1)(x-3)}$ First multiply out the brackets as it cannot be simplified as it is.

$$\frac{(x+1)(x+3)(x-2)}{(x-1)(x-3)}=\frac{(x^2+4x+3)(x-2)}{(x^2-4x+3)}=\frac{x^3+2x^2-5x-6}{x^2-4x+3}$$

Now perform algebraic long division.

$$\begin{array}{r|l} & \quad x \;+\; 6 \\ x^2-4x+3 & x^3+2x^2-5x-6 \\ & \underline{x^3-4x^2+3x} \\ & \quad 6x^2-8x-6 \\ & \quad \underline{6x^2-24x+18} \\ & \qquad 16x-24 \end{array}$$

So $\dfrac{(x+1)(x+3)(x-2)}{(x-1)(x-3)}=x+6+\dfrac{16x-24}{(x-1)(x-3)}$

Now express $\dfrac{16x-24}{(x-1)(x-3)}$ partial fractions.

$$\frac{16x-24}{(x-1)(x-3)}=\frac{A}{x-1}+\frac{B}{x-3}$$

Multiply through by $(x-1)(x-3)$.

$16x-24=A(x-3)+B(x-1)$

Let $x=1$ Let $x=3$

$-8=-2A$ $24=2B$

$A=4$ $B=12$

$$\therefore \frac{(x+1)(x+3)(x-2)}{(x-1)(x-3)}=x+6+\frac{4}{x-1}+\frac{12}{x-3}$$

Questions

Express each of these functions as a polynomial function and partial fractions.

(a) $\dfrac{x^3-x^2-5x-7}{x^2-2x-3}$ (b) $\dfrac{2x^2-7}{x^2-4}$ (c) $\dfrac{x^3+2}{x(x^2+3)}$ (d) $\dfrac{(x+3)(x+1)(x-4)}{(x+2)(x-1)}$

Past Paper Questions

2001

A5 (a): Obtain partial fractions for $\frac{x}{x^2-1}$, $x>1$. **(2 marks)**

2002

Ist part of A8: Express $\frac{x^2}{(x+1)^2}$ in the form $A+\frac{B}{x+1}+\frac{C}{(x+1)^2}$, $(x\neq -1)$ stating the values of the constants A, B and C. **(3 marks)**

2004

Ist part of Q5: Express $\frac{1}{x^2-x-6}$ in partial fractions. **(2 marks)**

2005

Ist part of Q13: Express $\frac{1}{x^3+x}$ in partial fractions. **(4 marks)**

2007

Ist part of Q4: Express $\frac{2x^2-9x-6}{x(x^2-x-6)}$ in partial fractions. **(3 marks)**

2008

Ist part of Q4: Express $\frac{12x^2+20}{x(x^2+5)}$ in partial fractions. **(3 marks)**

2009

Ist part of Q14: Express $\frac{x^2+6x-4}{(x+2)^2(x-4)}$ in partial fractions. **(4 marks)**

2011

Ist part of Q1: Express $\frac{13-x}{x^2+4x-5}$ in partial fractions. **(3 marks)**

2012

Q15(a): Express $\frac{1}{(x-1)(x+2)^2}$ in partial fractions. **(4 marks)**

2014

Ist part of Q14(b): Express $\frac{1}{3r^2-5r+2}$ in partial fractions. **(2 marks)**