The Binomial Theorem

Factorials

The calculations $5\times4\times3\times2\times1$, $3\times2\times1$, $6\times5\times4\times3\times2\times1$ etc. often appear in mathematics. They are called factorials and have been given the notation $n!$.

e.g. $6! = 6\times5\times4\times3\times2\times1$

$5! = 5\times4\times3\times2\times1$

$4! = 4\times3\times2\times1$

$3! = 3\times2\times1$

$2! = 2\times1$

$1! = 1$

We also define $0! = 1$

Combinatorics- Permutations and Combinations

Suppose you are asked to pick 3 different numbers between 1 and 5.

There are 10 ways of doing this:

$1,\ 2,\ 3$ $\quad 2,\ 3,\ 4$ $\quad 3,\ 4,\ 5$

$1,\ 2,\ 4$ $\quad 2,\ 3,\ 5$

$1,\ 2,\ 5$ $\quad 2,\ 4,\ 5$

$1,\ 3,\ 4$

$1,\ 3,\ 5$

$1,\ 4,\ 5$

The order in which we pick the numbers is not important e.g. $3,\ 4,\ 5$ is the same as $4,\ 3,\ 5$.

This is called a **combination.**

It is a selection without arrangement.

Combinations use the notation ${}^nC_r$ or $\binom{n}{r}$, where you are selecting r components from a total of n.

Formula

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

In the above example we are selecting 3 things from 5. This is ${}^{5}C_{3}$ or $\begin{pmatrix}5\\3\end{pmatrix}$.

$${}^{5}C_{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{120}{6\times 2} = 10$$

Learn how to calculate $n!$ and ${}^{n}C_{r}$ on you calculator.

If the order (arrangement) of the numbers is important this is a different calculation.

Suppose we are selecting 3 different numbers from 5 where the order does matter.
This time there are going to be more possibilities.

$1,2,3 \quad 1,2,4 \quad 1,2,5 \quad 1,3,2 \quad 1,3,4 \quad 1,3,5 \quad 1,4,2 \quad 1,4,3 \quad 1,4,5 \quad 1,5,2 \quad 1,5,3 \quad 1,5,4$

$2,1,3 \quad 2,1,4 \quad 2,1,5 \quad 2,3,4 \quad 2,3,5 \quad 2,4,5$……. and so on.

There are 60 possibilities altogether.

Think of it like this:-

For the first number there are 5 choices $-1,2,3,4$ or 5.

For the second number there are 4 choices as you have used one number already.

For the third number there are 3 choices as you have used 2 numbers already.

So in total you have $5\times 4\times 3 = 60$ possibilities.

$5\times 4\times 3$ is the same as doing $\dfrac{5\times 4\times 3\times 2\times 1}{2\times 1} = \dfrac{5!}{2!}$

This is known as a **permutation** when arrangement is important. It is denoted ${}^{n}P_{r}$.

Formula

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

In the above example ${}^{5}P_{3} = \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!} = \dfrac{120}{2} = 60$.

Permutations are not part of the Advanced Higher course but have been mentioned here to form a complete picture and for those who will study further mathematics.

We will concentrate on ${}^{n}C_{r}$.

Example

5 people have to be selected from 8 to form a committee.
How many ways are there to do this?

$$8 \quad \times \quad 7 \quad \times \quad 6 \quad \times \quad 5 \quad \times \quad 4 \qquad = 6720$$

This is the same as calculating $\dfrac{8\times7\times6\times5\times4\times3\times2\times1}{3\times2\times1}=\dfrac{8!}{3!}$.

But this includes all the possible arrangements. Arrangements don't matter here so we need to divide by $5\times4\times3\times2\times1=5!$ as this is the number of ways 5 things can be arranged.

So we have $\dfrac{8!}{5!3!}$.

It is easier to use our formula : $\quad {}^{8}C_{5}=\begin{pmatrix}8\\5\end{pmatrix}=\dfrac{8!}{5!(8-5)!}=\dfrac{8!}{5!3!}=\dfrac{40320}{120\times6}=56$

Example

How many ways can I place 2 discs into 5 empty boxes?

$${}^{5}C_{2}=\begin{pmatrix}5\\2\end{pmatrix}=\frac{5!}{2!(5-2)!}=\frac{5!}{2!3!}=\frac{120}{2\times6}=10$$

NB This is also the same as placing 3 empty boxes in 5.

$${}^{5}C_{3}=\begin{pmatrix}5\\3\end{pmatrix}=\frac{5!}{3!(5-3)!}=\frac{5!}{3!2!}=\frac{120}{6\times2}=10$$

This implies that ${}^{5}C_{2}={}^{5}C_{3}$ or $\begin{pmatrix}5\\2\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$.

Example

How many different ways can you place 3 swimmers in 8 lanes?

$${}^{8}C_{3}=\begin{pmatrix}8\\3\end{pmatrix}=\frac{8!}{3!(8-3)!}=\frac{8!}{3!5!}=\frac{40320}{6\times120}=56$$

This is the same as placing 5 gaps in 8 lanes.

$$^{8}C_{5}=\binom{8}{5}=\frac{8!}{5!(8-5)!}=\frac{8!}{5!3!}=\frac{40320}{120\times 6}=56$$

So $^{8}C_{3}={}^{8}C_{5}$ or $\binom{8}{3}=\binom{8}{5}$.

The general result is $\binom{n}{r}=\binom{n}{n-r}$

Proof

$$\binom{n}{n-r}=\frac{n!}{(n-r)!(n-(n-r))!}=\frac{n!}{(n-r)!r!}=\frac{n!}{r!(n-r)!}=\binom{n}{r}$$

Pascals Triangle

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array}$$

Notice that the results from combinations occur in Pascal's triangle.

$$\begin{array}{ccccccccc} &&&&\binom{0}{0}&&&& \\ &&&\binom{1}{0}&&\binom{1}{1}&&& \\ &&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&& \\ &\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}& \\ \binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4} \end{array}$$ etc.

From the triangle we can see another result:

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

Proof

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{n!r}{r!(n-r+1)!} + \frac{n!(n-r+1)}{r!(n-r+1)!}$$ common denominator of $r!(n-r+1)!$

$$= \frac{n!r + n!(n-r+1)}{r!(n-r+1)!}$$

$$= \frac{n!(r+(n-r+1))}{r!(n-r+1)!}$$

$$= \frac{n!(n+1)}{r!((n+1)-r)!}$$

$$= \frac{(n+1)!}{r!((n+1)-r)!}$$ since $(n+1)n! = (n+1)!$

$$= \binom{n+1}{r}$$ as required.

Question 5 from the 2010 paper required a working of this proof:-

Show that $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$

$$\binom{n+1}{3} - \binom{n}{3} = \frac{(n+1)!}{3!((n+1)-3)!} - \frac{n!}{3!(n-3)!}$$

$$= \frac{(n+1)!}{3!(n-2)!} - \frac{n!}{3!(n-3)!}$$

$$=\frac{(n+1)!}{3!(n-2)!}-\frac{n!(n-2)}{3!(n-2)!} \quad \text{common denominator of } 3!(n-2)!$$

$$=\frac{(n+1)!-n!(n-2)}{3!(n-2)!}$$

$$=\frac{(n+1)n!-n!(n-2)}{3!(n-2)!} \quad \text{since } (n+1)! \; = \; (n+1)n!$$

$$=\frac{n!((n+1)-(n-2))}{3!(n-2)!}$$

$$=\frac{3n!}{3!(n-2)!}$$

$$=\frac{n!}{2!(n-2)!}$$

$$=\binom{n}{2} \quad \text{as required.}$$

<u>Questions</u>

① Given that $\binom{10}{2}=45$, $\binom{10}{7}=120$ and $\binom{10}{4}=210$ write down the value of

(a) $\binom{10}{6}$ (b) $\binom{10}{8}$ (c) $\binom{10}{3}$

② Find (a) $\binom{8}{4}$ (b) $\binom{3}{1}$ (c) $\binom{6}{5}$ (d) $\binom{12}{4}$

③ Find another ${}^nC_r$ equivalent to (a) $\binom{6}{1}$ (b) $\binom{5}{3}$ (c) $\binom{9}{5}$ (d) $\binom{50}{4}$

④ Write down in $\binom{n}{r}$ form

(a) $\binom{8}{5}+\binom{8}{6}$ (b) $\binom{4}{2}+\binom{4}{3}$ (c) $\binom{12}{3}+\binom{12}{4}$ (d) $\binom{18}{8}+\binom{18}{9}$

Equations

① Suppose we know that $\binom{n}{2}=15$. Can we solve this for n?

$$\binom{n}{2}=15$$
$$\frac{n!}{2!(n-2)!}=15$$
$$\frac{n(n-1)(n-2)!}{2!(n-2)!}=15$$
$$\frac{n(n-1)}{2}=15$$
$$n^2-n=30$$
$$n^2-n-30=0$$
$$(n-6)(n+5)=0$$
$n=6$ or $n=-5$

$n=6, \quad n\in\mathbb{Z}^+$

② $\binom{n}{2}=66.$ What is the value of n?

$$\binom{n}{2}=66$$
$$\frac{n!}{2!(n-2)!}=66$$
$$\frac{n(n-1)(n-2)!}{2!(n-2)!}=66$$
$$\frac{n(n-1)}{2}=66$$
$$n^2-n=132$$
$$n^2-n-132=0$$
$$(n-12)(n+11)=0$$
$n=12$ or $n=-11$

$n=12, \quad n\in\mathbb{Z}^+$

③ Solve for n.

$$\binom{n}{1}+\binom{n}{2}=28$$
$$\binom{n+1}{2}=28 \text{ using } \binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$$
$$\frac{(n+1)!}{2!((n+1)-2)!}=28$$
$$\frac{(n+1)n(n-1)!}{2!(n-1)!}=28$$
$$n^2+n=56$$
$$n^2+n-56=0$$
$$(n-7)(n+8)=0$$
$$n=7 \text{ or } n=-8$$
$$n=7, \quad n\in\mathbb{Z}^+$$

Questions

① Find the value of $n, \quad n\in Z^+$. (a) $\binom{n}{2}=10$ (b) $\binom{n}{2}=36$ (c) $\binom{n}{2}=120$

② Solve (a) $\binom{n}{1}+\binom{n}{2}=21$ (b) $\binom{n+1}{1}+\binom{n+1}{2}=66$ (c) $\binom{n}{1}+\binom{n}{2}=190$

The Binomial Theorem

The Binomial Theorem helps us to multiply out brackets which we would otherwise have to complete longhand.

$$(x+y)^2=x^2+2xy+y^2$$
$$(x+y)^3=(x+y)(x^2+2xy+y^2)=x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4=(x+y)(x^3+3x^2y+3xy^2+y^3)=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

Look at the coefficients and compare with Pascal's Triangle.

1
1 1
1 2 1 → coefficients of $(x+y)^2$
1 3 3 1 → coefficients of $(x+y)^3$
1 4 6 4 1 → coefficients of $(x+y)^4$

So the coefficients are the same as $\binom{n}{r}$ or ${}^{n}C_{r}$ and are known as the **binomial coefficients**.

So $(x+y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}x^0y^4$

$(x+y)^5 = \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5$

In general

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots\ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}x^0y^n$$

for $x, y \in \mathbb{R}, n \in \mathbb{N}$

This is known as **<u>The Binomial Theorem.</u>**

It can also be written as $(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$ for $r, n \in \mathbb{N}$.

The general term of the expansion is given by $\binom{n}{r}x^{n-r}y^r$.

You may choose to use Pascal's triangle or $\binom{n}{r}$ to find the coefficients … it's up to you. At AH level, Pascal's triangle is usually sufficient.

<u>Examples</u>

① $(x+y)^5 = \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5$

$= x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + y^5$

② $(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

Be careful when there are coefficients within the bracket!

③ $(2x+y)^4 = (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4$

$= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

④ $(x+3y)^3 = x^3 + 3x^2(3y) + 3x(3y)^2 + (3y)^3$

$= x^3 + 9x^2y + 27xy^2 + 27y^3$

⑤ $(4a+3b)^5 = (4a)^5 + 5(4a)^4(3b) + 10(4a)^3(3b)^2 + 10(4a)^2(3b)^3 + 5(4a)(3b)^4 + (3b)^5$

$= 1024a^5 + 3840a^4b + 5760a^3b^2 + 4320a^2b^3 + 1620ab^4 + 243b^5$

Questions

Expand using the binomial theorem:

(a) $(x+y)^7$ (b) $(a+b)^6$ (c) $(x+4y)^6$ (d) $(3a+2b)^5$ (e) $(3x+5y)^4$

Examples involving negatives and fractions

Extra care must be taken here!

① $(x-y)^4 = x^4 + 4x^3(-y) + 6x^2(-y)^2 + 4x(-y)^3 + (-y)^4$

$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

② $(2x-y)^3 = (2x)^3 + 3(2x)^2(-y) + 3(2x)(-y)^2 + (-y)^3$

$= 8x^3 - 12x^2y + 6xy^2 - y^3$

③ $(3x-2y)^6$

$= (3x)^6 + 6(3x)^5(-2y) + 15(3x)^4(-2y)^2 + 20(3x)^3(-2y)^3 + 15(3x)^2(-2y)^4 + 6(3x)(-2y)^5 + (-2y)^6$

$= 729x^6 - 2916x^5y + 4860x^4y^2 - 4320x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6$

④ $\left(x+\frac{1}{y}\right)^4 = x^4 + 4x^3\left(\frac{1}{y}\right) + 6x^2\left(\frac{1}{y}\right)^2 + 4x\left(\frac{1}{y}\right)^3 + \left(\frac{1}{y}\right)^4$

$= x^4 + \frac{4x^3}{y} + \frac{6x^2}{y^2} + \frac{4x}{y^3} + \frac{1}{y^4}$

⑤ $\left(x-\dfrac{1}{2y}\right)^5 = x^5+5x^4\left(\dfrac{-1}{2y}\right)+10x^3\left(\dfrac{-1}{2y}\right)^2+10x^2\left(\dfrac{-1}{2y}\right)^3+5x\left(\dfrac{-1}{2y}\right)^4+\left(\dfrac{-1}{2y}\right)^5$

$$= x^5-\frac{5x^4}{2y}+\frac{10x^3}{4y^2}-\frac{10x^2}{8y^3}+\frac{5x}{16y^4}-\frac{1}{32y^5}$$

$$= x^5-\frac{5x^4}{2y}+\frac{5x^3}{2y^2}-\frac{5x^2}{4y^3}+\frac{5x}{16y^4}-\frac{1}{32y^5}$$

⑥ $\left(2x+\dfrac{3}{y}\right)^6$

$$=(2x)^6+6(2x)^5\left(\frac{3}{y}\right)+15(2x)^4\left(\frac{3}{y}\right)^2+20(2x)^3\left(\frac{3}{y}\right)^3+15(2x)^2\left(\frac{3}{y}\right)^4+6(2x)\left(\frac{3}{y}\right)^5+\left(\frac{3}{y}\right)^6$$

$$=64x^6+\frac{576x^5}{y}+\frac{2160x^4}{y^2}+\frac{4320x^3}{y^3}+\frac{4860x^2}{y^4}+\frac{2916x}{y^5}+\frac{729}{y^6}$$

⑦ $\left(3x-\dfrac{2}{x}\right)^5$

$$=(3x)^5+5(3x)^4\left(\frac{-2}{x}\right)+10(3x)^3\left(\frac{-2}{x}\right)^2+10(3x)^2\left(\frac{-2}{x}\right)^3+5(3x)\left(\frac{-2}{x}\right)^4+\left(\frac{-2}{x}\right)^5$$

$$=243x^5-\frac{810x^4}{x}+\frac{1080x^3}{x^2}-\frac{720x^2}{x^3}+\frac{240x}{x^4}-\frac{32}{x^5}$$

$$=243x^5-810x^3+1080x-\frac{720}{x}+\frac{240}{x^3}-\frac{32}{x^5}$$

⑧ $\left(x^2+3y\right)^4=\left(x^2\right)^4+4\left(x^2\right)^3(3y)+6\left(x^2\right)^2(3y)^2+4\left(x^2\right)(3y)^3+(3y)^4$

$$=x^8+12x^6y+54x^4y^2+108x^2y^3+81y^4$$

Questions

Expand using the binomial theorem.

(a) $(x-3y)^4$ (b) $(6x-2y)^5$ (c) $(y-4x)^5$ (d) $(3a-b)^6$

(e) $\left(a+\dfrac{1}{b}\right)^6$ (f) $\left(3a+\dfrac{1}{4b}\right)^4$ (g) $\left(x-\dfrac{2}{y}\right)^5$ (h) $\left(2x-\dfrac{3}{x}\right)^6$

Finding a Particular Term

You may be asked to find a particular term in an expansion or obtain its coefficient. This can be done by completing a whole expansion and picking out the required term but this can be time consuming and arithmetical errors are more likely to occur.

It helps if you remember the general formula $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$ for $r, n \in \mathbb{N}$.

Examples

① Find the coefficient of the x^3y^4 term in the expansion of $(x+y)^7$.

For $x^3y^4, n=7,\ r=4,\ n-r=3$.

The term is $\binom{7}{4} x^3y^4 = 35x^3y^4$

$\therefore$ coefficient $=35$

② Find the coefficient of the x^4y^2 term in the expansion of $(2x-y)^6$.

For $x^4y^2,\ n=6,\ r=2,\ n-r=4$.

The term is $\binom{6}{2}(2x)^4(-y)^2$

$=15(16x)^4 y^2$

$=240x^4y^2$

$\therefore$ coefficient $=240$

③ Find the term independent of x in the expansion of $\left(x-\frac{2}{x}\right)^{10}$.

Term independent of x requires $\frac{x^5}{x^5}$.

$n=10,\ r=5,\ n-r=5$

The term is $\binom{10}{5} x^5 \left(\frac{-2}{x}\right)^5$

$=252x^5\left(\frac{-32}{x^5}\right)$

$=-8064$

④ Find the x^2 term in the expansion of $\left(2x+\frac{4}{x}\right)^6$.

x^2 term requires $\frac{x^4}{x^2}$.

$n=6,\ r=2,\ n-r=4$

The term is $\binom{6}{2}(2x)^4\left(\frac{4}{x}\right)^2$

$=15\left(16x^4\right)\left(\frac{16}{x^2}\right)$

$=3840x^2$

Questions

① Find the coefficient of the x^2y^4 term in the expansion of $(x+2y)^6$.

② Find the coefficient of the x^9 term in the expansion of $\left(1+3x^3\right)^4$.

③ Find the y^{-1} term in the expansion of $\left(y-\frac{1}{y}\right)^5$.

④ Find the term independent of y in the expansion of $\left(3y+\frac{2}{y}\right)^8$.

⑤ Find the term independent of a in the expansion of $\left(a^2-\frac{2}{a}\right)^9$.

Writing down the General Term in an Expansion

Remember the general term of the expansion of $(x+y)^n$ is given by $\binom{n}{r}x^{n-r}y^r$.

Examples

① Write down and simplify the general term in the expansion of $\left(x^2+\frac{1}{x}\right)^{10}$.

Hence or otherwise obtain the term in x^{14}.

The rth term is given by $\binom{10}{r}\left(x^2\right)^{10-r}\left(\frac{1}{x}\right)^r$

$=\binom{10}{r}x^{20-2r}x^{-r}$

$=\binom{10}{r}x^{20-3r}$

$20-3r=14\Rightarrow r=2$ so term is $\binom{10}{2}x^{20-3(2)}=45x^{14}$.

② Write down and simplify the general term in the expansion of $\left(2x-\frac{1}{x^2}\right)^9$.

Hence or otherwise obtain the term independent of x.

The rth term is given by $\binom{9}{r}(2x)^{9-r}\left(\frac{-1}{x^2}\right)^r$

$$=\binom{9}{r}(2x)^{9-r}\left(-x^{-2}\right)^r$$
$$=\binom{9}{r}2^{9-r}x^{9-r}(-1)^r x^{-2r}$$
$$=\binom{9}{r}(-1)^r 2^{9-r}x^{9-3r}$$

$9-3r=0 \Rightarrow r=3$ so term is $=\binom{9}{3}(-1)^3 2^6=-2^6\binom{9}{3}=-5376$.

Questions

① Write down and simplify the general term in the expansion of $\left(x^2-\frac{2}{x}\right)^8$.

Hence or otherwise obtain the term in x^{10}.

② Write down and simplify the general term in the expansion of $\left(3x+\frac{1}{x^2}\right)^{12}$.

Hence or otherwise obtain the term independent of x.

Applications of the Binomial Theorem

We can use the binomial theorem to tackle other types of problems.

① Using the binomial theorem find $1\cdot02^4$.

$$1\cdot02^4=(1+0\cdot02)^4$$
$$=1^4+4(1)^3(0\cdot02)+6(1)^2(0\cdot02)^2+4(1)(0\cdot02)^3+(0\cdot02)^4$$
$$=1+0\cdot08+0\cdot0024+0\cdot000032+0\cdot00000016$$
$$=1\cdot08243216$$

② Using the binomial theorem find $0\cdot6^3$.

$$0\cdot6^3=(1-0\cdot4)^3$$
$$=1^3+3(1)^2(-0\cdot4)+3(1)(-0\cdot4)^2+(-0\cdot4)^3$$
$$=1-1\cdot2+0\cdot48-0\cdot064$$
$$=0\cdot216$$

③ Expand $(x-y)^3(x+y)^3$

The trick here is to notice it's a difference of 2 squares.

$$
\begin{aligned}
(x-y)^3(x+y)^3 &= \left((x-y)(x+y)\right)^3 \\
&= \left(x^2-y^2\right)^3 \qquad \text{Now use the binomial theorem.} \\
&= \left(x^2\right)^3 + 3\left(x^2\right)^2\left(-y^2\right) + 3\left(x^2\right)\left(-y^2\right)^2 + \left(-y^2\right)^3 \\
&= x^6 - 3x^4y^2 + 3x^2y^4 - y^6
\end{aligned}
$$

<u>Questions</u>

① Calculate (a) $2\cdot 03^4$ (b) $1\cdot 98^3$ (c) $9\cdot 9^5$

② Expand the following (a) $(a-b)^4(a+b)^4$

(b) $(2a+b)^3(2a-b)^3$

(c) $\left(\frac{1}{x}+3y\right)^4\left(\frac{1}{x}-3y\right)^4$

Past Paper Questions

2001 – A6

Expand $\left(x^2 - \frac{2}{x}\right)^4$, $x \neq 0$ and simplify as far as possible. **(5 marks)**

2004 – Q2

Obtain the binomial expansion of $\left(a^2 - 3\right)^4$. **(3 marks)**

2007 – Q1

Express the binomial expansion of $\left(x - \frac{2}{x}\right)^4$ in the form $ax^4 + bx^2 + c + \frac{d}{x^2} + \frac{e}{x^4}$

for integers a, b, c, d and e. **(4 marks)**

2008 – Q8

Write down and simplify the general term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$.

Hence or otherwise, obtain the term in x^{14}. **(3, 2 marks)**

2009 – Q8

(a) Write down the binomial expansion of $(1+x)^5$.

(b) Hence show that $0 \cdot 9^5$ is $0 \cdot 59049$. **(1, 2 marks)**

2010 – Q5

Show that

$$\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$$

where the integer n is greater than or equal to 3. **(4 marks)**

2011 – Q2

Use the binomial theorem to expand $\left(\frac{1}{2}x - 3\right)^4$ and simplify your answer. **(3 marks)**

2012 – Q4

Write down and simplify the general term in the expansion of $\left(2x - \frac{1}{x^2}\right)^9$.

Hence, or otherwise, obtain the term independent of x. **(3, 2 marks)**

2013 – Q1

Write down the binomial expansion of $\left(3x - \frac{2}{x^2}\right)^4$ and simplify your answer. **(4 marks)**

2014 – Q2

Write down and simplify the general term in the expression $\left(\frac{2}{x} + \frac{1}{4x^2}\right)^{10}$.

Hence, or otherwise, obtain the term in $\frac{1}{x^{13}}$. **(5 marks)**

2015 – Q1

① Use the binomial theorem to expand and simplify

$$\left(\frac{x^2}{3} - \frac{2}{x}\right)^5.$$

(4 marks)

② Show that $\binom{n+2}{3} - \binom{n}{3} = n^2$, for all integers, n, where $n \geq 3$. **(4 marks)**

2016 – Q3

Write down and simplify the general term in the binomial expansion of $\left(\frac{3}{x} - 2x\right)^{13}$.

Hence, or otherwise, find the term in x^9. **(5 marks)**

2017 – Q1

Write down the binomial expansion of $\left(\frac{2}{y^2} - 5y\right)^3$ and simplify your answer.

(4 marks)