Complex Numbers

We have already looked at sets of number, ending with $\mathbb{R}$, the real numbers. However if we have $x^2+1=0$, there is no solution for x in the real numbers. The number set $\mathbb{C}$ was created, the complex numbers, by defining $i^2=-1$.

Thus a complex number takes the form

$z=x+iy$ where x is the real part $\text{Re}(z)$
y is the imaginary part $\text{Im}(z)$.

So $z_1=3+2i, z_2=-4+6i, z_3=-4i$ are all complex numbers, furthermore z_3 is purely imaginary.

From $i^2=-1$, we have $i=\sqrt{-1}$, therefore $z^2=-1$ *and* $z=\pm i$

Now we can solve any quadratic equation by using the quadratic formula.

a) $x^2-2x+5=0$

$$\begin{aligned}x&=\frac{2\pm\sqrt{4-20}}{2}\\&=\frac{2\pm\sqrt{-16}}{2}\\&=\frac{2\pm\sqrt{16\times(-1)}}{2}\\&=\frac{2\pm\sqrt{16i^2}}{2}\\&=\frac{2\pm 4i}{2}\\&=1\pm 2i\end{aligned}$$

b) $x^2-6x+25=0$

$$\begin{aligned}x&=\frac{6\pm\sqrt{36-100}}{2}\\&=\frac{6\pm\sqrt{-64}}{2}\\&=\frac{6\pm\sqrt{64\times(-1)}}{2}\\&=\frac{6\pm\sqrt{64i^2}}{2}\\&=\frac{6\pm 8i}{2}\\&=3\pm 4i\end{aligned}$$

c) $x^2+2x+6=0$

$$\begin{aligned}x&=\frac{-2\pm\sqrt{4-24}}{2}\\&=\frac{-2\pm\sqrt{-20}}{2}\\&=\frac{-2\pm\sqrt{20\times(-1)}}{2}\\&=\frac{-2\pm\sqrt{20i^2}}{2}\\&=\frac{-2\pm i2\sqrt{5}}{2}\\&=-1\pm i\sqrt{5}\end{aligned}$$

The Arithmetic of Complex Numbers

Addition/Subtraction

To add or subtract complex numbers we simply perform the required calculation with the real part & with the imaginary part.

Examples

If $z_1=4-3i$ and $z_2=-1-4i$ evaluate

① z_1+z_2
$=(4+(-1))+(-3+(-4)i$
$=3-7i$

② z_1-z_2
$=(4-(-1))+(-3-(-4)i$
$=5+i$

③ z_2-z_1
$=(-1-4)+(-4-(-3)i$
$=-5-i$

Questions

Find the sum of

(a) $z_1 = 3 - i$ & $z_2 = 2 + 5i$ (b) $z_1 = -3 + 2i$ & $z_2 = -4 - 4i$ (c) $z_1 = -2i$ & $z_2 = \frac{1}{2} + \frac{1}{3}i$

Find $z_1 - z_2$ when

(d) $z_1 = 10 - 2i$ & $z_2 = 4 + 3i$ (e) $z_1 = -1 - 2i$ & $z_2 = \frac{1}{2} - \frac{3}{2}i$

(f) $z_1 = -4 + 6i$ & $z_2 = 6 - 2i$ (g) $z_1 = 4\sqrt{3} - i\sqrt{8}$ & $z_2 = \sqrt{3} + i\sqrt{2}$

Multiplication

To multiply two complex numbers we use the same method as we would use for multiplying out $(x+1)(2x-4)$. We must remember here that $i^2 = -1$.

① $(4+3i)(2-i)$

$$\begin{aligned} &= 8 - 4i + 6i - 3i^2 \\ &= 8 + 2i - 3(-1) \\ &= 11 + 2i \end{aligned}$$

② $(4-i)(3-2i)$

$$\begin{aligned} &= 12 - 8i - 3i + 2i^2 \\ &= 12 - 11i - 2 \\ &= 10 - 11i \end{aligned}$$

③ $\left(\frac{1}{2} + \frac{1}{4}i\right)\left(\frac{3}{2} + 4i\right)$

$$\begin{aligned} &= \frac{3}{4} + 2i + \frac{3}{8}i + i^2 \\ &= \frac{3}{4} + \frac{19}{8}i - 1 \\ &= -\frac{1}{4} + \frac{19}{8}i \end{aligned}$$

Questions

(a) $(-4+i)(3-i)$ (b) $(2+3i)(1-i)$ (c) $(5-6i)(3-2i)$ (d) $(-2-i)(-3-5i)$

(e) $(8-2i)(-1+9i)$ (f) $(4+5i)(4+5i)$ (g) $(3-2i)(3+2i)$ (h) $(-2+4i)(-2-4i)$

(g) & (h) show us important results. Here $\begin{matrix} z_1 = 3 - 2i \\ z_2 = 3 + 2i \end{matrix}$ & $\begin{matrix} z_1 = -2 + 4i \\ z_2 = -2 - 4i \end{matrix}$.

z_2 is said to be the **<u>complex conjugate</u>** of z_1 and is denoted by $\bar{z}$ $\Rightarrow$ if $z = x + iy$, $\bar{z} = x - iy$.

This definition is required for division.

Division

To perform division it is best to consider it as a fraction. We find the conjugate of the denominator and multiply the numerator and denominator by this. It is a little like the method for rationalising a denominator.

Find $z_1 \div z_2$ when $z_1 = 2+3i$ and $z_2 = 1-i$.

$\overline{z}_2 = 1+i$

$$\frac{z_1}{z_2} = \frac{2+3i}{1-i}$$

$$= \frac{2+3i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{2+2i+3i+3i^2}{1-i^2}$$

$$= \frac{-1+5i}{2}$$

$$= \frac{-1}{2} + \frac{5}{2}i$$

Further Examples

① $\dfrac{3-i}{2-2i}$

$$= \frac{(3-i)(2+2i)}{(2-2i)(2+2i)}$$

$$= \frac{6+6i-2i-2i^2}{4-4i^2}$$

$$= \frac{8+4i}{8}$$

$$= 1 + \frac{1}{2}i$$

② $\dfrac{-3+5i}{-3+i}$

$$= \frac{(-3+5i)(-3-i)}{(-3+i)(-3-i)}$$

$$= \frac{9+3i-15i-5i^2}{9-i^2}$$

$$= \frac{14-12i}{10}$$

$$= \frac{7}{5} - \frac{6}{5}i$$

③ $\dfrac{2+6i}{4+i}$

$$= \frac{(2+6i)(4-i)}{(4+i)(4-i)}$$

$$= \frac{8-2i+24i-6i^2}{16-i^2}$$

$$= \frac{14+22i}{17}$$

$$= \frac{14}{17} + \frac{22}{17}i$$

Questions

Find $z_1 \div z_2$ when

(a) $z_1 = -2+4i$ & $z_2 = 7+i$

(b) $z_1 = 4-2i$ & $z_2 = 3+i$

(c) $z_1 = 4-i$ & $z_2 = -3-2i$

(d) $z_1 = 2+\frac{1}{2}i$ & $z_2 = -6+4i$

Square Roots

By equating real and imaginary parts we can achieve the square root of a complex number. Clearly, as with the real numbers, there will be two solutions.

Examples

① $\sqrt{5+12i}$

If $x+iy=\sqrt{5+12i}$

Then $(x+iy)^2 = 5+12i$

$$x^2+2ixy+i^2y^2 = 5+12i$$
$$(x^2-y^2)+(2xy)i = 5+12i$$

Then $x^2-y^2=5$ $\quad 2xy=12$

$$\frac{36}{y^2}-y^2=5 \qquad x=\frac{6}{y}$$
$$36-y^4=5y^2$$
$$y^4+5y^2-36=0$$
$$(y^2+9)(y^2-4)=0$$

$y^2=4$, $y=\pm 2$ $\qquad$ $y^2=-9$, $y\in\mathbb{R}$ so no solns

When $y=2,\ x=\frac{6}{2}=3 \Rightarrow 3+2i$

When $y=-2,\ x=\frac{6}{-2}=-3 \Rightarrow -3-2i$

② $\sqrt{-3-4i}$

If $x+iy=\sqrt{-3-4i}$

Then $(x+iy)^2=-3-4i$

$$x^2+2ixy+i^2y^2=-3-4i$$
$$(x^2-y^2)+(2xy)i=-3-4i$$

Then $x^2-y^2=-3$ $\quad 2xy=-4$

$$\frac{4}{y^2}-y^2=-3 \qquad x=\frac{-2}{y}$$
$$4-y^4=-3y^2$$
$$y^4-3y^2-4=0$$
$$(y^2+1)(y^2-4)=0$$

$y^2=4$, $y=\pm 2$ $\qquad$ $y^2=-1$, $y\in\mathbb{R}$ so no solns

When $y=2,\ x=\frac{-2}{2}=-1 \Rightarrow -1+2i$

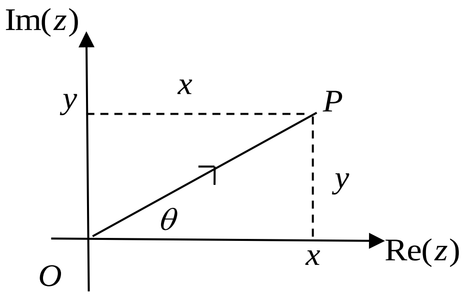
When $y=-2,\ x=\frac{-2}{-2}=1 \Rightarrow 1-2i$

Questions

Calculate (a) $\sqrt{15-8i}$ (b) $\sqrt{24-10i}$ (c) $\sqrt{-9+40i}$

Argand Diagrams

An argand diagram is used to geometrically represent a complex number. If $z = x + iy$ we can plot the point $P(x, y)$ in the complex plane.



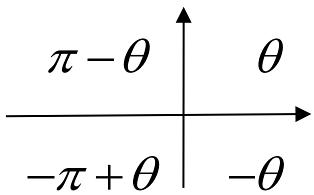
$\overrightarrow{OP}$ the position vector represents z.

The length of $\overrightarrow{OP}$ is called the **modulus** of z and is denoted $|z|$.

The angle is called the argument of z
$Arg(z) = \theta \pm 2\pi n, \quad n \in \mathbb{Z}.$

The principle $Arg(z), \ -\pi < \theta < \pi$

In order to help us find the principle argument the following diagram may help.

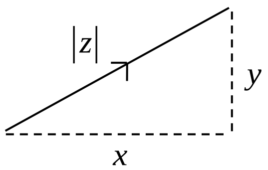


Therefore each time we are dealing with a complex number question, an Argand diagram will help us pick the correct principle argument, so it should always be drawn.

Once we know the modulus and principle argument we can change the number from Cartesian form $(x + iy)$ to another form known as polar form.

Modulus

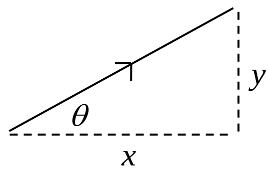
From Pythagoras:



$r = |z| = \sqrt{x^2 + y^2}$

Argument

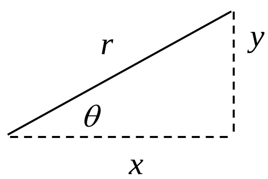
From Trigonometry:



$\tan\theta = \dfrac{y}{x}$

$Arg|z| \Rightarrow -\pi < \theta \le \pi$

The polar form, mentioned before, is then created as follows



$\cos\theta = \dfrac{x}{r}$ $\qquad$ $\sin\theta = \dfrac{y}{r}$

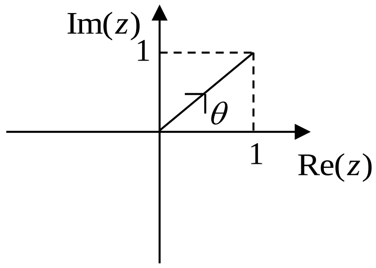
$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

Thus $z = x + iy$; $\quad z = r\cos\theta + ir\sin\theta$, i.e. $\boxed{z = r(\cos\theta + i\sin\theta)}$ **POLAR FORM**

Examples

For each of the following complex numbers, find the modulus, argument write in polar form.

① $1+i$



$r=|z|=\sqrt{1^2+1^2}$

$r=\sqrt{2}$

$\tan\theta=\dfrac{1}{1}$

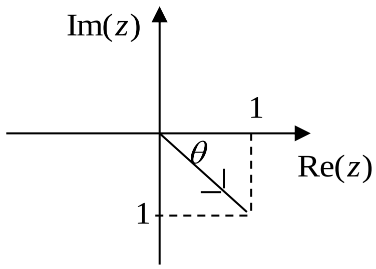
$\theta=\dfrac{\pi}{4}$

$Arg|z|=\dfrac{\pi}{4}$

Polar Form $z=r(\cos\theta+i\sin\theta)$

$=\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$

② $1-i$



$r=|z|=\sqrt{1^2+1^2}$

$r=\sqrt{2}$

$\tan\theta=\dfrac{1}{1}$

$\theta=\dfrac{\pi}{4}$

$Arg|z|=\dfrac{-\pi}{4}$

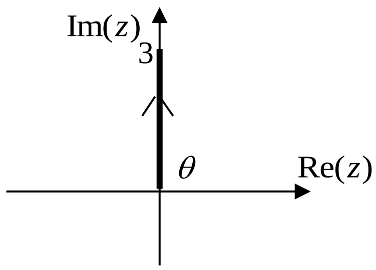
Polar Form $z=r(\cos\theta+i\sin\theta)$

$=\sqrt{2}\left(\cos\left(\dfrac{-\pi}{4}\right)+i\sin\left(\dfrac{-\pi}{4}\right)\right)$

$=\sqrt{2}\left(\cos\dfrac{\pi}{4}-i\sin\dfrac{\pi}{4}\right)$

(Recall $\cos(-\theta)=\cos\theta, \sin(-\theta)=-\sin(\theta)$.)

③ $3i$



$r=|z|=\sqrt{0^2+3^2}$

$r=3$

$\tan\theta=\dfrac{3}{0}$

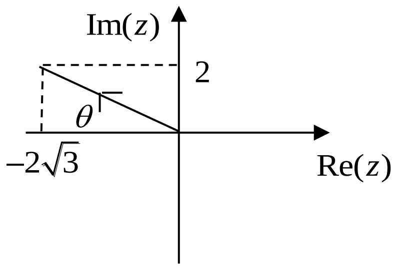
$\theta=\dfrac{\pi}{2}$ Read from

$Arg|z|=\dfrac{\pi}{2}$ Argand Diag.

Polar Form $z=r(\cos\theta+i\sin\theta)$

$=3\left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right)$

④ $-2\sqrt{3}+2i$



$$r=|z|=\sqrt{(2\sqrt{3})^2+2^2}$$

$$r=4$$

$$\tan\theta=\frac{2}{2\sqrt{3}}$$

$$\theta=\frac{\pi}{6}$$

$$Arg|z|=\pi-\theta$$
$$=\pi-\frac{\pi}{6}$$
$$=\frac{5\pi}{6}$$

Polar Form $z=r(\cos\theta+i\sin\theta)$

$$=4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$$

Questions

Express the following complex numbers in polar form.

(a) $z=-\sqrt{2}-i\sqrt{2}$ (b) $z=1+i\sqrt{3}$ (c) $z=-i$ (d) $z=-1-i\sqrt{3}$ (e) $z=-4$

Geometric Interpretations in the Complex Plane (Loci)

If we place a restriction on the modulus of a complex number we create a locus of the point which moves on the complex plane.

Example

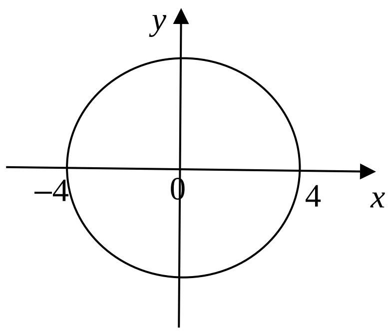
① Given $z=x+iy$ draw the locus of the point which moves on the complex plane so that

(a) $|z|=4$ (b) $|z|\le 4$.

$|z|=4$ $\Rightarrow\sqrt{x^2+y^2}=4$

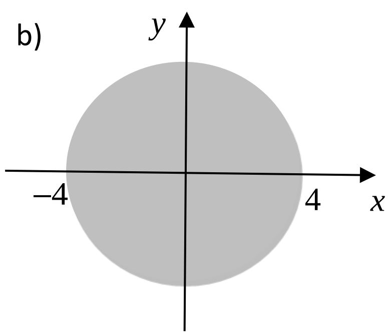
$|x+iy|=4$ $x^2+y^2=4^2$ Equation of a circle, centre origin and radius 4.

a)



$|z|=4$

b)



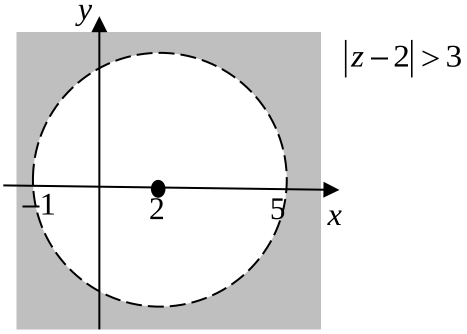
$|z|\le 4$

② If $z = x + iy$, find the equation of the locus $|z-2| > 3$ & draw on an Argand diagram.

$|z-2| > 3 \quad \Rightarrow \sqrt{(x-2)^2 + y^2} > 3$

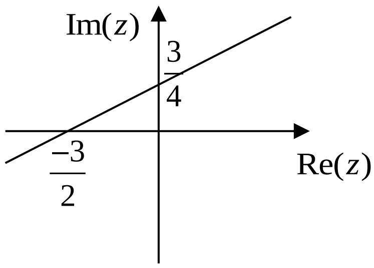
$|x+iy-2| > 3 \quad (x-2)^2 + y^2 > 3^2$ A circle, centre (2,0) and radius 3.

$|(x-2)+iy| > 3$



③ Find the equation of the locus when $|z-1| = |z-2i|$.

$$|z-1| = |z-2i|$$
$$|x+iy-1| = |x+iy-2i|$$
$$|(x-1)+iy| = |x+(y-2)i|$$
$$\sqrt{(x-1)^2 + y^2} = \sqrt{x^2 + (y-2)^2}$$
$$(x-1)^2 + y^2 = x^2 + (y-2)^2$$
$$x^2 - 2x + 1 + y^2 = x^2 + y^2 - 4y + 4$$
$$1 - 2x = 4 - 4y$$
$$4y = 2x + 3$$
$$y = \frac{1}{2}x + \frac{3}{4}$$



Questions

a) Given that $z = x + iy$ find the equation of each of the following loci and draw a representation of them on an Argand Diagram.

i) $|z| = 5$ ii) $|z| \le 3$ iii) $|z+1| = 4$

iv) $|z-2i| \ge 3$ v) $|z-2| = |z-i|$ vi) $|z-3| = |z-2i|$

The Fundamental Theorem of Algebra

The Fundamental Theorem of algebra stares that if $P(z)=z^n+a_{n-1}z^{n-1}+...+a_2z^2+a_1z+a_0$ is a polynomial of degree n (with real or complex coeffiecients) then $P(z)$ has n solutions $\alpha_1,\alpha_2,\alpha_3,...\alpha_n$ in the complex numbers & $P(z)=(z-\alpha_1)(z-\alpha_2)...(z-\alpha_n)$.

i.e. a cubic equation has 3 roots, a quartic equation has 4 roots…etc.

Also, if $z=x+iy$ is a solution of $P(z)$, $\bar{z}$ will also be a solution (conjugate pairs are roots).

Examples

① Find all solution of the equation $z^3-z^2-z-2=0$.

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -1 & -2 \\ & 0 & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$\therefore (z-2)$ is a factor, $z=2$ is a solution.

$f(z)=(z-2)(z^2+z+1)$

$$z=\frac{-1\pm\sqrt{1^2-4\times1\times1}}{2}$$

$$z=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$\therefore$ Solutions are $z=2, z=-\frac{1}{2}+i\frac{\sqrt{3}}{2}, z=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

② $f(z)=2z^3+5z^2+8z+20=0$. If $z-2i$ is a factor, find all roots of the equation.

$z-2i$ is a factor $\therefore z+2i$ is also a factor (the conjugate)

$(z-2i)(z+2i)$
$=z^2-4i^2$
$=z^2+4$

$\therefore f(z)=(z-2i)(z+2i)(\quad)$
$=(z^2+4)(\quad)$

Long Division will enable us to find the remaining factor.

$$\begin{array}{rrrrr} & & & 2z & +5 \\ z^2+4 & \big) \; 2z^3 & +5z^2 & +8z & +20 \\ - & 2z^3 & & 8z & \\ \hline & & 5z^2 & & 20 \\ - & & 5z^2 & & 20 \\ \hline & & & & 0 \end{array}$$

If the remainder is not 0 you have made a mistake.

$\therefore f(z)=(z-2i)(z+2i)(2z+5)$

$\therefore z=2i, z=-2i, z=\frac{-5}{2}$.

③ $f(z)=z^3-z^2-z-15$. If for $f(z)=0$, $z=3$ is a root, find the two remaining roots.

$$\begin{array}{c|cccc} 3 & 1 & -1 & -1 & -15 \\ & 0 & 3 & 6 & 15 \\ \hline & 1 & 2 & 5 & \boxed{0} \end{array}$$

$\therefore (z-3)$ is a factor, $z=3$ is a solution.

$f(z)=(z-3)(z^2+2z+5)$

$$z=\frac{-2\pm\sqrt{2^2-4\times1\times5}}{2}$$

$$z=-1\pm 2i$$

$\therefore$ Solutions are $z=3, z=-1+2i, z=-1-2i.$

④ Past Paper 2002.

Verify i is a solution of $z^4+4z^3+3z^2+4z+2=0$. Hence find all solutions.

To verify **either** do synthetic division, or sub in solution and check you get 0 out.

$$\begin{aligned} &\Rightarrow z^4+4z^3+3z^2+4z+2 \\ &=i^4+4i^3+3i^2+4i+2 \\ &=1+4(-i)+3(-1)+4(i)+2 \\ &=0 \end{aligned}$$

$\therefore i$ is a solution, $\therefore -i$ is also a solution (conjugate)

$$\begin{aligned} &(z-i)(z+i) \\ &=z^2-i^2 \\ &=z^2+1 \end{aligned}$$

$$\begin{aligned} \therefore f(z)&=(z-i)(z+i)(\quad)(\quad) \\ &=(z^2+1)(\quad)(\quad) \end{aligned}$$

$$\begin{array}{rrrrrr} & & & z^2 & +4z & +2 \\ \hline z^2+1 \,\big) & z^4 & +4z^3 & +3z^2 & +4z & +2 \\ - & z^4 & & +z^2 & & \\ \hline & & 4z^3 & +2z^2 & +4z & +2 \\ - & & 4z^3 & & +4z & \\ \hline & & & 2z^2 & & +2 \\ - & & & 2z^2 & & +2 \\ \hline & & & & & 0 \end{array}$$

$\therefore f(z)=(z-i)(z+i)(z^2+4z+2)$

$$z=\frac{-4\pm\sqrt{4^2-4\times1\times2}}{2}$$

$$z=-2\pm\sqrt{2}$$

$\therefore$ Solutions are $z=i, z=-i, z=-2+\sqrt{2}, z=-2-\sqrt{2}.$

Multiplying & Dividing Numbers in Polar Form

If we are multiplying two numbers in polar form we multiply the moduli & add arguments.

If we are dividing two numbers in polar form we divide the moduli & subtract arguments.

(Remember to always bring the argument back into the principle argument in the range $-\pi < \theta \leq \pi$)

Examples

① $z = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right); w = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$\begin{aligned} zw &= 3\times 4\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{3}+\frac{\pi}{2}\right)\right) \\ &= 12\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \end{aligned}$$

$$\begin{aligned} \frac{z}{w} &= \frac{3}{4}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{3}-\frac{\pi}{2}\right)\right) \\ &= \frac{3}{4}\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right) \\ &= \frac{3}{4}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) \end{aligned}$$

② $z = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right); w = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$\begin{aligned} zw &= 2\times 5\left(\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right) \\ &= 10\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) \end{aligned}$$

$$\begin{aligned} \frac{z}{w} &= \frac{2}{5}\left(\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right) \\ &= \frac{2}{5}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \end{aligned}$$

Questions

Find zw & $\frac{z}{w}$ if

(a) $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $w = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(b) $z = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, $w = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(c) $z = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, $w = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

De Moivre's Theorem

By extending the results previous, we obtain the result commonly known as De Moivre's Theorem.

If $z = r(\cos\theta + i\sin\theta)$ $\quad$ $z^n = r^n(\cos n\theta + i\sin n\theta)$

Again we must remember to ensure we return our answer to having the principle argument.

Examples

① If $z = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ find z^3.

$$\begin{aligned} z^3 &= 3^3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) \\ &= 27\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right) \\ &= 27\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right) \end{aligned}$$

Subtract 2π to return to the principle arg.

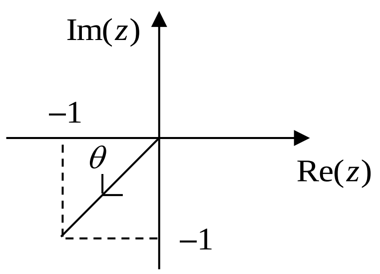
②If $z = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ find z^4.

$$\begin{aligned} z^4 &= 2^4\left(\cos 3\pi + i\sin 3\pi\right) \\ &= 16(\cos\pi + i\sin\pi) \end{aligned}$$

If we are given a complex number in Cartesian form, we can still use De Moivre's Theorem by converting it into polar form (Using an Argand diagram to help.)

Examples

① Find $(-1-i)^4$.



$$r = |z| = \sqrt{1^2 + (-1)^2}$$

$$r = \sqrt{2}$$

$$\tan\theta = \frac{-1}{-1}$$

$$\theta = \frac{\pi}{4}$$

$$\begin{aligned} Arg|z| &= -\pi + \frac{\pi}{4} \\ &= -\frac{3\pi}{4} \end{aligned}$$

Polar Form $z = r(\cos\theta + i\sin\theta)$

$$= \sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$$

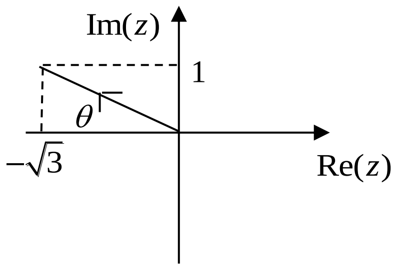
$$= \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$z^4 = \left(\sqrt{2}\right)^4\left(\cos\frac{12\pi}{4} - i\sin\frac{12\pi}{4}\right)$$

$$= 4(\cos 3\pi - i\sin 3\pi)$$

$$= 4(\cos\pi - i\sin\pi)$$

② Calculate $\left(-\sqrt{3}+i\right)^3$ by using De Moivre's Theorem.



$$r = |z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$$

$$r = 2$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

$$Arg|z| = \pi - \theta$$

$$= \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

Polar Form $z = r(\cos\theta + i\sin\theta)$

$$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$\therefore z = 2^3\left(\cos\frac{15\pi}{6} + i\sin\frac{15\pi}{6}\right)$$

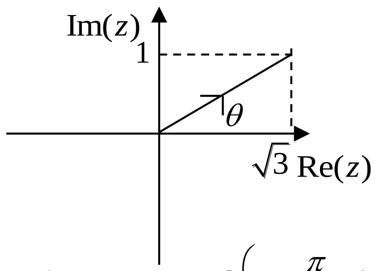
$$= 8\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$$

$$= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

③ Find $\dfrac{\left(\sqrt{3}+i\right)^4}{(1-i)^3}$.

$z_1 = \sqrt{3}+i$

$r = |z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$ $\qquad$ $\tan\theta = \dfrac{1}{\sqrt{3}}$



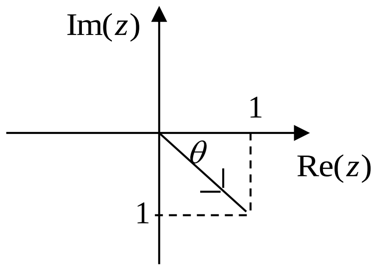
$r = 2$ $\qquad$ $\theta = \dfrac{\pi}{6}$

$Arg|z| = \dfrac{\pi}{6}$

Polar Form $z_1 = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$ $\qquad$ $z_1^{\;4} = 16\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

$z_2 = 1-i$

$r = |z| = \sqrt{1^2 + (-1)^2}$ $\qquad$ $\tan\theta = \dfrac{-1}{1}$



$r = \sqrt{2}$ $\qquad$ $\theta = \dfrac{\pi}{4}$

$Arg|z| = \dfrac{-\pi}{4}$

Polar Form $z = r(\cos\theta + i\sin\theta)$ $\qquad$ $z_2^{\;3} = 2\sqrt{2}\left(\cos\left(\dfrac{-3\pi}{4}\right) + i\sin\left(\dfrac{-3\pi}{4}\right)\right)$

$$= \sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right)$$

Leave the argument as negative at this stage.

$$\frac{z_1^{\;4}}{z_2^{\;3}} = \frac{16}{2\sqrt{2}}\left(\cos\left(\frac{2\pi}{3} - \left(-\frac{3\pi}{4}\right)\right) + i\sin\left(\frac{2\pi}{3} - \left(-\frac{3\pi}{4}\right)\right)\right)$$
$$= \frac{8}{\sqrt{2}}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$
$$= 4\sqrt{2}\left(\cos\left(\frac{-7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right)$$
$$= 4\sqrt{2}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right)$$

Fractional Indices

Fractional Indices may also be considered here, but we must be careful. We already know if we find $z^{\frac{1}{2}}$ we have 2 solutions; solving z^3 for z should give 3 solutions, etc.

To help achieve this we add multiples of 2π to the argument.

If we have $z^2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

Then $z = \left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right)^{\frac{1}{2}}$ where $k = 0,1$

(as this will give 2 solns)

<u>By De Moivre's Theorem</u>

$$z = 1^{\frac{1}{2}}\left(\cos\left(\frac{\frac{\pi}{3} + 2k\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{3} + 2k\pi}{2}\right)\right)$$

When $k = 0$

$z_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

When $k = 1$

$z_1 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$= \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$

$\therefore$ In general

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right) \text{ for } k = 0,1,2,\ldots,n-1.$$

You will need to remember this formula.

Examples

① Considering $z^3 = 8\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, find all possible results for z.

$$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{\frac{3\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4} + 2k\pi}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{3\pi + 8k\pi}{12}\right) + i\sin\left(\frac{3\pi + 8k\pi}{12}\right)\right) \text{ for } k = 0,1,2$$

When

$k = 0$ $\quad z_0 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$k = 1$ $\quad z_1 = 2\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

$k = 2$ $\quad z_2 = 2\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$

$$= 2\left(\cos\left(\frac{-5\pi}{12}\right) + i\sin\left(\frac{-5\pi}{12}\right)\right)$$

$$= 2\left(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right)$$

② Solve $z^3 = 8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

$$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{\frac{\pi}{3} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{3} + 2k\pi}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{\pi + 6k\pi}{9}\right) + i\sin\left(\frac{\pi + 6k\pi}{9}\right)\right)$$ for $k = 0,1,2$

When

$k = 0$ $\quad z_0 = 2\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$

$k = 1$ $\quad z_1 = 2\left(\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right)$

$k = 2$ $\quad z_2 = 2\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right)$

$$= 2\left(\cos\left(\frac{-5\pi}{9}\right) + i\sin\left(\frac{-5\pi}{9}\right)\right)$$

$$= 2\left(\cos\frac{5\pi}{9} - i\sin\frac{5\pi}{9}\right)$$

Nth Roots of Unity

Solving the equation $z^n = 1$ is a special case which gives solutions known as the n^{th} roots of unity. These & the other roots previously calculated can be drawn on an Argand Diagram and evaluated geometrically.

Examples

① Solve $z^3 = 1$ to find the cube roots of unity ($z^3 = 1(\cos 0 + i\sin 0)$.

$$z^3 = 1(\cos 0 + i\sin 0)$$
$$z = 1^{\frac{1}{3}}\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right) \text{ for } k = 0,1,2.$$

When

$k = 0$ $\quad z_0 = 1\left(\cos 0 + i\sin 0\right)$

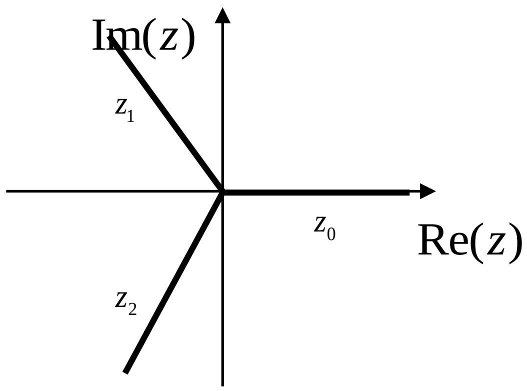
$(=1)$ for plotting

$k = 1$ $\quad z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$$\left(= -\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$k = 2$ $\quad z_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$$= \cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)$$
$$= \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$
$$\left(= -\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$



② Solve $z^4 = 1$ to find the fourth roots of unity.

$$z^4 = 1(\cos 0 + i\sin 0)$$
$$z = 1^{\frac{1}{4}}\left(\cos\frac{0 + 2k\pi}{4} + i\sin\frac{0 + 2k\pi}{4}\right)$$
$$= \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2} \text{ for } k = 0,1,2,3.$$

When

$k = 0$ $\quad z_0 = 1(\cos 0 + i\sin 0)$

$(=1)$ for plotting

$k = 1$ $\quad z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$(= i)$

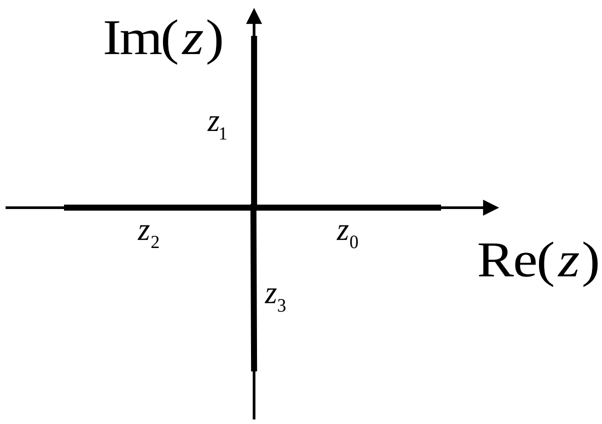
$k = 2$ $\quad z_2 = \cos\pi + i\sin\pi$

$(=-1)$

$k = 3$ $\quad z_3 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$(=-i)$



De Moivres' Theorem & Multiple Angle Trigonometric Formulae

Best shown by example.

Examples

① Using De Moivre's and the Binomial Theorem, express $\cos 4\theta$ as a polynomial in $\cos\theta$.

By De Moivre

$$\left(\cos\theta + i\sin\theta\right)^4 = \cos 4\theta + i\sin 4\theta$$

By Binomial Theorem

$$\begin{aligned}\left(\cos\theta + i\sin\theta\right)^4 &= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta \\ &= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta \\ &= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)\end{aligned}$$

Equating Real Parts gives us

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

(Note, equating imaginary parts would give $\sin 3\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$)

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \quad \text{replacing all } \sin\theta \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

② Find a formula for $\cos 3\theta$ in terms of powers of $\cos\theta$ by using De Moivre's Theorem.
Hence express $\cos^3\theta$ in terms of $\cos\theta$ & $\cos 3\theta$.

By De Moivre

$$\left(\cos\theta + i\sin\theta\right)^3 = \cos 3\theta + i\sin 3\theta$$

By Binomial Theorem

$$\begin{aligned}\left(\cos\theta + i\sin\theta\right)^3 &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

Equating Real Parts gives us

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

Hence $\cos^3\theta = \dfrac{\cos 3\theta + 3\cos\theta}{4}$

③ Express $\dfrac{\sin 5\theta}{\sin\theta}$ as a polynomial in $\sin\theta$.

By De Moivre

$$\left(\cos\theta + i\sin\theta\right)^5 = \cos 5\theta + i\sin 5\theta$$

By Binomial Theorem

$$\begin{aligned}\left(\cos\theta + i\sin\theta\right)^5 &= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$

Equating Imaginary Parts

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\begin{aligned}\frac{\sin 5\theta}{\sin\theta} &= 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= 5(1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^2\theta + \sin^4\theta \\ &= 5 - 10\sin^2\theta + 5\sin^4\theta - 10\sin^2\theta + 10\sin^4\theta + \sin^4\theta \\ &= 16\sin^4\theta - 20\sin^2\theta + 5\end{aligned}$$

Past Paper Questions

2001 – QA9

(a) Given that $-1 = \cos\theta + i\sin\theta,\ -\pi < \theta < \pi,$ state the value of θ.

(b) Use de Moivre's Theorem to find the non-real solutions, z_1 and z_2, of the equation $z^3 + 1 = 0$.

Hence show that $z_1^2 = -z_2$ and $z_2^2 = -z_1$.

(c) Plot all the solution of $z^3 + 1 = 0$ on an Argand diagram and state their geometrical significance. **(1,5,2,3 marks)**

2002 – Q2

Verify that i is a solution of $z^4 + 4z^3 + 3z^2 + 4z + 2 = 0$.
Hence find all the solutions **(5 marks)**

2003 – Q4

① Identify the locus in the complex plane given by $|z + i| = 2$ **(3 marks)**

2003 – Q9

② Given that $w = \cos\theta + i\sin\theta,$ show that $\frac{1}{w} = \cos\theta - i\sin\theta$.

Use de Moivre's Theorem to prove $w^k + w^{-k} = 2\cos k\theta,$ where k is a natural number.

Expand $\left(w + w^{-1}\right)^4$ by the binomial theorem and hence show that

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$

(1, 3, 5 marks)

2004 – Q4

Given $z = 1 + 2i,$ express $z^2(z+3)$ in the form $a + ib$.
Hence, or otherwise, verify that $1 + 2i$ is a root of the equation

$$z^3 + 3z^2 - 5z + 25 = 0.$$

Obtain the other roots of this equation. **(2, 2, 2 marks)**

2005 – Q9

① Given the equation $z + 2i\bar{z} = 8 + 7i,$ express z in the form $a + ib$. **(4 marks)**

2005 – Q12

② Let $z = \cos\theta + i\sin\theta$.

(a) Use the binomial expansion to express z^4 in the form $u + iv,$ where u and v are expression involving $\sin\theta$ and $\cos\theta$.

(b) Use de Moivre's theorem to write down a second expression for z^4.

(c) Using the results of (a) and (b), show that

$$\frac{\cos 4\theta}{\cos^2\theta} = p\cos^2\theta + q\sec^2\theta + r, \quad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

stating the values of p, q and r. **(3, 1, 6 marks)**

2006 – Q3

Express the complex number $z=-i+\frac{1}{1-i}$ in the form $z=x+iy,$ stating the values of x and y.

Find the modulus and argument of z and plot z and $\bar{z}$ on an Argand diagram.

(3, 4 marks)

2007 – Q3

① Show that $z=3+3i$ is a root of the equation $z^3-18z+108=0$ and obtain the remaining roots of the equation.

(4 marks)

2007 – Q11

② Given that $|z-2|=|z+i|,$ where $z=x+iy,$ show that $ax+by+c=0$ for suitable values of $a,$ b and c.
Indicate on an Argand diagram the locus of complex numbers z which satisfy $|z-2|=|z+i|$.

(3, 1 marks)

2008 – Q16

Given $z=\cos\theta+i\sin\theta,$ use de Moivre's theorem to write down an expression for z^k in terms of $\theta,$ where k is a positive integer.

Hence show that $\frac{1}{z^k}=\cos k\theta-i\sin k\theta.$

Deduce expression for $\cos k\theta$ and $\sin k\theta$ in terms of z.

Show that $\cos^2\theta\sin^2\theta=-\frac{1}{16}\left(z^2-\frac{1}{z^2}\right)^2.$

Hence show that $\cos^2\theta\sin^2\theta=a+b\cos 4\theta,$ for suitable constants a and b.

(3, 2, 3, 2 marks)

2009 – Q6

Express $z=\frac{(1+2i)^2}{7-1}$ in the form $a+ib$ where a and b are real numbers.

Show z on an Argand diagram and evaluate $|z|$ and $\arg(z)$.

(6 marks)

2010 – Q16

Given $z = r(\cos\theta + i\sin\theta)$, use de Moivre's theorem to express z^3 in polar form.

Hence obtain $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$ in the form $a+ib$.

Hence, or otherwise, obtain the roots of the equation $z^3 = 8$ in Cartesian form.

Denoting the roots of $z^3 = 8$ by z_1, z_2, z_3:

(a) state the value $z_1 + z_2 + z_3$;

(b) obtain the value of $z_1^6 + z_2^6 + z_3^6$.

(1, 2, 4, 3 marks)

2011 – Q10

Identify the locus in the complex plane given by

$$|z-1| = 3.$$

Show in a diagram the region given by $|z-1| \leq 3$. **(5 marks)**

2012 – Q16

① Given that $(-1+2i)$ is a root of the equation

$$z^3 + 5z^2 + 11z + 15 = 0,$$

obtain all roots.

Plot all the roots on an Argand diagram. **(4, 2 marks)**

② (a) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$.

(b) Show that the real part of $\dfrac{\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)^4}$ is zero.

(6, 4 marks)

2013 – Q10

① Given that $z = 1 - \sqrt{3}i$, write down $\bar{z}$ and express $\bar{z}^{2}$ in polar form. **(4 marks)**

② Describe the loci in the complex plane given by:

(a) $|z+i| = 1$;

(b) $|z-1| = |z+5|$. **(2, 3 marks)**

2014 – Q16

(a) Express -1 as a complex number in polar form and hence determine the solutions to the equation $z^4+1=0$.

(b) Write down the four solutions to the equation $z^4-1=0$.

(c) Plot the solutions of both equations on an Argand diagram.

(d) Show that the solutions of $z^4+1=0$ and the solutions of $z^4-1=0$ are also solutions of the equation $z^8-1=0$.

(e) Hence identify all the solutions to the equation

$$z^6+z^4+z^2+1=0.$$

(3, 2, 1, 2, 2 marks)

2015 – Q13

By writing z in the form $x+iy$:

(a) solve the equation $z^2=|z|^2-4$;

(b) find the solutions to the equation $z^2=i\left(|z|^2-4\right)$. **(3, 4 marks)**

2016 – Q8

Let $z=\sqrt{3}-i$.

(a) Plot z on an Argand diagram.

(b) Let $w=az$ where $a>0,\ a\in\mathbb{R}$.

Express w in polar form.

(c) Express w^8 in the form $ka^n(x+iy),\ \ k,x,y\in\mathbb{Z}$. **(1, 2, 3 marks)**

2017 - Q17

The complex number $z=2+i$ is a root of the polynomial equation

$z^4-6z^3+16z^2-22z+q=0$ where $q\in\boldsymbol{Z}$

(a) State a second root of the equation

(b) Find the value of q and the remaining roots

(c) Show the solutions to $z^4-6z^3+16z^2-22z+q=0$ on an argand diagram

(1, 6, 1 marks)