First Order Differential Equations

A differential equation is an equation involving one or more derivative

$\left(x^3-1\right)\frac{dy}{dx}+3xy=x$ is a **first-order differential equation** as the equation contains the first derivative only.

$\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y=0$ is a second-order differential equation as the equation contains a second derivative.

In this unit we will study first-order differential equations.

First-order Differential Equations With Variables Separable

If a first-order differential equation can be written in the form

$$f\left(y\right)dy=g\left(x\right)dx$$

where $f\left(y\right)$ is a function of y only and $g\left(x\right)$ is a function of x only we say that we have separated the variables x and y.

The general solution of the differential equation is then found by integrating each side with respect to the appropriate variable.

$$\boxed{\int f\left(y\right)dy=\int g\left(x\right)dx}$$

Examples

① $\frac{dy}{dx}=x^3-x$

$$dy=\left(x^3-x\right)dx$$

$$\int dy=\int\left(x^3-x\right)dx$$

$$y=\frac{x^4}{4}-\frac{x^2}{2}+C$$

This is the general solution where C is an arbitrary constant.

② $\dfrac{dy}{dx} = \dfrac{x^2}{y}$

$$y\ dy = x^2 dx$$

$$\int y\ dy = \int x^2 dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

$$3y^2 = 2x^2 + C$$

When solving differential equations it is convenient for C to denote the 'total' constant so far in each line of working, although strictly speaking a different letter should be used for each new constant.

③ $e^{4y}\dfrac{dy}{dx} - x = 5$

$$e^{4y}\frac{dy}{dx} = x + 5$$

$$e^{4y}dy = (x+5)dx$$

$$\int e^{4y}dy = \int (x+5)dx$$

$$\frac{1}{4}e^{4y} = \frac{1}{2}x^2 + 5x + C$$

$$e^{4y} = 2x^2 + 20x + C$$

$$4y = \ln(2x^2 + 20x + C)$$

$$y = \frac{1}{4}\ln(2x^2 + 20x + C)$$

④ $\dfrac{dy}{dx} = \dfrac{1}{16+4x^2}$

$$dy = \frac{1}{16+4x^2}dx$$

$$\int dy = \int \frac{1}{16+4x^2}dx$$

$$y = \int \frac{1}{4(4+x^2)}dx$$

$$y = \frac{1}{4}\int \frac{1}{2^2+x^2}dx$$

$$y = \frac{1}{4}\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$y = \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$$

⑤ $\dfrac{dy}{dx} = 3x\sqrt{9-y^2}$

$$\frac{dy}{\sqrt{9-y^2}} = 3x\ dx$$

$$\int \frac{dy}{\sqrt{9-y^2}} = \int 3x\ dx$$

$$\sin^{-1}\left(\frac{y}{3}\right) = \frac{3x^2}{2} + C$$

$$\frac{y}{3} = \sin\left(\frac{3x^2}{2} + C\right)$$

$$y = 3\sin\left(\frac{3x^2}{2} + C\right)$$

⑥ $\left(1+x^2\right)y^2\dfrac{dy}{dx} = 1$

$$y^2dy = \frac{dx}{1+x^2}$$

$$\int y^2dy = \int \frac{dx}{1+x^2}$$

$$\frac{1}{3}y^3 = \tan^{-1}x + C$$

$$y^3 = 3\tan^{-1}x + C$$

Questions

Find the general solution of these differential equations.

① $\frac{dy}{dx}=\frac{x}{y^2}$ ② $\frac{dy}{dx}=\frac{x+2}{y}$ ③ $\frac{dy}{dx}=\frac{\cos x}{y}$ ④ $\frac{dy}{dx}=\frac{4x}{e^y}$ ⑤ $x+4y\frac{dy}{dx}=0$

⑥ $e^{4y}\frac{dy}{dx}=x^2(x+1)$ ⑦ $y\frac{dy}{dx}=\frac{1}{\sqrt{x}}$ ⑧ $\frac{dy}{dx}=4x\sqrt{1-y^2}$ ⑨ $\frac{dy}{dx}=x\left(4+y^2\right)$

Sometimes the rules of log and exponential functions are needed to simplify general solutions of differential equations.

$$\ln a+\ln b=\ln(ab)$$

$$\ln a-\ln b=\ln\left(\frac{a}{b}\right)$$

$$n\ln a=\ln a^n$$

$$e^{(a+b)}=e^a e^b$$

Examples

⑦ $\frac{dy}{dx}=3y$

$\frac{dy}{y}=3dx$

$\int\frac{dy}{y}=\int 3dx$

$\ln y=3x+C$ Take exp of both sides

$y=e^{(3x+C)}$

$y=e^{3x}e^C$ where $A=e^C$

$y=Ae^{3x}$

⑧ $\frac{dy}{dx}=\frac{y-2}{x+3}$

$\frac{dy}{y-2}=\frac{dx}{x+3}$

$\int\frac{dy}{y-2}=\int\frac{dx}{x+3}$

$\ln(y-2)=\ln(x+3)+C$

$y-2=(x+3)e^C$

$y-2=A(x+3)$ where $A=e^C$

$y=A(x+3)+2$

⑨ $\frac{dy}{dx}=\frac{y}{2x+1}$

$\frac{dy}{y}=\frac{dx}{2x+1}$

$\int\frac{dy}{y}=\int\frac{dx}{2x+1}$

$\ln y=\frac{1}{2}\ln(2x+1)+C$

$\ln y=\ln(2x+1)^{\frac{1}{2}}+C$

$y=\sqrt{2x+1}\,e^C$

$y=A\sqrt{2x+1}$ where $A=e^C$

Questions

Find the general solution of each differential equation, expressing y explicitly in terms of x.

① $\frac{dy}{dx}=2y$ ② $\frac{dy}{dx}=\frac{y+1}{x+2}$ ③ $\frac{dy}{dx}=3x^2(y+1)$ ④ $\frac{dy}{dx}=\frac{y-1}{2x-1}$ ⑤ $\frac{dy}{dx}=x(y+1)$

Differential Equations Involving Partial Fractions

① Using partial fractions, find the general solution of the differential equation $\frac{dy}{dx}=\frac{1}{x(x-1)}$

Partial fractions $\frac{1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$

$1=A(x-1)+Bx$

Let $x=0, A=-1$ Let $x=1, B=1$

$$\frac{dy}{dx}=\frac{1}{x(x-1)}$$

$$\frac{dy}{dx}=\frac{1}{x-1}-\frac{1}{x}$$

$$\int dy=\int\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$$

$$y=\ln|x-1|-\ln|x|+C$$

$$y=\ln\left|\frac{x-1}{x}\right|+C$$

② Find the general solution of the differential equation $x(x+1)\frac{dy}{dx}=y(3x+1)$

$$x(x+1)\frac{dy}{dx}=y(3x+4)$$

$$\frac{dy}{y}=\frac{3x+4}{x(x+1)}dx$$

$$\int\frac{dy}{y}=\int\frac{3x+4}{x(x+1)}dx$$

$$\ln y=\int\left(\frac{4}{x}-\frac{1}{x+1}\right)dx$$

$$\ln y=4\ln|x|-\ln|x+1|+C$$

$$\ln y=\ln|x|^4-\ln|x+1|+C$$

$$\ln y=\ln\left|\frac{x^4}{x+1}\right|+C$$

$$y=\left(\frac{x^4}{x+1}\right)e^C$$

$$y=\frac{Ax^4}{x+1}\quad\text{where } A=e^C$$

Partial fractions

$$\frac{3x+4}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$$

$$3x+4=A(x+1)+Bx$$

Let $x=0,\ 4=A$ Let $x=-1,\ B=-1$

$$\frac{3x+4}{x(x+1)}=\frac{4}{x}-\frac{1}{x+1}$$

Questions

① Express $\dfrac{2x+1}{x(x+1)}$ in partial fractions and hence find the general solution of the differential equation $x(x+1)\dfrac{dy}{dx}=y(2x+1)$ expressing y explicitly in terms of x.

② Express $\dfrac{2x}{x^2-1}$ in partial fractions and hence find the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{2xy}{x^2-1}$ expressing y explicitly in terms of x.

③ Find the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{xy}{(x+2)(x+1)}$ expressing y explicitly in terms of x.

Particular Solutions of Differential Equations

If we are given initial conditions for x and y we can find the particular solution of a differential equation.

Examples

① Find the particular solution of the differential equation $y^3\dfrac{dy}{dx}=3x^2-1$ given that $y=2$ when $x=1$.

$$y^3\frac{dy}{dx}=3x^2-1$$

$$y^3dy=\left(3x^2-1\right)dx$$

$$\int y^3dy=\int\left(3x^2-1\right)dx$$

$$\frac{1}{4}y^4=x^3-x+C$$

$$y^4=4x^3-4x+C$$

$$y=2 \text{ when } x=1 \ \Rightarrow \quad 2^4=4\times1^3-4\times1+C$$

$$C=16$$

$y^4=4x^3-4x+16$ is the **particular solution**.

② Find the particular solution of the differential equation $\frac{dy}{dx} = e^{(3x+2)}$ given that $y = e^2$ when $x = 0$.

$$\frac{dy}{dx} = e^{(3x+2)}$$

$$dy = e^{(3x+2)}dx$$

$$\int dy = \int e^{(3x+2)}dx$$

$$y = \frac{1}{3}e^{(3x+2)} + C$$

Apply the initial conditions

$$e^2 = \frac{1}{3}e^2 + C$$

$$C = \frac{2}{3}e^2$$

$$\therefore\ y = \frac{1}{3}e^{(3x+2)} + \frac{2}{3}e^2$$

$y = \frac{1}{3}\left(e^{(3x+2)} + 2e^2\right)$ is the particular solution.

③ Find the particular solution of the differential equation $\frac{dy}{dx} = 4xy^2$ given that $y = \frac{2}{3}$ when $x = 2$.

$$\frac{dy}{dx} = 4xy^2$$

$$\frac{dy}{y^2} = 4x\ dx$$

$$\int \frac{dy}{y^2} = \int 4x\ dx$$

$$-\frac{1}{y} = 2x^2 + C$$

$$y = \frac{1}{C - 2x^2}$$

Apply the initial conditions

$$\frac{2}{3} = \frac{1}{C - 2(2)^2}$$

$$\frac{2}{3} = \frac{1}{C - 8}$$

$$2C - 16 = 3$$

$$C = \frac{19}{2} \qquad \therefore\quad y = \frac{1}{\frac{19}{2} - 2x^2}$$

$y = \frac{2}{19 - 4x^2}$ is the particular solution.

④ Find the particular solution of the differential equation $(x+3)\frac{dy}{dx}+1=y$ given that $y=5$ when $x=-1$.

$$(x+3)\frac{dy}{dx}+1=y$$
$$(x+3)\frac{dy}{dx}=y-1$$
$$\frac{dy}{y-1}=\frac{dx}{x+3}$$
$$\int\frac{dy}{y-1}=\int\frac{dx}{x+3}$$
$$\ln(y-1)=\ln(x+3)+C$$
$$y-1=(x+3)e^C$$
$$y-1=A(x+3) \quad \text{where } a=e^C$$
$$y=A(x+3)+1$$

Apply the initial conditions $5=A(2)+1$

$A=2$

Particular solution is $y=2(x+3)+1$

$y=2x+7$

⑤ (a) Express $\frac{2}{y(y+2)}$ in partial fractions.

(b) Hence find the particular solution of the differential equation $2x\frac{dy}{dx}=y(y+2)$ given that $y=2$ when $x=1$.

(a) $\frac{2}{y(y+2)}=\frac{A}{y}+\frac{B}{y+2}$

$$2=A(y+2)+By$$

Let $y=0,\ A=1$ Let $y=-2,\ B=-1$

$$\frac{2}{y(y+2)}=\frac{1}{y}-\frac{1}{y+2}$$

(b) $2x\frac{dy}{dx} = y(y+2)$

$$\frac{2dy}{y(y+2)} = \frac{dx}{x}$$

$$\int \frac{2dy}{y(y+2)} = \int \frac{dx}{x}$$

$$\int \left(\frac{1}{y} - \frac{1}{y+2}\right) dy = \int \frac{dx}{x}$$

$$\ln y - \ln(y+2) = \ln x + C$$

$$\ln\left(\frac{y}{y+2}\right) = \ln x + C$$

$$\frac{y}{y+2} = xe^C$$

$$\frac{y}{y+2} = Ax \quad \text{where } A = e^C$$

$$y = Axy + 2Ax$$

$$y - Axy = 2Ax$$

$$y(1 - Ax) = 2Ax$$

$$y = \frac{2Ax}{1 - Ax}$$

Apply initial conditions $2 = \frac{2A(1)}{1 - A(1)}$

$$2 - 2A = 2A$$

$$A = \frac{1}{2}$$

$$y = \frac{2\left(\frac{1}{2}\right)x}{1 - \left(\frac{1}{2}\right)x}$$

$$y = \frac{x}{\frac{2 - x}{2}}$$

$y = \frac{2x}{2 - x}$ is the particular solution.

Questions

① Find the particular solution of each differential equation, expressing y explicitly in terms of x.

(a) $y^2\dfrac{dy}{dx}=2x^2+1 \qquad y=1 \text{ when } x=1$

(b) $x\dfrac{dy}{dx}=y+2 \qquad y=7 \text{ when } x=3$

(c) $(x+4)\dfrac{dy}{dx}+3=y \qquad y=13 \text{ when } x=1$

(d) $e^y\dfrac{dy}{dx}+\sin x=0 \qquad y=0 \text{ when } x=\dfrac{\pi}{2}$

(e) $\dfrac{dy}{dx}-2xy=0 \qquad y=3 \text{ when } x=0$

(f) $x\dfrac{dy}{dx}+y^2=0 \quad (x>0) \qquad y=\dfrac{1}{2} \text{ when } x=1$

② (a) Express $\dfrac{1}{y(y+1)}$ in partial fractions.

(b) Hence solve the differential equation $\dfrac{dy}{dx}=\dfrac{y(y+1)}{x}$ given that $y=4$ when $x=2$, expressing y explicitly in terms of x.

③ (a) Express $\dfrac{2}{y^2-1}$ in partial fractions.

(b) Hence solve the differential equation $2x\dfrac{dy}{dx}+1=y^2$ given that $y=-3$ when $x=1$, expressing y explicitly in terms of x.

④ The gradient of the tangent to a curve is given by $\dfrac{dy}{dx}=\dfrac{x^2+1}{y^2}$, $y\neq 0$.

The point $P(-1,1)$ lies on the curve.

(a) Find the equation of the tangent to the curve at P.

(b) Solve the differential equation to find the equation of the curve in the form $y=f(x)$.

⑤ The gradient of the tangent to a curve is given by $\dfrac{dy}{dx}=\dfrac{y}{x(x+1)}$.

The point $P(3,6)$ lies on the curve.

(a) Find the equation of the tangent to the curve at P.

(b) Solve the differential equation to find the equation of the curve in the form $y=f(x)$.

Differential Equations As Mathematical Models

Examples

① In a culture of bacteria, the rate of increase of bacteria at any given time is proportional to the number of bacteria present at that time. If x denotes the number of bacteria present after t days, the growth of bacteria in the culture is given by a differential equation of the form

$$\frac{dx}{dt} = kx \text{ where } k \text{ is a constant.}$$

(a) Find the general solution of this differential equation, expressing x as a function of t.

(b) Given that there are 350 bacteria present initially and that there are 1700 bacteria after 4 days, find the number of bacteria present after 1 week.

(a) The rate of increase of the bacteria is $\frac{dx}{dt}$. The number of bacteria present is x.

As $\frac{dx}{dt}$ is proportional to x we write this as $\frac{dx}{dt} = kx$ where k is a constant.

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = k\ dt$$

$$\int \frac{dx}{x} = \int k\ dt$$

$$\ln x = kt + C$$

$$x = e^{(kt+C)}$$

$$x = e^{kt}e^{C}$$

$x = Ae^{kt}$ is the general solution.

(b) Apply initial conditions : $t = 0,\ x = 350 \quad \Rightarrow \quad 350 = Ae^{k\times 0}$

$$350 = Ae^{0}$$

$$A = 350$$

$t = 4,\ x = 1700 \quad \Rightarrow \quad 1700 = Ae^{k\times 4}$

$$1700 = 350e^{4k}$$

$$e^{4k} = \left(\frac{1700}{350}\right)$$

$$4k = \ln\left(\frac{1700}{350}\right)$$

$$k = \frac{1}{4}\ln\left(\frac{1700}{350}\right)$$

$$k = 0\cdot 3951$$

So the particular solution is $x = 350e^{0\cdot3951t}$

When $t = 7,\quad x = 350e^{0\cdot3951\times7}$

$$x = 350e^{2\cdot7657}$$

$$x = 5562$$

So after 1 week there will be 5562 bacteria present.

② The mass m (grams) of a radioactive substance at time t (years) decreases at a rate which is proportional to the mass at that time.
The half-life of a radioactive substance is the time taken for half of the mass to decay.
The original mass of a radioactive substance is 700 grams and after 8 years it has decayed to 550 grams.
Calculate the half-life of this radioactive substance.

The rate of increase of the mass is $\frac{dm}{dt}$. The mass is m.

As $\frac{dm}{dt}$ is proportional to m we write this as $\frac{dm}{dt} = -km$. This time k is negative because the mass is decreasing.

$$\frac{dm}{dt} = -km$$

$$\frac{dm}{m} = -k\ dt$$

$$\int \frac{dm}{m} = \int -k\ dt$$

$$\ln m = -kt + C$$

$$m = e^{(-kt+C)}$$

$$m = e^{-kt}e^{C}$$

$m = Ae^{-kt}$ is the general solution.

Apply initial conditions : $t = 0,\ m = 700 \ \Rightarrow\ 700 = Ae^{-k\times0}$

$$700 = Ae^{0}$$

$$A=700$$

$t = 8,\ m = 550 \ \Rightarrow\ 550 = Ae^{-k\times8}$

$$550 = 700e^{-8k}$$

$$e^{-8k} = \frac{550}{700}$$

$$-8k = \ln\left(\frac{550}{700}\right)$$

$$k = \frac{-1}{8}\ln\left(\frac{550}{700}\right)$$

$$k = 0\cdot03015$$

So the particular solution is $m = 700e^{-0\cdot 03015t}$.

To find the half-life, make $m = 350$ and solve the equation for t.

$$350 = 700e^{-0\cdot 03015t}$$
$$e^{-0\cdot 03015t} = \frac{1}{2}$$
$$-0\cdot 03015t = \ln\left(\frac{1}{2}\right)$$
$$t \approx 23$$

So it will take approximately 23 years for half of the mass of the radioactive substance to decay.

③ Newton's law of cooling states that the rate at which an object cools is proportional to the difference between its temperature and the temperature of the surrounding environment.
When an object cools surrounded by air at a temperature of $25°\text{C}$, the cooling of the object is given by a differential equation of the form

$$\frac{dT}{dt} = -k\left(T - 25\right)$$

where $T°\text{C}$ is the temperature of the object after t minutes of cooling and k is a constant.

(a) Find the general solution of this differential equation, expressing T as a function of t.
(b) Given that a cup of liquid cools from $110°\text{C}$ to $80°\text{C}$ after 10 minutes in a room whose air temperature is $25°\text{C}$, find the
(i) the temperature of the liquid after another 15 minutes
(ii) the time taken for the liquid to fall to $40°\text{C}$.

$$\frac{dT}{dt} = -k\left(T - 25\right)$$
$$\frac{dT}{T - 25} = -k\ dt$$
$$\int \frac{dT}{T - 25} = \int -k\ dt$$
$$\ln\left(T - 25\right) = -kt + C$$
$$T - 25 = e^{-kt+C}$$
$$T - 25 = e^{-kt}e^{C}$$
$$T - 25 = Ae^{-kt} \qquad \text{where } A = e^{C}$$
$$T = 25 + Ae^{-kt} \qquad \text{is the general solution.}$$

(b) When $t = 0,\ \ T = 110 \quad \Rightarrow \quad 110 = 25 + Ae^{-k\times 0}$

$$110 = 25 + A$$

$$A = 85$$

When $t = 10,\ \ T = 80 \quad \Rightarrow \quad 80 = 25 + Ae^{-k\times 10}$

$$80 = 25 + 85e^{-10k}$$

$$55 = 85e^{-10k}$$

$$e^{-10k} = \frac{55}{85}$$

$$-10k = \ln\left(\frac{11}{17}\right)$$

$$k = \frac{-1}{10}\ln\left(\frac{11}{17}\right)$$

$$k = 0\cdot 0435$$

So $T = 25 + 85e^{-0\cdot 0435t}$

(i) When $t = 25,\ \ T = 25 + 85e^{-0\cdot 0435\times 25}$

$$T = 53.6°\text{C}$$

(ii) When $T = 40,\ \ 40 = 25 + 85e^{-0\cdot 0435\times t}$

$$15 = 85e^{-0\cdot 0435\times t}$$

$$e^{-0\cdot 0435\times t} = \frac{15}{85}$$

$$-0\cdot 0435t = \ln\left(\frac{15}{85}\right)$$

$$t = \frac{-1}{0\cdot 0435}\ln\left(\frac{15}{85}\right)$$

$$t = 39\cdot 9$$

It will take 39·9 minutes for the liquid to cool to 40°C.

Past Paper Questions

2001

A chemical plant food loses effectiveness at a rate proportional to the amount present in the soil. The amount M grams of plant food effective after t days satisfies the differential equation

$\frac{dM}{dt} = kM$, where k is a constant.

(a) Find the general solution for M in terms of t where the initial amount of food is M_0 grams.

(b) Find the value of k if, after 30 days, only half the initial amount of plant food is effective.

(c) What percentage of the original amount of plant food is effective after 35 days?

(d) The plant food has to be renewed when its effectiveness falls below 25%.
Is the manufacturer of the plant food justified in calling its product "sixty day super food"? **(3, 3, 2, 2 marks)**

2003

The volume $V(t)$ of a cell at time t changes according to the law

$\frac{dV}{dt} = V(10-V)$ for $0 < V < 10$

Show that $\frac{1}{10}\ln V - \frac{1}{10}\ln(10-V) = t + C$ for some constant C.

Given that $V(0) = 5$, show that $V(t) = \frac{10e^{10t}}{1+e^{10t}}$.

Obtain the limiting value of $V(t)$ as $t \to \infty$. **(4, 3, 2 marks)**

2007

A garden centre advertises young plants to be used as hedging.
After planting, the growth, G metres (ie the increase in height) after t years is modelled by the differential equation

$$\frac{dG}{dt}=\frac{25k-G}{25}$$

where k is a constant and $G=0$ when $t=0$.

(a) Express G in terms of t and k.

(b) Given that a plant grows $0\cdot6$ metres by the end of 5 years, find the value of k correct to 3 decimal places.

(c) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified?

(d) Given that the initial height of the plants was $0\cdot3$ m, what is the likely long-term height of the plants? **(4, 2, 2, 2 marks)**

2009

Given that $x^2e^y\frac{dy}{dx}=1$ and $y=0$ when $x=1$, find y in terms of x. **(4 marks)**

2011

Given that $y>-1$ and $x>-1$, obtain the general solution of the differential equation

$$\frac{dy}{dx}=3(1+y)\sqrt{1+x}$$

expressing your answer in the form $y=f(x)$. **(5 marks)**

2013

In an environment without enough resources to support a population greater than 1000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{dP}{dt}=P(1000-P)$$

Show that $\ln\frac{P}{1000-P}=1000t+C$ for some constant C.

Hence show that $P(t)=\frac{1000K}{K+e^{-1000t}}$ for some constant K.

Given that $P(0)=200$, determine at what time t, $P(t)=900$. **(4, 3, 3 marks)**

2015

Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli's Law states that the rate of change of volume, V , of the water in the tank is proportional to the sqare root of the height, h , of the water above the hole.

This is given by the differential equation:

$$\frac{dV}{dt} = -k\sqrt{h},\ k > 0$$

(a) For a cylindrical tank with constant cross-sectional area, A , show that the rate of change of the height of the water in the tank is given by

$$\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}\ .$$

(b) Initially, when the height of the water is 144cm, the rate at which the height is changing is -3 cm/hr.

By solving the differential equation in part (a), show that $h = \left(12 - \frac{1}{80}t\right)^2$.

(c) How many days will it take for the tank to empty?

(d) Given that the tank has radius 20cm, find the rate at which the water was being delivered to the vegetation (in cm^3/hr) at the end of the fourth day.

(2, 4, 2, 3 marks)

2016

A beaker of liquid was placed in a fridge.

The rate of cooling is given by $\frac{dT}{dt} = -k\left(T - T_F\right)$, where T_F is the constant temperature in the fridge and T is the temperature of the liquid at time t.

- The temperature in the fridge is constant at $4^\circ C$.
- When first placed in the fridge, the temperature of the liquid was $25^\circ C$.
- At 12 noon, the temperature of the liquid was $9\cdot 8^\circ C$.
- At 12:15pm the temperature of the liquid had dropped to $6\cdot 5^\circ C$.

At what time, to the nearest minute, was the liquid placed in the fridge?

(9 marks)