Differentiation

Chain Rule Revision

It is crucial to not forget about chain rule! While we will learn other rules for differentiation, chain rule is still a huge part of what will be done & you need to recognise it. The following questions will serve as reminders:

Questions

Differentiate

(a) $y=(5x+1)^3$ (b) $y=\cos(4x+1)$ (c) $y=(x^2+3x+1)^4$

(d) $y=\dfrac{3}{(1-4x)^2}$ (e) $y=\sin^3 x$ (f) $y=\cos(\cos x)$

We may also see some more difficult applications of chain rule.

Examples

① Differentiate $y=\cos^3 3x$

$$\begin{aligned} y &= \cos^3 3x \\ \frac{dy}{dx} &= 3(\cos 3x)^2 \times \frac{d}{dx}(\cos 3x) \\ &= 3\cos^2 3x \times (-3\sin 3x) \\ &= -9\cos^2 3x \sin 3x \end{aligned}$$

② Differentiate $f(x)=(x+\sin 3x)^2$

$$\begin{aligned} f(x) &= (x+\sin 3x)^2 \\ f'(x) &= 2(x+\sin 3x)\times\frac{d}{dx}(x+\sin 3x) \\ &= 2(x+\sin 3x)(1+\cos 3x) \end{aligned}$$

③ If $y=\cos^3(2x+4)$, find $\dfrac{dy}{dx}$.

$$y=\cos^3(2x+4)$$

$$\begin{aligned} \frac{dy}{dx} &= 3(\cos(2x+4))^2 \times \frac{d}{dx}(\cos(2x+4)) \\ &= 3(\cos(2x+4))^2 \times (-2\sin(2x+4)) \\ &= -6\cos^2(2x+4)\sin(2x+4)) \end{aligned}$$

④ If $f(x)=\dfrac{1}{\sin^2(3x+1)}$, find $f'(x)$.

$$f(x)=\left(\sin(3x+1)\right)^{-2}$$

$$\begin{aligned} f'(x) &= -2\left(\sin(3x+1)\right)^{-3}\times\frac{d}{dx}(\sin(3x+1)) \\ &= -2\sin^{-3}(3x+1)\times(3\cos(3x+1)) \\ &= \frac{-6\cos(3x+1)}{\sin^3(3x+1)} \end{aligned}$$

Product Rule

If we have two functions $f(x)$ and $g(x)$ which are both differentiable then if we have $h(x) = f(x)g(x)$ we can differentiate this using the product rule.

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

In Leibniz notation we have

$$y = u(x)v(x)$$

$$\frac{dy}{dx} = u'v + uv'$$

Examples

Differentiate

① $y = x^2 \cos x$ $\quad u = x^2 \quad v = \cos x$ $\quad u' = 2x \quad v' = -\sin x$

$$\frac{dy}{dx} = u'v + uv'$$
$$= 2x\cos x + x^2(-\sin x)$$
$$= 2x\cos x - x^2 \sin x$$

② $y = 4x^2(3+x)^5$ $\quad u = 4x^2 \quad v = (3+x)^5$ $\quad u' = 8x \quad v' = 5(3+x)^4$

$$\frac{dy}{dx} = u'v + uv'$$
$$= 8x(3+x)^5 + 4x^2(5(3+x)^4)$$
$$= 8x(3+x)^5 + 20x^2(3+x)^4$$

③ $y = (x+1)^2(2x-3)^3$ $\quad u = (x+1)^2 \quad v = (2x-3)^3$ $\quad u' = 2(x+1) \quad v' = 3(2x-3)^2.2 = 6(2x-3)^2$

$$\frac{dy}{dx} = u'v + uv'$$
$$= 2(x+1)(2x-3)^3 + 6(x+1)^2(2x-3)^2$$

④ $y = (x^2+x)\sin 2x$ $\quad u = x^2 + x \quad v = \sin 2x$ $\quad u' = 2x+1 \quad v' = 2\cos 2x$

$$\frac{dy}{dx} = u'v + uv'$$
$$= (2x+1)\sin 2x + (x^2+x)2\cos 2x$$
$$= (2x+1)\sin 2x + 2(x^2+x)\cos 2x$$

⑤ $y = \sin 3x \cos 2x$ $\quad u = \sin 3x \quad v = \cos 2x$ $\quad u' = 3\cos 3x \quad v' = -2\sin 2x$

$$\frac{dy}{dx} = u'v + uv'$$
$$= 3\cos 3x \cos 2x - 2\sin 3x \sin 2x$$

⑥ $y = x^5(x^3 - 2x^2)^2$

$u = x^5 \qquad v = (x^3 - 2x^2)^2$

$u' = 5x^4 \qquad v' = 2(x^3 - 2x^2)(3x^2 - 4x)$

$$\begin{aligned}\frac{dy}{dx} &= u'v + uv' \\ &= 5x^4(2(x^3 - 2x^2)(3x^2 - 4x)) + x^5(2(x^3 - 2x^2)(3x^2 - 4x)) \\ &= x^6(x-2)(11x-18)\end{aligned}$$

<u>Quotient Rule</u>

Similarly, if $f(x)$ and $g(x)$ are both differentiable then if we have $h(x) = \frac{f(x)}{g(x)}$ we can differentiate this using the quotient rule.

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

Or, in Leibniz notation

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

<u>Examples</u>

Differentiate

① $y = \frac{x^2}{x+1}$

$u = x^2 \qquad v = x+1$

$u' = 2x \qquad v' = 1$

$$\begin{aligned}\frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \\ &= \frac{2x(x+1) - x^2(1)}{(x+1)^2} \\ &= \frac{x(x+2)}{(x+1)^2}\end{aligned}$$

② $y = \frac{\sin x}{3x}$

$u = \sin x \qquad v = 3x$

$u' = \cos x \qquad v' = 3$

$$\begin{aligned}\frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \\ &= \frac{\cos x(3x) - \sin x(3)}{(3x)^2} \\ &= \frac{3x\cos x - 3\sin x}{9x^2} \\ &= \frac{x\cos x - \sin x}{3x^2}\end{aligned}$$

③ $y=\frac{3x-1}{x^2-x+1}$

$u=3x-1$ $\quad v=x^2-x+1$

$u'=3$ $\quad v'=2x-1$

$$\frac{dy}{dx}=\frac{u'v-uv'}{v^2}$$
$$=\frac{3(x^2-x+1)-(3x-1)(2x-1)}{(x^2-x+1)^2}$$
$$=\frac{-3x^2+2x+2}{(x^2-x+1)^2}$$

④ $y=\frac{2x^2}{\cos^2 x}$

$u=2x^2$ $\quad v=\cos^2 x$

$u'=4x$ $\quad v'=2\cos x(-\sin x)$

$\quad\quad =-\sin 2x$

$$\frac{dy}{dx}=\frac{u'v-uv'}{v^2}$$
$$=\frac{4x\cos^2 x-2x^2(-\sin 2x)}{(\cos^2 x)^2}$$
$$=\frac{4x\cos^2 x+2x^2\sin 2x}{\cos^4 x}$$

⑤ $y=\frac{2x+1}{\sqrt{2x+1}}$

$u=2x+1$ $\quad v=(2x+1)^{\frac{1}{2}}$

$u'=2$ $\quad v'=\frac{1}{2}(2x+1)^{\frac{-1}{2}}.2$

$$\frac{dy}{dx}=\frac{u'v-uv'}{v^2}$$
$$=\frac{2(2x+1)^{\frac{1}{2}}-(2x+1)(2x+1)^{\frac{-1}{2}}}{\left((2x+1)^{\frac{1}{2}}\right)^2}$$
$$=\frac{2(2x+1)^{\frac{1}{2}}-(2x+1)^{\frac{1}{2}}}{(2x+1)}$$
$$=(2x+1)^{\frac{-1}{2}}$$

Product Rule or Quotient Rule?

Some questions can be tackled by either rule, depending on how they are presented & your own preference. These questions can be tackled by either method - although one may prove more straightforward than the other.

Questions

(a) $y=\frac{1}{2x}\sin x$ (b) $y=\frac{(x^5+2)^2}{3x}$ (c) $y=\frac{3x^2}{\sin^2 x}$

Question c) by the product rule would be simpler has we been aware of three other functions which exist in mathematics.

Secant, Cosecant & Cotangent

These three functions are defined as follows;

Secant : $\sec x = \dfrac{1}{\cos x}$ Cosecant : $\operatorname{cosec} x = \dfrac{1}{\sin x}$ Cotangent : $\cot x = \dfrac{1}{\tan x}$

Remembering these three functions along with their derivatives can prove useful and make calculations quicker.

They will also allow us to add $\tan x$ to our list of trigonometric functions which we can differentiate.

$y = \sec x$ $u = 1$ $v = \cos x$

$= \dfrac{1}{\cos x}$ $u' = 0$ $v' = -\sin x$

$$\frac{d}{dx}(\sec x) = \frac{u'v - uv'}{v^2}$$
$$= \frac{0 - (-\sin x)}{(\cos x)^2}$$
$$= \frac{\sin x}{\cos x \cos x}$$
$$= \sec x \tan x$$

$y = \operatorname{cosec} x$ $u = 1$ $v = \sin x$

$= \dfrac{1}{\sin x}$ $u' = 0$ $v' = \cos x$

$$\frac{d}{dx}(\operatorname{cosec} x) = \frac{u'v - uv'}{v^2}$$
$$= \frac{0 - \cos x}{(\sin x)^2}$$
$$= \frac{-\cos x}{\sin x \sin x}$$
$$= -\operatorname{cosec} x \cot x$$

$y = \cot x$ $u = \cos x$ $v = \sin x$

$= \dfrac{1}{\tan x}$ $u' = -\sin x$ $v' = \cos x$

$= \dfrac{1}{\frac{\sin x}{\cos x}}$

$= \dfrac{\cos x}{\sin x}$

$$\frac{d}{dx}(\cot x) = \frac{u'v - uv'}{v^2}$$
$$= \frac{-\sin x(\sin x) - \cos x(\cos x)}{(\sin x)^2}$$
$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$
$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$
$$= \frac{-1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x$$

If you forget these they can be derived by quotient rule as shown.

These definitions & our knowledge of the quotient rule now allow us to calculate the derivative of $\tan x$.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \qquad u = \sin x \qquad v = \cos x$$

$$= \frac{u'v - uv'}{v^2} \qquad u' = \cos x \qquad v' = -\sin x$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

These are important results to remember

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cos ecx) = -\cos ecx \cot x \qquad \frac{d}{dx}(\cot x) = -\cos ec^2 x$$

Derivatives of Exponentials and Logarithms

Two definitions

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Again, we must remember about chain rule and ensure we apply it correctly. We can also use product and quotient rule with all of these new derivatives.

Examples

① $y = e^{4x}$

$$\frac{dy}{dx} = e^{4x} \times \frac{d}{dx}(4x)$$
$$= 4e^{4x}$$

② $y = \ln 6x$

$$\frac{dy}{dx} = \frac{1}{6x} \times \frac{d}{dx}(6x)$$
$$= \frac{1}{x}$$

③ $y = \sec(3x+1)$

$$\frac{dy}{dx} = \sec(3x+1)\tan(3x+1) \times \frac{d}{dx}(3x+1)$$
$$= 3\sec(3x+1)\tan(3x+1)$$

④ $y = e^{\cos x}$

$$\frac{dy}{dx} = e^{\cos x} \times \frac{d}{dx}(\cos x)$$
$$= -\sin x e^{\cos x}$$

⑤ $y = \ln(\tan x)$

$$\frac{dy}{dx} = \frac{1}{\tan x} \times \frac{d}{dx}(\tan x)$$
$$= \frac{\sec^2 x}{\tan x}$$
$$= \sec x \cos ecx$$

⑥ $y = \sin(\sec x)$

$$\frac{dy}{dx} = \cos(\sec x) \times \frac{d}{dx}(\sec x)$$
$$= \sec x \tan x \cos(\sec x)$$

⑦ $y = xe^{-2x}$ $\quad u = x \quad v = e^{-2x}$

$\frac{dy}{dx} = e^{-2x} - 2xe^{-2x}$ $\quad u' = 1 \quad v' = -2e^{-2x}$

$= (1-2x)e^{-2x}$

⑧ $y = \tan x \ln(3x+1)$ $\quad u = \tan x \quad v = \ln(3x+1)$

$\frac{dy}{dx} = \sec^2 x \ln(3x+1) + \frac{3\tan x}{3x+1}$ $\quad u' = \sec^2 x \quad v' = \frac{3}{3x+1}$

⑨ $y = \frac{\cos ecx}{e^{3x}}$ $\quad u = \cos ecx \quad v = e^{3x}$

$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ $\quad u' = -\cos ecx \cot x \quad v' = 3e^{3x}$

$$= \frac{-\cot x \cos ecxe^{3x} - \cos ecx(3e^{3x})}{(e^{3x})^2}$$

$$= \frac{-\cot x \cos ecx - 3\cos ecx}{e^{3x}}$$

$$= \frac{-\cos ecx(\cot x + 3)}{e^{3x}}$$

⑩ $y = \frac{\ln(2x\cos x)}{\tan x}$ $\quad u = \ln 2x + \ln\cos x \quad v = \tan x$

$y = \frac{\ln 2x + \ln\cos x}{\tan x}$ $\quad u' = \frac{1}{x} - \frac{\sin x}{\cos x} \quad v' = \sec^2 x$

$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ $\quad u' = \frac{1}{x} - \tan x$

$$= \frac{\left(\frac{1}{x} - \tan x\right)\tan x - \ln(2x\cos x)\sec^2 x}{\tan^2 x}$$

$$= \frac{\left(\frac{\tan x}{x} - \frac{\tan^2 x}{x}\right) - \frac{x\ln(2x\cos x)\sec^2 x}{x}}{\tan^2 x}$$

$$= \frac{\tan x - \tan^2 x - x\sec^2 x\ln(2x\cos x)}{x\tan^2 x}$$

⑪ $y = \frac{\tan x}{\sqrt{x}}$ $u = \tan x$ $v = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ $u' = \sec^2 x$ $v' = \frac{1}{2\sqrt{x}}$

$$= \frac{\sec^2 x(\sqrt{x}) - \frac{\tan x}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{x\sec^2 x - \tan x}{x^{\frac{3}{2}}}$$

Higher Derivatives

In mathematics we can sometimes be required or find it more useful in taking more than one derivative of a function. This will aid us in many areas this year, including curve sketching, Maclaurin series and differential equations. It is probably best shown by example.

Examples

① $f(x) = x^5$

$f'(x) = 5x^4$

$f''(x) = 20x^3$

$f'''(x) = 60x^2$

$f^{IV}(x) = 120x$

$f^{V}(x) = 120$

$f^{VI}(x) = 0$

And each subsequent derivative = 0

② $y = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x$

$\frac{d^2y}{dx^2} = -4\sin 2x$

$\frac{d^3y}{dx^3} = -8\cos 2x$

$\frac{d^4y}{dx^4} = 16\sin 2x$

etc........

③ Find the first to fourth derivatives of the function $f(x) = \sqrt{x}$

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$

$f''(x) = \frac{-1}{4}x^{\frac{-3}{2}} = \frac{-1}{4\sqrt{x^3}}$

$f'''(x) = \frac{3}{8}x^{\frac{-5}{2}} = \frac{3}{8\sqrt{x^5}}$

$f^{IV}(x) = \frac{-15}{16}x^{\frac{-7}{2}} = \frac{-15}{16\sqrt{x^7}}$

④ Calculate $\frac{dy}{dx}, \frac{d^2y}{dx^2} \& \frac{d^3y}{dx^3}$ if $y = \ln(1+x^2)$.

$y = \ln(1+x^2)$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \times \frac{d}{dx}(1+x^2)$$
$$= \frac{2x}{1+x^2}$$

Now we need to use quotient rule to go further.

$$\frac{d^2y}{dx^2} = \frac{u'v - uv'}{v^2}$$
$$= \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2}$$
$$= \frac{2-2x^2}{(1+x^2)^2}$$

$u = 2x \qquad v = 1+x^2$

$u' = 2 \qquad v' = 2x$

$$\frac{d^3y}{dx^3} = \frac{u'v - uv'}{v^2}$$
$$= \frac{-4x(1+x^2)^2 - 2(1-x^2)4x(1+x^2)}{\left((1+x^2)^2\right)^2}$$
$$= \frac{-4x - 4x^3 - 8x + 8x^3}{(1+x^2)^3}$$
$$= \frac{-12x + 4x^3}{(1+x^2)^3}$$
$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

$u = 2-2x^2 \qquad v = (1+x^2)^2$

$u' = -4x \qquad v' = 4x(1+x^2)$

⑤ Calculate $\frac{dy}{dx} \& \frac{d^2y}{dx^2}$ if $y = e^{\cos x}$.

$y = e^{\cos x}$

$$\frac{dy}{dx} = -\sin x e^{\cos x}$$

$$\frac{d^2y}{dx^2} = u'v + uv'$$
$$= -\cos x e^{\cos x} + \sin^2 x e^{\cos x}$$

$u = -\sin x \qquad v = e^{\cos x}$

$u' = -\cos x \qquad v' = -\sin x e^{\cos x}$

Rectinlinear Motion (Motion in a straight line)

This considers a body moving in a straight line (along an *x* axis which we set).

Its displacement from the origin if denoted either by *x* or *s* as a function of *t* {*x(t); s(t)*}

Velocity is the rate of change of displacement with time, thus $v = \frac{dx}{dt}; v(t) = x'(t)$.

Acceleration (which is not necessarily constant), is the rate of change of velocity with respect to time.

i.e. $a = \frac{dv}{dt} = \frac{d^2v}{dt^2}$ $a(t) = v'(t) = x''(t).$

displacement → velocity → acceleration

differentiate w.r.t. time

Questions

① A particle travels along the *x* axis such that $x(t) = 2t^3 - 3t + 6$ where x represents its displacement in metres from the origin t seconds after observations began.

(a) Where is the particle initially?

(b) After 4 seconds what are the velocity and acceleration of the particle?

② Find the position and acceleration of a particle when it comes to rest if its displacement is given by $x = 10t - e^t$.

③ Part of a journey an object made was observed. The displacement, x metres, of the object travelling in a straight line at time t seconds is given as $x = \frac{t^3}{3} + t^2 - 8t + 10$

(a) How far from the origin was the object when its observation was started?

(b) At what time was the object stationary?

(c) Comment on the motion of the object after 5 seconds.

(d) Does the object ever reach a constant velocity or decelerate during the journey? Justify your answer.

Differentiation Sheet 3 contains more questions for practise.

Past Paper Questions

2001

Differentiate with respect to x $g(x) = e^{\cot 2x},\ o < x < \frac{\pi}{2}.$ **(2 marks)**

2002

Given that $f(x) = \sqrt{x}e^{-x},\ x \geq 0$, obtain and simplify $f'(x)$. **(4 marks)**

2003

Given $f(x) = x(1+x)^{10}$, obtain $f'(x)$ and simplify your answer. **(3 marks)**

2004

Given $f(x) = \cos^2 x e^{\tan x}, \frac{-\pi}{2} < x < \frac{\pi}{2}$, obtain $f'(x)$ and evaluate $f'\left(\frac{\pi}{4}\right)$. **(3,1 marks)**

2005

(a) Given $f(x) = x^3 \tan 2x$, where $0 < x < \frac{\pi}{4}$, obtain $f'(x)$. **(3 marks)**

(b) For $y = \frac{1+x^2}{1+x}$, where $x \neq -1$, determine $\frac{dy}{dx}$ in simplified form. **(3 marks)**

2006

Differentiate, simplifying your answer: $\frac{1+\ln x}{3x}$, where $x > 0$. **(3 marks)**

2007

Obtain the derivative of the function $f(x) = \exp(\sin 2x)$. **(3 marks)**

2009

Given $f(x) = (x+1)(x-2)^3$, obtain the values of x for which $f'(x) = 0$. **(3 marks)**

2010

Differentiate the following functions

(a) $f(x) = e^x \sin x^2$. **(3 marks)**

(b) $g(x) = \frac{x^3}{1+\tan x}$. **(3 marks)**

2011

Given $f(x) = \sin x \cos^3 x$, obtain $f'(x)$. **(3 marks)**

2012

(a) Given $f(x) = \frac{3x+1}{x^2+1}$, obtain $f'(x)$. **(3 marks)**

(b) Let $g(x) = \cos^2 x \exp(\tan x)$. Obtain an expression for $g'(x)$ and simplify your answer. **(4 marks)**

2013

Differentiate $f(x) = e^{\cos x} \sin^2 x$. **(3 marks)**

2014

Given $f(x) = \frac{x^2-1}{x^2+1}$, obtain $f'(x)$ and simplify your answer. **(3 marks)**

2015

(a) For $y = \frac{5x+1}{x^2+2}$, find $\frac{dy}{dx}$. Express your answer as a single, simplified fraction. **(3 marks)**

(b) Given $f\left(x\right) = e^{2x}\sin^2 3x$, obtain $f'\left(x\right)$. **(3 marks)**

2016

(a) Differentiate $y = x\tan^{-1} 2x$ **(3 marks)**

(b) Given $f(x) = \frac{1-x^2}{1+4x^2}$, find $f'(x)$, simplifying your answer. **(3 marks)**