Further Differential Equations

We already know how to solve first order separable differential equations.

We will now study more complex differential equations.

First Order Linear Differential Equations

A first order linear differential equation takes the form $\frac{dy}{dx}+P(x)y=f(x)$

e.g. $\frac{dy}{dx}+2xy=e^{x}$ $\quad \left(P(x)=2x, \quad f(x)=e^{x}\right)$

$\frac{dy}{dx}+(\sin x)y=x^{2}$ $\quad \left(P(x)=\sin x, \quad f(x)=x^{2}\right)$

We may need to perform some manipulation to get the equation in this form.

e.g.
$$x\frac{dy}{dx}=y+3x^{2}$$
$$\frac{dy}{dx}=\frac{y+3x^{2}}{x}$$
$$\frac{dy}{dx}=\frac{y}{x}+3x$$
$$\frac{dy}{dx}-\frac{1}{x}y=3x \qquad \left(P(x)=\frac{-1}{x}, \quad f(x)=3x\right)$$

The method we use to solve this type of differential equation is called the **integrating factor method.**

The integrating factor is $I(x)=e^{\int P(x)dx}$.

The strategy for solving these comes from the following.

$$I(x)=e^{\int P(x)dx}$$
$$\frac{d}{dx}\left(I(x)\right)=\frac{d}{dx}\left(e^{\int P(x)dx}\right)$$
$$=P(x)e^{\int P(x)dx} \quad \text{(by the Chain Rule)}$$
$$=P(x)I(x)$$

We now take the equation $\frac{dy}{dx}+P(x)y=f(x)$.

If we multiply both sides of our equation by $I(x)$ we have

$$I(x)\frac{dy}{dx}+I(x)P(x)y=I(x)f(x)$$

The LHS is $\frac{d}{dx}\left(I(x)y\right)$ by the product rule as $\frac{d}{dx}\left(I(x)y\right)=\frac{d}{dx}\left(I(x)\right)\cdot y+I(x)\frac{dy}{dx}$

$$=P(x)I(x)y+I(x)\frac{dy}{dx}$$

$$\therefore \frac{d}{dx}\left(I(x)y\right)=I(x)f(x)$$

Now we can integrate both sides of this equation.

$$\int\left(I(x)\frac{dy}{dx}+I(x)P(x)y\right)dx=\int\left(I(x)f(x)\right)dx$$

$$\int\left(\frac{d}{dx}\left(I(x)y\right)\right)=\int\left(I(x)f(x)\right)dx$$

$$I(x)y=\int\left(I(x)f(x)\right)dx$$

This can now be solved for y.

This is why the integrating factor method works. It will be much clearer how it works by example.

Examples

① Find the general solution of $\frac{dy}{dx}+2y=e^{3x}$

Here $P(x)=2$ so the integrating factor is $e^{\int 2dx}=e^{2x}$.

Multiply both sides of the differential equation by the integrating factor.

$$e^{2x}\left(\frac{dy}{dx}+2y\right)=e^{2x}e^{3x}$$

$$\frac{d}{dx}\left(e^{2x}y\right)=e^{5x}$$

Now integrate both sides. $\int\frac{d}{dx}\left(e^{2x}y\right)dx=\int e^{5x}dx$

$$e^{2x}y=\frac{1}{5}e^{5x}+C$$

$$y=\frac{1}{5}e^{3x}+Ce^{-2x}$$

The general solution is $y=\frac{1}{5}e^{3x}+Ce^{-2x}$

② Find the general solution of the differential equation $\frac{dy}{dx}+\frac{y}{x}=4x^2$.

$P(x)=\frac{1}{x}$ so the integrating factor is $e^{\int\frac{1}{x}dx}=e^{\ln x}=x$.

Multiply both sides of the differential equation by the integrating factor.

$$x\left(\frac{dy}{dx}+\frac{y}{x}\right)=4x^3$$

$$\frac{d}{dx}(xy)=4x^3$$

Integrate both sides. $$\int \frac{d}{dx}(xy)dx=\int 4x^3dx$$

$$xy=x^4+C$$

$$y=x^3+\frac{C}{x}$$

The general solution is $y=x^3+\frac{C}{x}$.

③ Find the general solution of $\frac{dy}{dx}=\frac{xy+4}{x^2}$.

The equation must firstly be rearranged into the correct form.

$$x^2\frac{dy}{dx}=xy+4$$

$$\frac{dy}{dx}-\frac{1}{x}y=\frac{4}{x^2}$$

$P(x)=\frac{-1}{x}$ so I.F.$=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=e^{\ln x^{-1}}=\frac{1}{x}$

Multiply both sides of the differential equation by the integrating factor.

$$\frac{1}{x}\left(\frac{dy}{dx}-\frac{1}{x}y\right)=\frac{4}{x^3}$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right)=4x^{-3}$$

Integrate both sides. $$\int \frac{d}{dx}\left(\frac{1}{x}y\right)dx=\int 4x^{-3}dx$$

$$\frac{y}{x}=-2x^{-2}+C$$

$$y=\frac{-2}{x^2}+Cx$$

The general solution is $y=Cx-\frac{2}{x}$.

④ Find the general solution of the differential equation $\frac{dy}{dx}+2xy=5x$.

$P(x)=2x$ so the I.F.$=e^{\int 2xdx}=e^{x^2}$.

Multiply both sides of the differential equation by the integrating factor.

$$e^{x^2}\left(\frac{dy}{dx}+2xy\right)=5xe^{x^2}$$

$$\frac{d}{dx}\left(e^{x^2}y\right)=5xe^{x^2}$$

Integrate both sides $\int\frac{d}{dx}\left(e^{x^2}y\right)dx=\int 5xe^{x^2}dx$ Let $u=x^2$, $\frac{du}{dx}=2x$

$$e^{x^2}y=\frac{5}{2}\int e^u du \qquad \frac{5}{2}du=5xdx$$

$$e^{x^2}y=\frac{5}{2}e^{x^2}+C$$

The general solution is $y=\frac{5}{2}+\frac{C}{e^{x^2}}$.

Questions

Find the general solution of the following differential equations.

① $\frac{dy}{dx}+2y=1$ ② $\frac{dy}{dx}-2y=1-2x$ ③ $x\frac{dy}{dx}-2y=6x^4$ ④ $\frac{dy}{dx}-4xy=10x$

Particular Solutions of Differential Equations

If we are given initial conditions for x and y we can find the particular solution of a differential equation.

Examples

① Find the unique solution of the differential equation $x^2\frac{dy}{dx}-x^3+x^2y=0$ if when $x=1$, $y=1$.

Rearrange into correct form...... $\frac{dy}{dx}-x+y=0$

$$\frac{dy}{dx}+y=x$$

$P(x)=1$ so I.F. $=e^{\int 1dx}=e^x$

Multiply both sides of the differential equation by the integrating factor.

$$e^x\left(\frac{dy}{dx}+y\right)=xe^x$$

$$\frac{d}{dx}\left(e^xy\right)=xe^x$$

Integrate both sides $\int \frac{d}{dx}\left(e^x y\right)dx = \int xe^x dx$ $\quad u = x \quad v' = e^x \quad u' = 1 \quad v = e^x$

$$e^x y = xe^x - \int e^x dx$$

$$e^x y = xe^x - e^x + C$$

$$y = x - 1 + Ce^{-x}$$

Apply the initial conditions $1 = 1 - 1 + Ce^{-1}$

$$C = e$$

$$y = x - 1 + ee^{-x}$$

So the unique solution is $y = x - 1 + e^{1-x}$

② Find the particular solution of the differential equation $x\frac{dy}{dx} - y = x^2$ given that $x = 1$ when $y = 0$.

Rearrange into correct form ……$\frac{dy}{dx} - \frac{y}{x} = x$

$P(x) = \frac{-1}{x}$ so I.F. $= e^{\int \frac{-1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

Multiply both sides of the differential equation by the integrating factor.

$$\frac{1}{x}\left(\frac{dy}{dx} - \frac{y}{x}\right) = 1$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 1$$

Integrate both sides $\int \frac{d}{dx}\left(\frac{1}{x}y\right)dx = \int 1 dx$

$$\frac{1}{x}y = x + C$$

$$y = x^2 + Cx$$

Apply the initial conditions $0 = 1^2 + C$

$$C = -1$$

So the particular solution is $y = x^2 - x$.

Questions

Find the particular solutions of the following differential equations.

① $\frac{dy}{dx} - 3y = x$ given $x = 0$ when $y = 1$

② $\frac{dy}{dx} + y\tan x = \sec x$ given $y(0) = 1$

③ $\frac{dy}{dx} = \frac{-3}{x^2}y + e^{\frac{3}{x}}$ given $x = 3$ when $y = 5e$

④ $\frac{dy}{dx} - y\cos x = 2xe^{\sin x}$ given $y = 0$ when $x = \pi$

Second Order Linear Differential Equations

A second order differential equation takes the form

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where a, b and c are constants and $a \neq 0$.

There are 2 different cases.

If $f(x) = 0$, this is a homogenous equation.

If $f(x) \neq 0$, this is a non-homogenous equation.

Homogenous Equations

As these equations involve a second derivative the general solution will contain two arbitrary constants.

The method we use to find the general solution is to consider a solution of the form $y = Ae^{mx}$ for some values of m (to be calculated) and A (to be calculated if given initial conditions).

$y = Ae^{mx}$

$\frac{dy}{dx} = mAe^{mx}$

$\frac{d^2y}{dx^2} = m^2Ae^{mx}$

Substitute these into $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

$$am^2Ae^{mx} + bmAe^{mx} + cAe^{mx} = 0$$ divide through by Ae^{mx} to get

$$am^2 + bm + c = 0$$

This is known as the auxiliary equation. It can be factorised to solve for m.

There are 3 possible cases when solving the auxiliary equation-

$b^2 - 4ac > 0$ two real distinct roots

$b^2 - 4ac = 0$ repeated real roots

$b^2 - 4ac < 0$ complex roots.

Example - two real distinct roots

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$.

The auxiliary equation is $m^2 - 4m + 3 = 0$

$$(m-1)(m-3) = 0$$

$$m = 1 \text{ or } m = 3$$

The general solution is $y = Ae^x + Be^{3x}$

Example - real repeated roots

If the roots are equal we must consider $y = Ae^{mx}$ as one solution and $y = Bxe^{mx}$ as the other.

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

The auxiliary equation is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2 \quad \text{(twice)}$$

The general solution is $y = Ae^{2x} + Bxe^{2x}$.

Example - complex roots

If the roots are complex we will need to use the quadratic formula and the roots will be in the form $p \pm qi$. The general solution is given by $y = e^{px}\left(A\cos qx + B\sin qx\right)$.

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$.

The auxiliary equation is $m^2 + 2m + 5 = 0$.

$$m = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2}$$

$$m = \frac{-2 \pm \sqrt{-16}}{2}$$

$$m = -1 \pm \frac{1}{2}\sqrt{16i^2}$$

$$m = -1 \pm 2i$$

The general solution is $y = e^{-x}\left(A\cos 2x + B\sin 2x\right)$.

Questions

Find the general solutions of the following differential equations.

① $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=0$ ② $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$ ③ $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$

④ $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=0$ ⑤ $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+17y=0$ ⑥ $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}=3y$

Particular Solutions to Second Order Differential Equations

In a similar way to before we can apply initial conditions to find the values of the constants A and B. We have to differentiate the general solution to be able to do this.

Examples

① Find the particular solution of $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$ given that when $x=0$, $y=0$ and $\frac{dy}{dx}=2$.

Auxiliary equation is $m^2-3m+2=0$.

$$(m-1)(m-2)=0$$

$$m=1 \text{ or } m=2$$

General solution is $y=Ae^x+Be^{2x}$.

Apply initial conditions : $y=Ae^x+Be^{2x} \quad \to 0=A+B$

$$\frac{dy}{dx}=Ae^x+2Be^{2x} \quad \to 2=A+2B \quad \Rightarrow A=-2, \quad B=2$$

Particular solution is $y=2e^{2x}-2e^x$

② Find the particular solution of $\frac{d^2y}{dx^2}-10\frac{dy}{dx}+25y=0$ given that when $x=0$, $y=1$ and $\frac{dy}{dx}=2$.

Auxiliary equation is $m^2-10m+25=0$

$$(m-5)^2=0$$

$$m=5 \quad \text{(twice)}$$

General solution is $y=Ae^{5x}+Bxe^{5x}$.

Apply initial conditions: $y=Ae^{5x}+Bxe^{5x} \qquad \rightarrow 1=A$

$$\frac{dy}{dx}=5Ae^{5x}+Be^{5x}+5Bxe^{5x} \quad \rightarrow 2=5+B$$

$$B=-3$$

Particular solution is $y=e^{5x}-3xe^{5x}$.

③ Find the particular solution of the differential equation $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}+9y=0$ given that when $x=0$, $y=6$ and $\frac{dy}{dx}=1$.

Auxiliary equation is $4m^2+4m+9=0$

$$m=\frac{-4\pm\sqrt{4^2-4\times4\times9}}{8}$$

$$m=\frac{-1}{2}\pm\frac{1}{8}\sqrt{-128}$$

$$m=\frac{-1}{2}\pm\sqrt{2}i$$

General solution is $y=e^{\frac{-x}{2}}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right)$

Apply initial conditions: $y=e^{\frac{-x}{2}}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right) \quad \rightarrow 6=e^0\left(A\cos 0+B\sin 0\right)$

$$A=6$$

$$\frac{dy}{dx}=\frac{-e^{\frac{-x}{2}}}{2}\left(A\cos\sqrt{2}x+B\sin\sqrt{2}x\right)+e^{\frac{-x}{2}}\left(-\sqrt{2}A\sin\sqrt{2}x+\sqrt{2}B\cos\sqrt{2}x\right)$$

$$\rightarrow 1=\frac{-1}{2}\left(6+B(0)\right)+\left(-\sqrt{2}A(0)+\sqrt{2}B\right)$$

$$1=-3+\sqrt{2}B$$

$$B=\frac{4}{\sqrt{2}}=2\sqrt{2}$$

Particular solution is $y=e^{\frac{-x}{2}}\left(6\cos\sqrt{2}x+2\sqrt{2}\sin\sqrt{2}x\right)$

Questions

Find the particular solution of the following differential equations.

① $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=0$ given that when $x=0$, $y=1$ and $\frac{dy}{dx}=3$

② $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$ given that when $x=0$, $y=2$ and $\frac{dy}{dx}=0$

③ $4\frac{d^2y}{dx^2}-20\frac{dy}{dx}+25y=0$ given that when $x=0$, $y=2$ and when $x=2$, $y=3e^5$

④ $16\frac{d^2y}{dx^2}-9y=0$ given that when $x=0$, $y=1$ and $\frac{dy}{dx}=3$

⑤ $\frac{d^2y}{dx^2}-\frac{dy}{dx}+y=0$ given that when $x=0$, $y=1$ and $\frac{dy}{dx}=4$

Non-Homogenous Equations

When we are dealing with a non-homogenous equation we initially deal with it as if it were homogenous.
We set the RHS=0 to obtain the auxiliary equation which we then solve for m. We then use these values of m to obtain the **complementary function**.
To deal with the RHS we need something called a **particular integral** which should take the form (or similar form) to the RHS.
Finally-

general solution = complementary function + particular integral

Some examples will make this clearer.

Examples

① Find the general solution of the differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=4e^{2x}$.

The corresponding homogenous equation is $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$.

The auxiliary equation is $m^2+m-2=0$

$$(m-1)(m+2)=0$$

$$m=1 \text{ or } m=-2$$

The complementary function is $y=Ae^{x}+Be^{-2x}$.

The particular integral is of the form $y=ke^{2x}$.

$$\frac{dy}{dx}=2ke^{2x}$$

$$\frac{d^2y}{dx^2}=4ke^{2x}$$

Substitute these into the differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=4e^{2x}$

$$4ke^{2x}+2ke^{2x}-2ke^{2x}=4e^{2x}$$

$$4ke^{2x}=4e^{2x}$$

$$k=1$$

So the particular integral is $y=e^{2x}$.

The general solution is $y=Ae^{x}+Be^{-2x}+e^{2x}$.

② Find the general solution of the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=2x$.

The corresponding homogenous equation is $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$.

The auxiliary equation is $m^2-3m+2=0$

$$(m-1)(m-2)=0$$

$$m=1 \text{ or } m=2$$

The complementary function is $y=Ae^{2x}+Be^{x}$

The particular integral is of the form $y=ax+b$

$$\frac{dy}{dx}=a$$

$$\frac{d^2y}{dx^2}=0$$

Substitute these into the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=2x$

$$0-3a+2(ax+b)=2x$$

$$2ax+2b-3a=2x$$

Equating coeffiecients: $2a=2,\quad 2b-3a=0$

$$a=1 \qquad 2b-3=0$$

$$b = \frac{3}{2}$$

So the particular integral is $y = x + \frac{3}{2}$.

The general solution is $y = Ae^{2x} + Be^{x} + x + \frac{3}{2}$.

③ Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x^2 + 3$.

The corresponding homogenous equation is $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$

The auxiliary equation is $m^2 - 5m + 4 = 0$

$$(m-1)(m-4) = 0$$

$$m = 1 \text{ or } m = 4$$

The complementary function is $y = Ae^{4x} + Be^{x}$.

The particular integral is of the form $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

Substitute these into the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x^2 + 3$

$$2a - 5(2ax + b) + 4(ax^2 + bx + c) = 8x^2 + 3$$

$$4ax^2 + (4b - 10a)x + (2a - 5b + 4c) = 8x^2 + 3$$

Equating coefficients: $4a = 8$ $\quad 4b - 10a = 0$ $\quad 2a - 5b + 4c = 3$

$a = 2$ $\quad 4b = 20$ $\quad 4 - 25 + 4c = 3$

$b = 5$ $\quad c = 6$

So the particular integral is $y = 2x^2 + 5x + 6$.

The general solution is $y = Ae^{4x} + Be^{x} + 2x^2 + 5x + 6$

④ Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5\sin 2x$

The corresponding homogenous equation is $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$.

The auxiliary equation is $m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 2}}{2}$$

$$m = -1 \pm \frac{1}{2}\sqrt{4i^2}$$

$$m = -1 \pm i$$

The complementary function is $y = e^{-x}\left(A\cos x + B\sin x\right)$

The particular integral is of the form $y = p\sin 2x + q\cos 2x$

$$\frac{dy}{dx} = 2p\cos 2x - 2q\sin 2x$$

$$\frac{d^2y}{dx^2} = -4p\sin 2x - 4q\cos 2x$$

Substitute these into the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5\sin 2x$

$$-4p\sin 2x - 4q\cos 2x + 2\left(2p\cos 2x - 2q\sin 2x\right) + 2\left(p\sin 2x + q\cos 2x\right) = 5\sin 2x$$

$$\left(-4q - 2p\right)\sin 2x + \left(4p - 2q\right)\cos 2x = 5\sin 2x$$

Equating coefficients: $4p - 2q = 0$ $\quad$ $-4q - 2p = 5$

$2q = 4p$ $\quad$ $-2\left(4p\right) - 2p = 5$

$$-10p = 5$$

$$p = \frac{-1}{2}, \quad q = -1$$

So the particular integral is $y = \frac{-1}{2}\sin 2x - \cos 2x$

The general solution is $y = e^{-x}\left(A\cos x + B\sin x\right)\frac{-1}{2}\sin 2x - \cos 2x$

⑤ Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{4x} + \cos x$.

The corresponding homogenous equation is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

The auxiliary equation is $m^2 - 3m + 2 = 0$.

$$\left(m-1\right)\left(m-2\right) = 0$$

$m = 1$ or $m = 2$

The complementary function is $y = Ae^{2x} + Be^{x}$.

The particular integral is of the form $y = ae^{4x} + p\cos x + q\sin x$

$$\frac{dy}{dx} = 4ae^{4x} - p\sin x + q\cos x$$

$$\frac{d^2y}{dx^2} = 16ae^{4x} - p\cos x - q\sin x$$

Substitute these into the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{4x} + \cos x$

$$16ae^{4x} - p\cos x - q\sin x - 3\left(4ae^{4x} - p\sin x + q\cos x\right) + 2\left(ae^{4x} + p\cos x + q\sin x\right) = e^{4x} + \cos x$$

$$6ae^{4x} + (p - 3q)\cos x + (3p + q)\sin x = e^{4x} + \cos x$$

Equating coefficients: $6a = 1$ $\quad 3p + q = 0$ $\quad p - 3q = 1$

$a = \frac{1}{6}$ $\quad q = -3p$ $\quad p + 9p = 1$

$p = \frac{1}{10}$, $\quad q = \frac{-3}{10}$

So the particular integral is $y = \frac{1}{6}e^{4x} + \frac{1}{10}\cos x - \frac{3}{10}\sin x$.

The general solution is $y = Ae^{2x} + Be^{x} + \frac{1}{6}e^{4x} + \frac{1}{10}\cos x - \frac{3}{10}\sin x$

Questions

Find the general solution of each of the following

① $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = \sin x$

② $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 22e^{2x}$

③ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 9x^2 - 3x$

④ $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = x$

When the 'Normal' Particular Integral is in the Complementary Function

The particular integral cannot have the same form as either term of the complementary function. If it is, we should multiply the particular integral by x.
If this is also part of the complementary function (e.g. in the case of a repeated root) we should multiply the particular integral by x^2.

Examples

① Find the general solution of the equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=8e^{2x}$.

The corresponding homogenous equation is $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$

The auxiliary equation is $m^2-5m+6=0$

$$(m-2)(m-3)=0$$

$$m=2 \text{ or } m=3$$

The complementary function is $y=Ae^{2x}+Be^{3x}$

For the particular integral we cannot use $y=ke^{2x}$ as this is already part of the complementary function so we use $y=kxe^{2x}$.

$$\frac{dy}{dx}=ke^{2x}+2kxe^{2x}$$

$$\frac{d^2y}{dx^2}=2ke^{2x}+2\left(ke^{2x}+2kxe^{2x}\right)$$

$$=4ke^{2x}+4kxe^{2x}$$

Substitute these into the differential equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=8e^{2x}$.

$$4ke^{2x}+4kxe^{2x}-5\left(ke^{2x}+2kxe^{2x}\right)+6kxe^{2x}=8e^{2x}$$

$$-ke^{2x}=8e^{2x}$$

$$k=-8$$

So the particular integral is $y=-8xe^{2x}$.

The general solution is $y=Ae^{2x}+Be^{3x}-8xe^{2x}$.

NB If you pick the wrong particular integral a contradiction will occur at some stage in the calculation and this should flag it up.

② Find the general solution of the differential equation $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=6e^{4x}$.

The corresponding homogenous equation is $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=0$.

The auxiliary equation is $m^2-8m+16=0$.

$$(m-4)^2=0$$

$$m = 4 \text{ (twice)}$$

The complementary function is $y = Ae^{4x} + Bxe^{4x}$.

For the particular integral we cannot use $y = ke^{4x}$ or $y = kxe^{4x}$ as these are both already in the complementary function so use $y = kx^2e^{4x}$.

$$\frac{dy}{dx} = 2kxe^{4x} + 4kx^2e^{4x}$$

$$\frac{d^2y}{dx^2} = 2ke^{4x} + 8kxe^{4x} + 8kxe^{4x} + 16kx^2e^{4x}$$

$$= 2ke^{4x} + 16kxe^{4x} + 16kx^2e^{4x}$$

Substitute these into the differential equation $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 6e^{4x}$.

$$2ke^{4x} + 16kxe^{4x} + 16kx^2e^{4x} - 8\left(2kxe^{4x} + 4kx^2e^{4x}\right) + 16kx^2e^{4x} = 6e^{4x}$$

$$2ke^{4x} = 6e^{4x}$$

$$2k = 6$$

$$k = 3$$

So the particular integral is $y = 3x^2e^{4x}$.

The general solution is $y = Ae^{4x} + Bxe^{4x} + 3x^2e^{4x}$.

③ Find the general solution of the differential equation $4\frac{d^2y}{dx^2} + 36y = \sin 3x$.

The corresponding homogenous equation is $4\frac{d^2y}{dx^2} + 36y = 0$.

The auxiliary equation is $4m^2 + 36 = 0$.

$$m^2 = -9$$

$$m = \pm 3i$$

The complementary function is $y = e^0\left(A\cos 3x + B\sin 3x\right)$

$$y = A\cos 3x + B\sin 3x$$

For the particular integral try $y = x\left(p\cos 3x + q\sin 3x\right)$.

$$\frac{dy}{dx} = p\cos 3x + q\sin 3x + x\left(-3p\sin 3x + 3q\cos 3x\right)$$

$$\frac{d^2y}{dx^2} = -6p\sin 3x + 6q\cos 3x - 9xp\cos 3x - 9xq\sin 3x$$

Substitute these into the equation $4\frac{d^2y}{dx^2} + 36y = \sin 3x$.

$$4\left(-6p\sin 3x + 6q\cos 3x - 9xp\cos 3x - 9xq\sin 3x\right) + 36x\left(p\cos 3x + q\sin 3x\right) = \sin 3x$$

$$-24p\sin 3x + 24q\cos 3x = \sin 3x$$

Equating terms: $-24p = 1$ $\quad 24q = 0$

$$p = \frac{-1}{24} \qquad q = 0$$

So the particular integral is $y = \frac{-x}{24}\cos 3x$.

The general solution is $y = A\cos 3x + B\sin 3x \frac{-x}{24}\cos 3x$.

Past Paper Questions

2001

① Find the solution of the following differential equation:

$$\frac{dy}{dx} + \frac{y}{x} = x, \qquad x > 0$$ **(4 marks)**

② Find the general solution of the following differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x - 1$$ **(5 marks)**

2002

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4\cos x$$

Hence determine the solution which satisfies $y(0) = 0$ and $y'(0) = 1$. **(6, 4 marks)**

2003

Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$$

given that $y = 2$ and $\frac{dy}{dx} = 1$, when $x = 0$. **(10 marks)**

2004

(a) A mathematical biologist believes that the differential equation $x\frac{dy}{dx} - 3y = x^4$ models a process. Find the general solution of the differential equation.

Given that $y = 2$ when $x = 1$, find the particular solution , expressing y in terms of x.

(5, 2 marks)

(b) The biologist subsequently decides that a better model is given by the equation

$$y\frac{dy}{dx} - 3x = x^4.$$

Given that $y = 2$ when $x = 1$, obtain y in terms of x. **(4 marks)**

2005

Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin x$$

Hence find the particular solution for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$. **(7, 3 marks)**

2006

Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 2$. **(6 marks)**

2007

Obtain the general solution of the equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{2x}$ **(6 marks)**

2008

Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2$$

Given that $y = \frac{1}{2}$ and $\frac{dy}{dx} = 1$, when $x = 0$, find the particular solution. **(7, 3 marks)**

2009

(a) Solve the differential equation

$$(x+1)\frac{dy}{dx}-3y=(x+1)^4$$

given that $y=16$ when $x=1$, expressing the answer in the form $y=f(x)$.

(b) Hence find the area enclosed by the graphs of $y=f(x)$, $y=(1-x)^4$ and the x-axis.

(6, 4 marks)

2010

Obtain the general solution of the equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0.$$

Hence obtain the solution for which $y=3$ when $x=0$ and $y=e^{-\pi}$ when $x=\frac{\pi}{2}$.

(4, 3 marks)

2011

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=e^x+12$$

Find the particular solution for which $y=\frac{-3}{2}$ and $\frac{dy}{dx}=\frac{1}{2}$ when $x=0$. **(7, 3 marks)**

2012

(a) Express $\frac{1}{(x-1)(x+2)^2}$ in partial fractions.

(b) Obtain the general solution of the differential equation

$$(x-1)\frac{dy}{dx}-y=\frac{x-1}{(x+2)^2}$$

expressing your answer in the form $y=f(x)$. **(4, 7 marks)**

2013

Solve the differential equation

$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}$, given that $y = 1$ and $\frac{dy}{dx} = -1$ when $x = 0$. **(11 marks)**

2014

Find the solution $y = f(x)$ to the differential equation

$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0,$ given that $y = 4$ and $\frac{dy}{dx} = 3$ when $x = 0$. **(6 marks)**

2015

Solve the second order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x}$$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 0$. **(10 marks)**

2015

Solve the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12x^2 + 2x - 5$$

given $y = -6$ and $\frac{dy}{dx} = 3,$ when $x = 0.$ **(10 marks)**