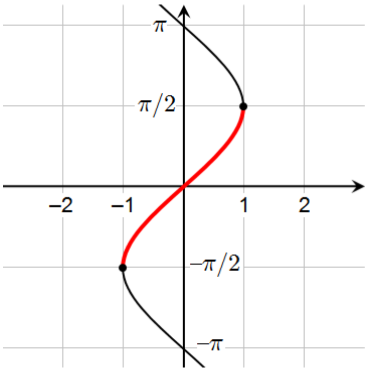
Further Differentiation

Differentiating Inverse Functions

We will first look at inverse trigonometric functions.

$y = \sin^{-1} x$



Here we can see that every value of x has an infinite number of y values but for an inverse function to exist it has to be one-to-one.

If we restrict the y values to $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ we have a one to one function.

$$y = \sin^{-1} x \Leftrightarrow x = \sin y$$

We need to be able to differentiate this function.

$$y = \sin^{-1} x$$

$$x = \sin y$$

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\cos^2 y = 1 - \sin^2 y$$

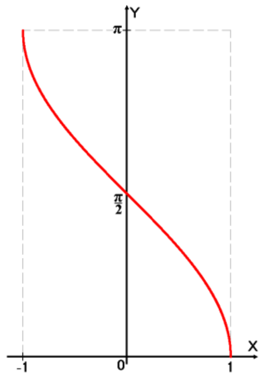
$$\cos y = \sqrt{1-\sin^2 y}$$

$$\cos y = \sqrt{1-x^2}$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

We can now look at cos and tan in a similar way.

$\underline{y = \cos^{-1} x}$



The values of y have to be restricted to $0 \le y \le \pi$ so that the function is one to one.

$y = \cos^{-1} x$

$x = \cos y$

$\frac{dx}{dy} = -\sin y$

$\frac{dy}{dx} = \frac{-1}{\sin y}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$ ←

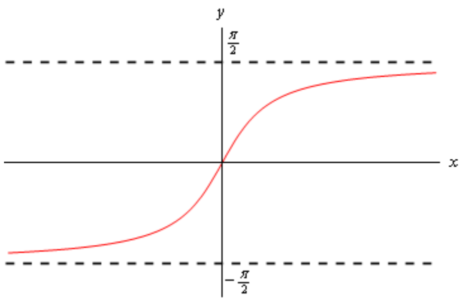
$\sin^2 y = 1 - \cos^2 y$

$\sin y = \sqrt{1-\cos^2 y}$

$\sin y = \sqrt{1-x^2}$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$\underline{y = \tan^{-1} x}$



The values of y are restricted to $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ so that we have a one to one function.

$y = \tan^{-1} x$

$x = \tan y$

$$\frac{dx}{dy} = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \longleftarrow \quad \begin{array}{l} \sec^2 y = 1+\tan^2 y \\ \sec^2 y = 1+x^2 \end{array}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

These 3 results (and their inverses for integration) must be memorised.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

We should also be prepared to use the chain rule, product rule and quotient rule with these results.

Examples

① $y = \tan^{-1}(4x)$

$$\frac{dy}{dx} = \frac{1}{1+(4x)^2} \cdot \frac{d}{dx}(4x)$$

$$\frac{dy}{dx} = \frac{4}{1+16x^2}$$

② $y = \cos^{-1}(2x)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x)^2}} \cdot \frac{d}{dx}(2x)$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}}$$

③ $y = \sin^{-1}(3x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \cdot \frac{d}{dx}(3x)$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-9x^2}}$$

④ $y=\tan^{-1}(3x+1)$

$$\frac{dy}{dx}=\frac{1}{1+(3x+1)^2}\bullet\frac{d}{dx}(3x+1)$$

$$\frac{dy}{dx}=\frac{3}{(3x+1)^2+1}$$

⑤ $y=\sin^{-1}(e^x)$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(e^x)^2}}\bullet\frac{d}{dx}(e^x)$$

$$\frac{dy}{dx}=\frac{e^x}{\sqrt{1-e^{2x}}}$$

⑥ $y=\cos^{-1}\left(\frac{x}{3}\right)$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-\left(\frac{x^2}{9}\right)}}\bullet\frac{d}{dx}\left(\frac{x}{3}\right)$$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{9-x^2}}$$

⑦ $y=2x\cos^{-1}x$ $\qquad u=2x$ $\qquad v=\cos^{-1}x$

$$\frac{dy}{dx}=u'v+uv' \qquad u'=2 \qquad v'=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx}=2\cos^{-1}x-\frac{2x}{\sqrt{1-x^2}}$$

⑧ $y=\frac{\sin^{-1}(2x)}{x}$ $\qquad u=\sin^{-1}(2x)$ $\qquad v=x$

$$\frac{dy}{dx}=\frac{u'v-uv'}{v^2} \qquad u'=\frac{2}{\sqrt{1-4x^2}} \qquad v'=1$$

$$\frac{dy}{dx}=\frac{\frac{2x}{\sqrt{1-4x^2}}-\sin^{-1}(2x)}{x^2}$$

$$\frac{dy}{dx}=\frac{2x-\sqrt{1-4x^2}\sin^{-1}(2x)}{x^2\sqrt{1-4x^2}}$$

Questions

Differentiate

① $y=\cos^{-1}(3x)$ ② $y=\tan^{-1}(2x)$ ③ $y=\sin^{-1}(5x)$

④ $y=\tan^{-1}(x+1)$ ⑤ $y=\cos^{-1}(3x-1)$ ⑥ $y=\sin^{-1}(x^2)$

⑦ $y=\tan^{-1}(e^x)$ ⑧ $y=\tan^{-1}(3\cos x)$ ⑨ $y=\sin^{-1}(\ln 3x)$

⑩ $y=\tan^{-1}\left(\frac{x}{5}\right)$ ⑪ $y=\cos^{-1}\left(\frac{x}{2}\right)$ ⑫ $y=\frac{\tan^{-1}x}{1+x^2}$

⑬ $y=x^3\sin^{-1}x$ ⑭ $y=\sqrt{x}\cos^{-1}(2x)$ ⑮ $y=(1+x^3)\tan^{-1}x$

⑯ $y=\ln x\tan^{-1}x$ ⑰ $y=\frac{\tan^{-1}(3x)}{x}$ ⑱ $y=\frac{\sin^{-1}x}{\sqrt{x}}$

⑲ $y=\frac{x}{\sin^{-1}(5x)}$ ⑳ $y=\frac{\ln x}{\tan^{-1}x}$

Differentiating Other Inverse Functions

We have already used the result $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$.

If $x=f(y)$ then $f^{-1}(x)=y$ and $\frac{dx}{dy}=f'(y)$.

then $\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'(y)}$ since $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$.

$y=f^{-1}(x)\Leftrightarrow f(y)=x$

so $$\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'\left(f^{-1}(x)\right)}$$ or $$\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'(y)} \quad \text{where } y=f^{-1}(x)$$

Examples will hopefully make this clearer.

① Find the derivative of $f^{-1}(x)$ when $f(x)=x^3$.

$f(x)=x^3\Rightarrow f'(x)=3x^2$

$f'(y)=3y^2$

$y=f^{-1}(x)$

$f(y)=x$

$y^3=x$

$y=x^{\frac{1}{3}}$

$$\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'(y)}$$
$$=\frac{1}{3y^2}$$
$$=\frac{1}{3\left(x^{\frac{1}{3}}\right)^2}$$
$$=\frac{1}{3x^{\frac{2}{3}}}$$

② Find the derivative of $f^{-1}(x)$ when $f(x)=3e^{2x}+4$.

$$f(x)=3e^{2x}+4 \Rightarrow f'(x)=6e^{2x}$$
$$f'(y)=6e^{2y}$$

$$y=f^{-1}(x)$$
$$f(y)=x$$
$$3e^{2y}+4=x$$
$$e^{2y}=\frac{1}{3}x-\frac{4}{3}$$

$$\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'(y)}$$
$$=\frac{1}{6e^{2y}}$$
$$=\frac{1}{6\left(\frac{1}{3}x-\frac{4}{3}\right)}$$
$$=\frac{1}{2x-8}$$

③ Find the derivative of $f^{-1}(x)$ when $f(x)=\ln(3x+1)$.

$$f(x)=\ln(3x+1) \Rightarrow f'(x)=\frac{3}{3x+1}$$
$$f'(y)=\frac{3}{3y+1}$$

$$y=f^{-1}(x)$$
$$f(y)=x$$
$$\ln(3y+1)=x$$
$$3y+1=e^x$$

$$\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'(y)}$$

$$=\frac{1}{\frac{3}{3y+1}}$$

$$=\frac{3y+1}{3}$$

$$=\frac{1}{3}e^x$$

④ A function is defined as $f(x)=x^2+4x+5$, $x\geq 5$.

Find the derivative of the inverse function.

$f(x)=x^2+4x+5 \Rightarrow f'(x)=2x+4$

$f'(y)=2y+4$

$y=f^{-1}(x)$

$f(y)=x$

$y^2+4y+5=x$

$(y+2)^2+1=x$ by completing the square

$y=\sqrt{x-1}-2$ taking the positive root since $x\geq 5$ is the domain of $f(x)$.

$$\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'(y)}$$

$$=\frac{1}{2y+4}$$

$$=\frac{1}{2\left(\sqrt{x-1}-2\right)+4}$$

$$=\frac{1}{2\sqrt{x-1}}$$

⑤ A function is defined as $f(x)=x^3+3x-4$.

(a) Find the derivative of the inverse in terms of y if $y=f^{-1}(x)$.

(b) Find the equation of the tangent to the inverse at the point $(10,2)$.

(a) $f(x)=x^3+3x-4 \Rightarrow f'(x)=3x^2+3$

$f'(y)=3y^2+3$

$y=f^{-1}(x)$

$f(y)=x$

$y^3+3y-4=x$

$$\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'(y)}$$
$$=\frac{1}{3y^2+3}$$

(b) At $(10,2)$, $m=\frac{1}{3\times2^2+3}=\frac{1}{15}$

$$y-2=\frac{1}{15}(x-10)$$
$$15y-x-20=0$$

Questions

① Find in terms of x the derivative of $f^{-1}(x)$.

(a) $f(x)=5e^{3x}$ (b) $f(x)=2\ln x+6$ (c) $f(x)=\ln(2x+5)$

② A function is defined as $f(x)=x^3+4x+2$.

(a) Find the derivative of the inverse in terms of y if $y=f^{-1}(x)$.

(b) Find the equation of the tangent to the inverse at the point $(7,1)$.

③ A function is defined as $f(x)=4x+\sin(2x)$.

(a) Find the derivative of the inverse in terms of y if $y=f^{-1}(x)$.

(b) Find the equation of the tangent to the inverse at the point $(0,0)$.

④ A function is defined as $f(x)=2x^2-2x+1$, $x\geq1$.

Find the derivative of the inverse function.

⑤ Find $\frac{dy}{dx}$ when $x=e^{y^2}$. Give your answer in terms of x.

Implicit and Explicit Functions

Some functions have y defined **explicitly** in terms of x.

e.g. $y = \frac{1}{2}x^2 + 2, \quad y = x^3 - 3\sin x + \sqrt{3}x$

These are clearly defined functions of x with y as the subject of the equation.

Some functions are **implicit**. In this case, y cannot be clearly defined as a function of x.

e.g. $xy^2 + 3y - 2x^2 = 0, \qquad x^2 + y^2 = 9$

Differentiating Implicit Functions

We already know how to differentiate explicit functions. To differentiate implicit functions we must make use of the chain rule.

Examples

① Find $\frac{dy}{dx}$ in terms of x and y when $4y^2 + 2x^2 - 16x = 0$.

$$4y^2 + 2x^2 - 16x = 0$$

Differentiating $\frac{d(4y^2)}{dx} + \frac{d(2x^2)}{dx} - \frac{d(16x)}{dx} = 0$

Using the chain rule $8y\frac{dy}{dx} + 4x - 16 = 0$

$$8y\frac{dy}{dx} = 16 - 4x$$

$$\frac{dy}{dx} = \frac{4-x}{2y}$$

② Differentiate $x^4 = x^2 - y^2$

$$\frac{d(x^4)}{dx} = \frac{d(x^2)}{dx} - \frac{d(y^2)}{dx}$$

$$4x^3 = 2x - 2y\frac{dy}{dx}$$

$$4x^3 - 2x = -2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x-4x^3}{2y}$$

$$\frac{dy}{dx} = \frac{x-2x^3}{y}$$

③ Differentiate xy

$$\frac{d(xy)}{dx} = \frac{d(x)}{dx}y + x\frac{d(y)}{dx} \qquad \text{using product rule} \qquad u = x \qquad v = y$$

$$= y + x\frac{dy}{dx} \qquad u' = 1 \qquad v' = \frac{dy}{dx}$$

We can use the results from ③ to aid our next question.

④ Differentiate $2xy + 4y^2 - x^2 = 5x$

$$2y + 2x\frac{dy}{dx} + 8y\frac{dy}{dx} - 2x = 5 \qquad \text{using product rule} \qquad u = 2x \qquad v = y$$

$$(2x + 8y)\frac{dy}{dx} = 5 + 2x - 2y \qquad u' = 2 \qquad v' = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{5+2x-2y}{2x+8y}$$

⑤ Find an expression in terms of x and y for the gradient at the point (x, y) on the curve with equation $x^2 + y^2 = \frac{y}{x}$.

$$x^2 + y^2 = \frac{y}{x}$$

$$x^3 + y^2x = y$$

$$\frac{d(x^3+xy^2)}{dx} = \frac{d(y)}{dx}$$

$$3x^2 + y^2 + 2xy\frac{dy}{dx} = \frac{dy}{dx}$$

$$3x^2 + y^2 = (1 - 2xy)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2+y}{1-2xy}$$

So gradient at the point $(x, y) = \frac{3x^2+y}{1-2xy}$

⑥ Find $\frac{dy}{dx}$ for the function implicitly defined by $\sin(2x + 3y) = 9x$.

$$\sin(2x + 3y) = 9x$$

$$\frac{d}{dx}(\sin(2x+3y)) = \frac{d}{dx}(9x)$$

$$\cos(2x+3y) \cdot \frac{d}{dx}(2x+3y) = 9$$

$$\cos(2x+3y) \cdot \left(2 + 3\frac{dy}{dx}\right) = 9$$

$$2\cos(2x+3y) + 3\cos(2x+3y)\frac{dy}{dx} = 9$$

$$\frac{dy}{dx} = \frac{9 - 2\cos(2x+3y)}{3\cos(2x+3y)}$$

Questions

① Differentiate

(a) $5x^2 - 7x = 2y^2 + 5y$ (b) $x^2 = \ln y$ (c) $2x^2 + 2y^2 - 5x + 4y - 9 = 0$

(d) $x^2 + xy + y^2 = 10$ (e) $5x^2 - 4xy + 3y^2 = 2$ (f) $6xy - x^2 = 2y$

② Find the equation of the tangent to the curve with equation $xy^4 + 3x^2y^2 = 28$ at the point (1,2).

③ Find the gradient of the curve at the given point and the equation of the tangent of the curve at that point.

(a) $3x^2 + 5y^2 = 17$ at $(-2,1)$ (b) $2x\sin y + x^2 = 1$ at $(1, \pi)$

Implicit Differentiation – Second Derivatives

These have seldom appeared but are equally part of the course.

Examples

① Differentiate twice $xy - x = 4$

$$y + x\frac{dy}{dx} - 1 = 0 \text{ -------------①}$$

$$\frac{dy}{dx} = \frac{1-y}{x} \text{ -------------②}$$

We have 2 options – differentiate ① again or differentiate ② using the quotient rule.

① $\frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2} = \frac{-2\frac{dy}{dx}}{x}$

$\frac{d^2y}{dx^2} = \frac{-2(1-y)}{x^2}$

② $\frac{d^2y}{dx^2} = \frac{\frac{(y-1)x}{x} - (1-y)}{x^2}$ $u = 1 - y$ $v = x$

$\frac{d^2y}{dx^2} = \frac{-2(1-y)}{x^2}$ $u' = -\frac{dy}{dx}$ $v' = 1$

② Differentiate twice $3x - 6y + xy = 10$

$3 - 6\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-y-3}{x-6}$

$-6\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\frac{-2(-y-3)}{x-6}}{x-6} = \frac{2y+6}{(x-6)^2}$

Questions

① Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y only for each of these implicit functions.

(a) $x^3 + y^2 = 5$ (b) $xy = y^2 + 2$ (c) $x^2 = y \ln y$

② Show that if $x^2 = e^y$, then $e^y\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 2$

Logarithmic Differentiation

Logarithmic differentiation is used to differentiate functions with awkward powers.

We take the natural log of both sides and then differentiate. We need to use the log rules from Higher.

Examples

① $y = 10^x$

$\ln y = \ln 10^x$

$\ln y = x \ln 10$ (Remember $\ln 10$ is a number.)

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln 10)$

$\frac{1}{y}\frac{dy}{dx} = \ln 10$

$\frac{dy}{dx} = y \ln 10$ (Substitute back in $y = 10^x$)

$\frac{dy}{dx} = 10^x \ln 10$

② $y = 3^x$

$\ln y = \ln 3^x$

$\ln y = x \ln 3$ (Remember $\ln 3$ is a number.)

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\mathrm{x} \ln 3)$

$\frac{1}{y}\frac{dy}{dx} = \ln 3$

$\frac{dy}{dx} = y \ln 3$ (Substitute back in $y = 3^x$)

$\frac{dy}{dx} = 3^x \ln 3$

③ $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\mathrm{x} \ln x)$ $u = x$ $v = \ln x$

$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$ $u' = 1$ $v' = \frac{1}{x}$

$\frac{dy}{dx} = y (\ln x + 1)$

$\frac{dy}{dx} = x^x (\ln x + 1)$

④ $y = \frac{(x+1)^3(x-5)^4}{(2x+1)^2}$

$\ln y = \ln\left(\frac{(x+1)^3(x-5)^4}{(2x+1)^2}\right)$

$\ln y = \ln(x+1)^3 + \ln(x-5)^4 - \ln(2x+1)^2$

$\ln y = 3\ln(x+1) + 4\ln(x-5) - 2\ln(2x+1)$

Now differentiate

$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x+1} + \frac{4}{x-5} - \frac{2}{2x+1} \cdot 2$ ----- Remember chain rule!

$\frac{dy}{dx} = y\left(\frac{3}{x+1} + \frac{4}{x-5} - \frac{4}{2x+1}\right)$

$\frac{dy}{dx} = \frac{(x+1)^3(x-5)^4}{(2x+1)^2}\left(\frac{3}{x+1} + \frac{4}{x-5} - \frac{4}{2x+1}\right)$

This can be simplified further but should only be done if required by the question.

Questions

Differentiate

① $y = 8^x$ ② $y = (\cos x)^x$ ③ $y = e^{\sin^2 x}$ ④ $y = x^{x^2}$

⑤ $y = \frac{e^x \sin x}{x}$ ⑥ $y = \frac{2^x}{2x+1}$ ⑦ $y = (1 + x)(2 + 3x)(x - 5)$

⑧ $y = \frac{(3-x)(4x-3)^2}{(7-2x)^{1/2}}$

Parametric Curves

Sometimes x and y are defined in terms of a parameter (usually t or θ).

e.g. $x = t + 3$ $x = \cos\theta$

$y = t^2$ $y = \sin\theta$

Each value of t/θ will give a point on the curve, although the values of t/θ may be restricted.

If we eliminate this parameter, we find the Cartesian Equation of the curve.

e.g. When we have $x = t + 3 \Rightarrow t = x - 3$

Substituting this into $y = t^2$ gives $y = (x - 3)^2$ which is the equation of a parabola.

When $\cos/\sin$ are involved we need to square both equations.

e.g. $x = \cos\theta \Rightarrow x^2 = \cos^2\theta$

$y = \sin\theta \Rightarrow y^2 = \sin^2\theta$

If we sum these $x^2 + y^2 = \cos^2\theta + \sin^2\theta$

$x^2 + y^2 = 1$ which is the equation of a circle with centre(0,0) and radius 1.

This method of representing a curve parametrically is useful for studying motion in a plane.

Changing from Parametric Equations to a Cartesian Equation

Examples

① $\left.\begin{array}{l} x = 3t \\ y = \frac{3}{t} \end{array}\right\}$

$t = \frac{x}{3}$ Substitute this into $y = \frac{3}{t}$

$y = \frac{3}{\frac{x}{3}} = \frac{9}{x}$ $\therefore y = \frac{9}{x}$ is the Cartesian Equation (a hyperbola)

② $\left.\begin{array}{l} x = 5p^2 \\ y = 10p \end{array}\right\}$

$p = \frac{y}{10}$ Substitute into $x = 5p^2$

$x = 5\left(\frac{y}{10}\right)^2$

$x = \frac{5y^2}{100}$

$\therefore\ 5y^2 = 100x$

$y^2 = 20x$ is the Cartesian Equation.

③ $\left.\begin{array}{l} x = 4\cos\theta \\ y = 3\sin\theta \end{array}\right\}$

$x^2 = 16\cos^2\theta \ \Rightarrow\ \cos^2\theta = \frac{x^2}{16}$

$y^2 = 9\sin^2\theta \ \Rightarrow\ \sin^2\theta = \frac{y^2}{9}$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$ is the Cartesian Equation (an ellipse).

④ $\left.\begin{array}{l} x = 4 + \mathrm{r}\cos\theta \\ y = 3 + \mathrm{r}\sin\theta \end{array}\right\}$

$x - 4 = \mathrm{r}\cos\theta \ \Rightarrow\ (x-4)^2 = r^2\cos^2\theta$

$y - 3 = \mathrm{r}\sin\theta \ \Rightarrow\ (y-3)^2 = r^2\sin^2\theta$

$(x-4)^2 + (y-3)^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$(x-4)^2 + (y-3)^2 = r^2(\cos^2\theta + \sin^2\theta)$

$(x-4)^2 + (y-3)^2 = r^2$ is the Cartesian Equation (a circle, centre (4,3), radius r).

⑤ $\left.\begin{array}{l} x = 4\sec\theta \\ y = 4\tan\theta \end{array}\right\}$

$x^2 = 16\sec^2\theta$

$y^2 = 16\tan^2\theta$

$x^2 - y^2 = 16(\sec^2\theta - \tan^2\theta)$

$x^2 - y^2 = 16$ (since $\sec^2\theta = 1 + \tan^2\theta$)

⑥ $\left.\begin{array}{l} x = \frac{2+3t}{4} \\ y = \frac{3-4t}{5} \end{array}\right\}$

$4x = 2 + 3t$

$t = \frac{4x - 2}{3}$ substitute this into equation for y

$y = \frac{3 - 4\left(\frac{4x-2}{3}\right)}{5}$

$y = \frac{9 - 16x + 8}{15}$

$y = \frac{17 - 16x}{15}$ is the Cartesian Equation (a straight line).

Questions

Change these parametric equations into Cartesian form.

① $\left.\begin{matrix} x = 2t \\ y = 8t^2 \end{matrix}\right\}$ ② $\left.\begin{matrix} x = 3\cos\theta \\ y = -3\sin\theta \end{matrix}\right\}$ ③ $\left.\begin{matrix} x = t \\ y = \sqrt{1 - t^2} \end{matrix}\right\}$ ④ $\left.\begin{matrix} x = 5 + \text{r}\cos\theta \\ y = 7 + \text{r}\sin\theta \end{matrix}\right\}$

Differentiating Parametric Equations

If we have 2 separate equations, we can differentiate them both with respect to the parameter.

i.e. $\frac{dx}{dt}$ and $\frac{dy}{dt}$ or $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$

We also know from the Chain Rule that $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$.

This can also be written as $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

Once we can perform these differentiations, we can do all the usual questions on finding tangents and coordinates of points on curves.

Examples

① Find $\frac{dy}{dx}$ in terms of the parameter t when $x = \frac{2}{t}$ and $y = \sqrt{t^2 - 3}$.

$\frac{dx}{dt} = -2t^{-2}$

$= \frac{-2}{t^2}$

$\frac{dy}{dt} = \frac{1}{2}(t^2 - 3)^{-\frac{1}{2}} \cdot 2t$

$= \frac{t}{\sqrt{t^2 - 3}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{t}{\sqrt{t^2-3}} \times \frac{t^2}{-2}$

$= \frac{-t^3}{2\sqrt{t^2-3}}$

remember that $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$

② Find $\frac{dy}{dx}$ when $x = 4\cos t,\ y = 4\sin t$.

$\frac{dx}{dt} = -4\sin t$ $\qquad$ $\frac{dy}{dt} = 4\cos t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \frac{4\cos t}{-4\sin t}$$
$$= -\cot t$$

③ Find $\frac{dy}{dx}$ given $x = 3t^2 + 4t,\ \ y = t^3 + t^2$.

$\frac{dx}{dt} = 6t + 4$ $\qquad$ $\frac{dy}{dt} = 3t^2 + 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \frac{3t^2+2t}{6t+4}$$
$$= \frac{t(3t+2)}{2(3t+2)}$$
$$= \frac{t}{2}$$

④ Find $\frac{dy}{dx}$ in terms of x if $x = 4 - 3t$ and $y = \frac{3}{t}$.

$\frac{dx}{dt} = -3$ $\qquad$ $\frac{dy}{dt} = -\frac{3}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= -\frac{3}{t^2} \times -\frac{1}{3}$$
$$= \frac{1}{t^2}$$

substitute $t = \frac{4-x}{9}$ to get in terms of x.

$$= \frac{1}{\frac{(4-x)^2}{9}}$$
$$= \frac{9}{(4-x)^2}$$

⑤ Find $\frac{dy}{dx}$ in terms of x if $x = \sqrt{t}$ and $y = t^2 - 3$.

$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$ $\qquad\qquad$ $\frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 2t \times 2\sqrt{t}$

$= 4t^{\frac{3}{2}}$ $\qquad\qquad$ substitute $t = x^2$ to get in terms of x.

$= 4(x^2)^{\frac{3}{2}}$

$= 4x^3$

Questions

Find $\frac{dy}{dx}$ in terms of the parameter t, for the following pairs of parametric equations.

① $\left.\begin{array}{r} x = t^2 + 1 \\ y = 4t^3 \end{array}\right\}$ $\qquad$ ② $\left.\begin{array}{r} x = t^2 \\ y = t(t^2 - 1) \end{array}\right\}$ $\qquad$ ③ $\left.\begin{array}{r} x = 2t \\ y = \frac{2}{t} \end{array}\right\}$ $\qquad$ ④ $\left.\begin{array}{r} x = 4\cos t \\ y = 2\sin t \end{array}\right\}$

⑤ $\left.\begin{array}{r} x = e^t \cos t \\ y = e^t \sin t \end{array}\right\}$ $\qquad$ ⑥ $\left.\begin{array}{r} x = \frac{1}{t} \\ y = \sqrt{t^2 + 1} \end{array}\right\}$ $\qquad$ ⑦ $\left.\begin{array}{r} x = \frac{t}{t+1} \\ y = \frac{t^2}{t+1} \end{array}\right\}$ $\qquad$ ⑧ $\left.\begin{array}{r} x = \sin t + \cos t \\ y = \sin t - \cos t \end{array}\right\}$

Second Derivatives of Parametric Equations

Example

Find $\frac{d^2y}{dx^2}$ for the parametric equation $x = t^2 + t$, $y = t^3 + 3t$.

$\frac{dx}{dt} = 2t + 1$ $\qquad\qquad$ $\frac{dy}{dt} = 3t^2 + 3$

$\frac{dy}{dx} = \frac{dy}{dt} \text{x} \frac{dt}{dx}$

$= \frac{3t^2+3}{2t+1}$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$. However $\frac{dy}{dx}$ is currently a function of t.

So $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$ $\qquad$ by the Chain Rule

$\frac{dy}{dx} = \frac{3t^2+3}{2t+1}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{3t^2+3}{2t+1}\right) \times \frac{dt}{dx}$

$= \frac{6t(2t+1)-2(3t^2+3)}{(2t+1)^2} \times \frac{1}{2t+1}$ using the quotient rule and $\frac{dx}{dt} = 2t + 1$ from above

$= \frac{12t^2+6t-6t^2-6}{(2t+1)^3}$

$= \frac{6t^2+6t-6}{(2t+1)^3}$

$= \frac{6(t^2+t-1)}{(2t+1)^3}$

Questions

Find $\frac{d^2y}{dx^2}$ for the following pairs of parametric equations.

① $\left.\begin{array}{c} x = t^2 \\ y = \ln t \end{array}\right\}$ ② $\left.\begin{array}{c} x = t + \sin t \\ y = t - \cos t \end{array}\right\}$ ③ $\left.\begin{array}{c} x = 3t^3 - t \\ y = 4t^2 \end{array}\right\}$ ④ $\left.\begin{array}{c} x = \theta - \sin\theta \\ y = 1 + \cos\theta \end{array}\right\}$

Further Examples

① given a curve defined by $x = t^2 + \frac{2}{t}$ and $y = t^2 - \frac{2}{t}$

(a) Find the coordinates of the turning point on the curve.
(b) Establish the nature of the turning point by considering the concavity of the curve.

(a) $x = t^2 + \frac{2}{t}$ $\qquad$ $y = t^2 - \frac{2}{t}$

$\frac{dx}{dt} = 2t - \frac{2}{t^2}$ $\qquad$ $\frac{dy}{dt} = 2t + \frac{2}{t^2}$

$= \frac{2t^3-2}{t^2}$ $\qquad$ $= \frac{2t^3+2}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{2t^3+2}{t^2} \times \frac{t^2}{2t^3-2}$

$= \frac{t^3+1}{t^3-1}$

Stationary points occur when $\frac{dy}{dx} = 0$

$\frac{t^3+1}{t^3-1} = 0$

$t^3 + 1 = 0$

$t^3 = -1$

$t = -1$

When $t = -1$, $x = -1$ and $y = 3$ so stationary point is at (1,3).

(b) $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{t^3+1}{t^3-1}\right) \times \frac{dt}{dx}$

$= \frac{3t^2(t^3-1)-3t^2(t^3+1)}{(t^3-1)^2} \times \frac{t^2}{2(t^3-1)}$ by the quotient rule and $\frac{dx}{dt} = \frac{2(t^3-1)}{t^2}$ from above

$= \frac{3t^2\left(t^3-1-(t^3+1)\right)t^2}{2(t^3-1)^3}$

$= \frac{-6t^4}{2(t^3-1)^3}$

$= \frac{-3t^4}{(t^3-1)^3}$

When $t = -1$, $\frac{d^2y}{dx^2} = \frac{-3}{(-2)^3} = \frac{3}{8}$

$\frac{d^2y}{dx^2} > 0$ so curve is concave up. (1,3) is a local minimum.

② Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the curve defined by $x = t^2 - \frac{1}{t^2}$, $y = t^2 + \frac{1}{t^2}$

Determine the coordinates of the turning point on the curve and prove that the curve is always concave up.

$x = t^2 - \frac{1}{t^2}$ $\quad\quad$ $y = t^2 + \frac{1}{t^2}$

$\frac{dx}{dt} = 2t + \frac{2}{t^3}$ $\quad\quad$ $\frac{dy}{dt} = 2t - \frac{2}{t^3}$

$= \frac{2t^4+2}{t^3}$ $\quad\quad$ $= \frac{2t^4-2}{t^3}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{2t^4-2}{t^3} \times \frac{t^3}{2t^4+2}$

$= \frac{t^4-1}{t^4+1}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{t^4-1}{t^4+1}\right) \times \frac{dt}{dx}$

$= \frac{4t^3(t^4+1)-4t^3(t^4-1)}{(t^4+1)^2} \times \frac{t^3}{2(t^4+1)}$ by the quotient rule and $\frac{dx}{dt} = \frac{2(t^4+1)}{t^3}$ from above

$= \frac{4t^6\left(t^4+1-(t^4-1)\right)}{2(t^4+1)^3}$

$$= \frac{8t^6}{2(t^4+1)^3}$$

$$= \frac{4t^6}{(t^4+1)^3}$$

Stationary points occur when $\frac{dy}{dx} = 0$

$\frac{t^4-1}{t^4+1} = 0$

$t^4 - 1 = 0$

$t^4 = 1$

$t = 1$ or $t = -1$

When $t = 1, x = 0$ and $y = 2$ so stationary point is at (0,2).

When $t = -1, x = 0$ and $y = 2$ so this also gives the stationary point at (0,2)

Curve is concave up if $\frac{d^2y}{dx^2} > 0$.

$\frac{4t^6}{(t^4+1)^3} > 0$ as $4t^6 > 0$ and $(t^4 + 1)^3 > 0$ so curve is always concave up.

③ Find the equation of the tangent at $t = \frac{\pi}{4}$ on the curve $x = 5\cos t$, $y = 3\sin t$.

$\frac{dx}{dt} = -5\sin t$ $\qquad \frac{dy}{dt} = 3\cos t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{3\cos t}{-5\sin t}$$

$$= \frac{-3}{5}\cot t$$

When $t = \frac{\pi}{4}$,

$\frac{dy}{dx} = \frac{-3}{5\tan\frac{\pi}{4}} = \frac{-3}{5}$, $\quad x = 5\cos\frac{\pi}{4} = \frac{5}{\sqrt{2}}$, $y = 3\sin\frac{\pi}{4} = \frac{3}{\sqrt{2}}$

Equation of tangent $\quad y - b = m(x - a)$

$$y - \frac{3}{\sqrt{2}} = \frac{-3}{5}\left(x - \frac{5}{\sqrt{2}}\right)$$

$$5y - \frac{15}{\sqrt{2}} = \frac{15}{\sqrt{2}} - 3x$$

$$5y = \frac{30}{\sqrt{2}} - 3x$$

Questions

① A curve is defined by the parametric equations $x = 2t^2 + t - 5,\ \ y = t^2 + 3t + 1.$

Find the equation of the tangent to the curve at the point where $t = 1$.

② Find the equation of the tangent at $t = \frac{\pi}{3}$ on the curve defined parametrically by the equations $x = 3\sin t,\ \ y = 4\cos t$.

③ A curve is defined by the parametric equations $x = t,\ y = t^3 - 3t$.

Find the coordinates and nature of the stationary points on the curve.

④ A curve is defined by the parametric equations $x = t^2 + t,\ y = t^3 - 12t$.

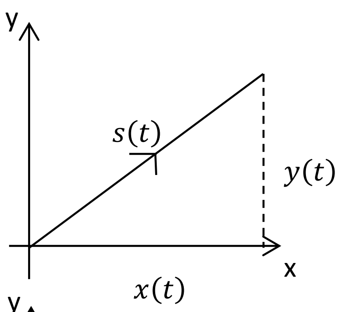
Find the coordinates and nature of the stationary points on the curve.

Motion In A Plane

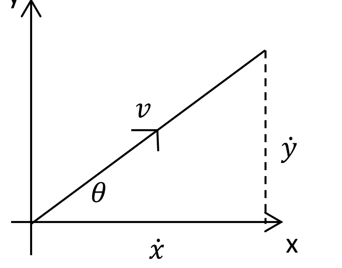
We have already considered rectilinear motion (motion in a straight line) but a particle is more likely to move in a plane.

It is therefore defined by parametric equations.

A particle in motion on a plane is at position $\big(x(t), y(t)\big)$ at a time t. The displacement of the particle from the origin is denoted by $s(t)$.



The distance from the origin is the magnitude of the displacement = $|s| = \sqrt{\big(x(t)\big)^2 + \big(y(t)\big)^2}$



Horizontal velocity component is $\frac{dx}{dt}$ or $\dot{x}$.

Vertical velocity component is $\frac{dy}{dt}$ or $\dot{y}$.

The speed of the particle is the magnitude of the velocity = $|v| = \sqrt{(\dot{x})^2 + (\dot{y})^2}$

The direction of motion at any instant is given by $\tan\theta = \frac{\dot{y}}{\dot{x}}$. In Leibniz notation this is given by $\frac{dy}{dx}$.

The acceleration of the particle is the rate of change of its velocity with respect to time.

$$a(t) = v'(t) = s''(t)$$

Acceleration has horizontal component $\frac{d^2x}{dt^2}$ or $\ddot{x}$ and vertical component $\frac{d^2y}{dt^2}$ or $\ddot{y}$.

The magnitude of the acceleration is $|a| = \sqrt{(\ddot{x})^2 + (\ddot{y})^2}$.

The direction of acceleration is given by tan α = $\frac{\ddot{y}}{\ddot{x}}$.

Examples

① A particle is moving along a path determined by parametric equations

$x = 4t - 1,\ y = t^2 + 3t$

where t represents time in seconds and distance is measured in metres.

(a) How far is the particle from the origin at $t = 0$?

When $t = 0,\ \ x = -1,\ \ y = 0$

$|s| = \sqrt{(-1)^2 + (0)^2}$

$= 1$

$\therefore$ $1m$ from the origin.

(b) Calculate the horizontal and vertical velocity components.

Horizontal velocity component = $\dot{x} = 4$ Vertical velocity component = $\dot{y} = 2t + 3$

(c) When $t = 2$, calculate the horizontal and vertical velocity and the speed of the particle.

Horizontal velocity =4m/s Vertical velocity = $2 \times 2 + 3 = 7$m/s

Speed = $|v| = \sqrt{(4)^2 + (7)^2}$

$= \sqrt{65}$

$= 8 \cdot 1$m/s

(c) Derive a formula for the instantaneous direction of motion of the particle and calculate the direction the particle is moving in when $t = 2$.

Instantaneous direction of motion = $\frac{dy}{dx}$ = $\frac{\dot{y}}{\dot{x}}$ = $\frac{2t+3}{4}$ = $\frac{1}{2}t + \frac{3}{4}$

When $t = 2$, $\frac{dy}{dx} = \frac{7}{4}$

$\therefore \ \tan\theta = \frac{7}{4}$

$\theta = 60 \cdot 3°$

So the particle is moving on a bearing of $030°$.

② The equations of motion of a particle moving in a plane are $x = t^3 - 3t$ and $y = t^2 + t$ where x and y are measured in metres and t is measured in seconds.
Find the magnitude and direction of the velocity and the acceleration of the particle after 3 seconds.

Horizontal velocity component = $\dot{x} = 3t^2 - 3$

Vertical velocity component = $\dot{y} = 2t + 1$

At $= 3$, $\quad \dot{x} = 3(3)^2 - 3 = 24m/s$. $\qquad$ At $t = 3$, $\quad \dot{y} = 2(3) + 1 = 7m/s$.

$|v| = \sqrt{(24)^2 + (7)^2}$ $\qquad$ $\tan\theta = \frac{\dot{y}}{\dot{x}}$

$\quad = \sqrt{625}$ $\qquad$ $\theta = \tan^{-1}\left(\frac{7}{24}\right)$

$\quad = 25$m/s $\qquad$ $\theta = 16 \cdot 26°$ so particle is moving on a bearing of $074°$.

Horizontal acceleration component = $\ddot{x} = 6t$. $\quad$ Vertical acceleration component = $\ddot{y} = 2$

At $t = 3$, $\quad \ddot{x} = 6 \times 3 = 18m/s^2$ $\qquad$ At $t = 3$, $\quad \ddot{y} = 2m/s^2$

$|a| = \sqrt{(18)^2 + (2)^2} = \sqrt{328} = 18 \cdot 1m/s^2$

Related Rates

This part of the course is an application of the chain rule.

Examples

① The radius of a circular ripple is increasing at a rate of 30cm per second.
How fast is the area increasing when the radius is 50cm?

This question is asking us to calculate the rate at which the area is increasing from the rate at which the radius is increasing. This is a related rates problem.

$A =$ area, $r =$ radius, $t =$ time

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r,$ $\frac{dr}{dt} = 30$ (as we are told the radius of the ripple is increasing at a rate of 30cm/s)

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 2\pi r \times 30$

$= 60\pi r$

When $r = 50$, $\frac{dA}{dt} = 60 \times \pi \times 50 = 3000\pi$

Area is increasing at a rate of $3000\pi\ cm^2/s$.

② The volume of a spherical balloon is increasing at a rate of 0·05 m^3/s.

How fast is the surface area increasing when the radius is 0·4m?

$v =$ volume, $s =$ surface area, $r =$ radius

$v = \frac{4}{3}\pi r^3$ $s = 4\pi r^2$

$\frac{dv}{dr} = 4\pi r^2$ $\frac{ds}{dr} = 8\pi r$ $\frac{dv}{dt} = 0{\cdot}05$

$\frac{ds}{dt} = \frac{ds}{dr} \times \frac{dr}{dv} \times \frac{dv}{dt}$

$= 8\pi r \times \frac{1}{4\pi r^2} \times 0{\cdot}05$

$= \frac{0{\cdot}1}{r}$

When $r = 0{\cdot}4$, $\frac{ds}{dt} = \frac{0{\cdot}1}{0{\cdot}4} = 0{\cdot}25$ Surface area is increasing at $0{\cdot}25 m^2/s$.

③ A cube has edge length $x$cm and is expanding as time passes. If the length of the edge is increasing at 1 cm/s, how fast is the volume changing when the edge length is 20cm?

$V = x^3$

$\frac{dV}{dx} = 3x^2,$ $\frac{dx}{dt} = 1$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$= 3x^2$

When $x = 20$, $\frac{dV}{dt} = 3 \times 20^2 = 1200cm^3/s.$

④ Ink is dropped onto blotting paper forming a circular stain which increases at the rate of $5cm^2/s$.

Find the rate of change of radius when the area is $30cm^2$.

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r, \quad \frac{dA}{dt} = 5$

$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$

$= \frac{1}{2\pi r} \times 5$

$= \frac{5}{2\pi r}$

When $A = 30$, $30 = \pi r^2$ so $r = \sqrt{\frac{30}{\pi}}$

$\frac{dr}{dt} = \frac{5}{2\pi\sqrt{\frac{30}{\pi}}} = 0{\cdot}26$

Rate of change of radius = $0{\cdot}26\text{cm/s}$.

Implicitly Defined Related Rates

Examples

① A point moves on the curve $x^2 - 4y^2 = 32$ so that the y-coordinate increases at the constant rate of 9m/s. This means that $\frac{dy}{dt} = 9$.

(a) At what rate is the x-coordinate changing at the point $(6, -1)$?

(b) What is the slope of the curve at the point $(6, -1)$?

(a) The rate of change of the x-coordinate is $\frac{dx}{dt}$.

We need to differentiate $x^2 - 4y^2 = 32$ with respect to t then rearrange for $\frac{dx}{dt}$.

This gives $2x\frac{dx}{dt} - 8y\frac{dy}{dt} = 0$

$$2x\frac{dx}{dt} = 8y\frac{dy}{dt}$$

$\frac{dx}{dt} = \frac{8y}{2x}\frac{dy}{dt}$ Now substitute $x = 6, y = -1, \frac{dy}{dt} = 9$

At $(6, -1)$ $\frac{dx}{dt} = \frac{-8}{12} \times 9$

$= 6\text{m/s}$

(b) The slope of the curve is given by $\frac{dy}{dx}$.

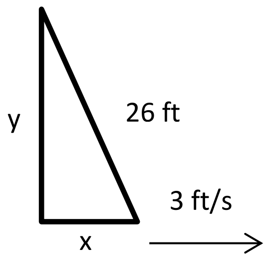
At $(6, -1)$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 9 \times \frac{1}{6}$

$= \frac{3}{2}$

② A ladder 26 ft long is leaning against a wall. The foot of the ladder is pulled away from the wall at a rate of 3 ft/second.

How fast is the top of the ladder sliding down the wall when the foot is 10 ft away from the bottom of the wall?



When $x = 10$ ft, $y = \sqrt{26^2 - 10^2} = 24$ ft

We need to find $\frac{dy}{dt}$.

$x^2 + y^2 = 26^2$

$x^2 + y^2 = 676$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-2x}{2y}\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{-20}{48} \times 3$

$\frac{dy}{dt} = -1 \cdot 25$ ft/second

The top of the ladder is sliding down the wall at a rate of $1 \cdot 25$ ft/second.

Try some of the questions from Scholar Unit 2, topic 6, exercise 16.

Past Paper Questions

2001

① Differentiate with respect to x.

(a) $f(x)=(2+x)\tan^{-1}\sqrt{x-1}, \quad x>1$

(b) $g(x)=e^{\cot 2x}, \; 0<x<\frac{\pi}{2}$ **(4, 2 marks)**

② A curve has equation $xy+y^2=2$.

(a) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

(b) Hence find an equation of the tangent to the curve at the point $(1,1)$.

(3, 2 marks)

2002

① A curve is defined by the parametric equations

$$x=t^2+t-1, \quad y=2t^2-t+2$$

for all t. show that the point $A(-1,5)$ lies on the curve and obtain an equation of the tangent to the curve at the point A.

(6 marks)

② Given $y=(x+1)^2(x+2)^{-4}$ and $x>0$, use logarithmic differentiation to show that $\frac{dy}{dx}$ can be expressed in the form $\left(\frac{a}{x+1}+\frac{b}{x+2}\right)y$, stating the values of the constants a and b. **(3 marks)**

2003

① (a) Given $f(x) = x(1+x)^{10}$, obtain $f'(x)$ and simplify your answer.

(b) Given $y = 3^x$, use logarithmic differentiation to obtain $\frac{dy}{dx}$ in terms of x.

(3, 3 marks)

② The equation $y^3 + 3xy = 3x^2 - 5$ defines a curve passing through the point $A(2,1)$.

Obtain an equation for the tangent to the curve at A.

(4 marks)

2004

① (a) Given $f(x) = \cos^2 x \, e^{\tan x}$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$, obtain $f'(x)$ and evaluate $f'\left(\frac{\pi}{4}\right)$.

(b) Differentiate $g(x) = \frac{\tan^{-1} 2x}{1+4x^2}$. **(4, 3 marks)**

② A curve is defined by the equations $x = 5\cos\theta$, $y = 5\sin\theta$, $(0 \le \theta \le 2\pi)$.

Use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ.

Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$.

(2, 3 marks)

2005

Given the equation $2y^2 - 2xy - 4y + x^2 = 0$ of a curve, obtain the x coordinate of each point at which the curve has a horizontal tangent.

(4 marks)

2006

① Differentiate, simplifying your answers:

(a) $2\tan^{-1}\sqrt{1+x}$, where $x > -1$;

(b) $\frac{1+\ln x}{3x}$, where $x > 0$. **(3, 3 marks)**

② Given $xy - x = 4$, use implicit differentiation to obtain $\frac{dy}{dx}$ in terms of x and y.

Hence obtain $\frac{d^2y}{dx^2}$ in terms of x and y. **(2, 3 marks)**

2007

A curve is defined by the parametric equations $x = \cos 2t$, $y = \sin 2t$, $0 < t < \frac{\pi}{2}$.

(a) Use parametric differentiation to find $\frac{dy}{dx}$.

Hence find the equation of the tangent when $t = \frac{\pi}{8}$.

(b) Obtain an expression for $\frac{d^2y}{dx^2}$ and hence show that $\sin 2t \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = k$,

where k is an integer. State the value of k.

(5, 5 marks)

2008

① (a) Differentiate $f(x) = \cos^{-1}(3x)$ where $\frac{-1}{3} < x < \frac{1}{3}$.

(b) Given $x = 2\sec\theta$, $y = 3\sin\theta$, use parametric differentiation to find $\frac{dy}{dx}$

in terms of θ. **(2, 3 marks)**

② A curve is defined by the equation $xy^2 + 3x^2y = 4$ for $x > 0$ and $y > 0$.

Use implicit differentiation to find $\frac{dy}{dx}$.

Hence find an equation of the tangent to the curve where $x = 1$.

(3, 3 marks)

2009

① (a) Given $f(x) = (x+1)(x-2)^3$, obtain the values of x for which $f'(x) = 0$.

(b) Calculate the gradient of the curve defined by $\frac{x^2}{y} + x = y - 5$ at the point $(3, -1)$.

(3, 4 marks)

② The curve $y = x^{2x^2+1}$ is defined for $x > 0$. Obtain the values of y and $\frac{dy}{dx}$ at the point where $x = 1$.

(5 marks)

2010

Given $y = t^3 - \frac{5}{2}t^2$ and $x = \sqrt{t}$ for $t > 0$, use parametric differentiation to express $\frac{dy}{dx}$ in terms of t in simplified form.

Show that $\frac{d^2y}{dx^2} = at^2 + bt$, determining the values of the constants a and b.

Obtain an equation for the tangent to the curve which passes through the point of inflexion.

(4, 3, 3 marks)

2011

Obtain $\frac{dy}{dx}$ when y is defined as a function of x by the equation $y + e^y = x^2$.

(3 marks)

2012

① The radius of a cylindrical column of liquid is decreasing at the rate of $0\cdot02ms^{-1}$, while the height is increasing at the rate of $0\cdot01ms^{-1}$.
Find the rate of change of the volume when the radius is $0\cdot6$ metres and the height is 2 metres.
(Recall that the volume of a cylinder is given by $V = \pi r^2 h$.) **(5 marks)**

② A curve is defined parametrically, for all t, by the equations

$$x = 2t + \frac{1}{2}t^2, \qquad y = \frac{1}{3}t^3 - 3t.$$

Obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t.
Find the values of t at which the curve has stationary points and determine their nature.
Show that the curve has exactly two points of inflexion.

(5, 3, 2 marks)

2013

A curve has equation $x^2 + 4xy + y^2 + 11 = 0$.

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-2,3)$. **(6 marks)**

2014

① Differentiate $y = \tan^{-1}(3x^2)$. **(3 marks)**

② Given $x = \ln(1+t^2)$, $y = \ln(1+2t^2)$ use parametric differentiation to find $\frac{dy}{dx}$ in terms of t. **(3 marks)**

③ Given $e^y = x^3\cos^2 x$, $x > 0$, show that $\frac{dy}{dx} = \frac{a}{x} + b\tan x$, for some constants a and b

State the values of a and b. **(3 marks)**

2015

① The equation $x^4 + y^4 + 9x - 6y = 14$ defines a curve passing through the point A$(1,2)$.
Obtain the equation of the tangent to the curve at A. **(4 marks)**

② For $y = 3^{x^2}$, obtain $\frac{dy}{dx}$. **(3 marks)**

③ Given $x = \sqrt{t+1}$ and $y = \cot t$, $0 < t < \pi$,

obtain $\frac{dy}{dx}$ in terms of t. **(3 marks)**

2016

① Differentiate $y = x\tan^{-1} 2x$. **(3 marks)**

② A curve is given by the parametric equations $x = 6t$ and $y = 1 - \cos t$.

Find $\frac{dy}{dx}$ in terms of t. **(2 marks)**

③ The height of a cube is increasing at the rate of $5cms^{-1}$.

Find the rate of increase of the volume when the height of the cube is $3cm$. **(4 marks)**