Further Integration

Integration Resulting In Inverse Trig Results

Remember from differentiation

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1) \qquad \text{and} \qquad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \quad (x \in \mathbb{R})$$

This gives us the following standard integrals.

$$\boxed{\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C} \qquad \text{and} \qquad \boxed{\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C}$$

This can be extended as follows.

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2-x^2}} \qquad \text{and} \qquad \frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{a^2+x^2} \qquad \text{by using the chain rule.}$$

From these we also have

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C} \qquad \text{and} \qquad \boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

These are results which we can prove by integration by substitution.

We need to manipulate integrands into this form so that we can use these results.

Examples

① $\int \frac{dx}{\sqrt{9-x^2}}$

$= \sin^{-1}\left(\frac{x}{3}\right) + C$

② $\int \frac{dx}{25+x^2}$

$= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C$

③ $\int_0^2 \frac{dx}{4+x^2}$

$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$= \frac{1}{2}\tan^{-1} 1 - \frac{1}{2}\tan^{-1} 0$

$= \frac{\pi}{8}$

④ $\int_3^6 \frac{dx}{\sqrt{36-x^2}}$

$= \left[\sin^{-1}\left(\frac{x}{6}\right)\right]_3^6$

$= \sin^{-1}\left(\frac{6}{6}\right) - \sin^{-1}\left(\frac{3}{6}\right)$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$

Some harder questions….

⑤

$$\int \frac{5dx}{25+4x^2}$$

$$= \int \frac{5dx}{4\left(\frac{25}{4}+x^2\right)}$$

$$= \frac{5}{4}\int \frac{dx}{\left(\frac{5}{2}\right)^2+x^2}$$

$$= \frac{5}{4}\times\frac{2}{5}\tan^{-1}\left(\frac{2x}{5}\right)+C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2x}{5}\right)+C$$

⑥

$$\int \frac{4dx}{\sqrt{16-9x^2}}$$

$$= \int \frac{4dx}{\sqrt{9\left(\frac{16}{9}-x^2\right)}}$$

$$= \frac{4}{3}\int \frac{dx}{\sqrt{\left(\frac{4}{3}\right)^2-x^2}}$$

$$= \frac{4}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$$

⑦

$$\int_0^{\frac{3}{4}} \frac{dx}{\sqrt{9-4x^2}}$$

$$= \int_0^{\frac{3}{4}} \frac{dx}{2\sqrt{\left(\frac{9}{4}\right)-x^2}}$$

$$= \int_0^{\frac{3}{4}} \frac{dx}{2\sqrt{\left(\frac{3}{2}\right)^2-x^2}}$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^{\frac{3}{4}}$$

$$= \frac{1}{2}\left(\sin^{-1}\frac{2}{3}\left(\frac{3}{4}\right)-\sin^{-1}0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{12}$$

⑧

$$\int_0^4 \frac{9dx}{3x^2+16}$$

$$= \int_0^4 \frac{9dx}{3\left(x^2+\frac{16}{3}\right)}$$

$$= 3\int_0^4 \frac{dx}{x^2+\left(\frac{4}{\sqrt{3}}\right)^2}$$

$$= 3\left[\frac{\sqrt{3}}{4}\tan^{-1}\left(\frac{x\sqrt{3}}{4}\right)\right]_0^4$$

$$= 3\left(\frac{\sqrt{3}}{4}\tan^{-1}\sqrt{3}-\frac{\sqrt{3}}{4}\tan^{-1}0\right)$$

$$= \frac{\pi\sqrt{3}}{4}$$

⑨

$$\int_2^5 \frac{dx}{9+(x-2)^2}$$

Let $u = x-2$, $\frac{du}{dx}=1$, $du = dx$

$x=2$, $u=0$

$x=5$, $u=3$

$$= \int_0^3 \frac{du}{9+u^2}$$

$$= \left[\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_0^3$$

$$= \frac{1}{3}\left(\tan^{-1}(1)-\tan^{-1}(0)\right)$$

$$= \frac{1}{3}\left(\frac{\pi}{4}-0\right)$$

$$= \frac{\pi}{12}$$

If we have an integral of the form $\int \frac{dx}{ax^2+bx+c}$ where the quadratic in the denominator does not factorise we have to complete the square in the denominator and then use substitution as in example ⑨.

⑩ $\int_2^7 \frac{dx}{x^2-4x+29}$

$$= \int_2^7 \frac{dx}{(x-2)^2+25}$$

Let $u = x-2$ $\quad x=2 \quad x=7$

$\frac{du}{dx}=1 \quad u=0 \quad u=5$

$du = dx$

$$= \int_0^5 \frac{du}{u^2+5^2}$$

$$= \frac{1}{5}\left[\tan^{-1}\left(\frac{u}{5}\right)\right]_0^5$$

$$= \frac{1}{5}\left[\tan^{-1}1-\tan^{-1}0\right]$$

$$= \frac{1}{5}\left[\frac{\pi}{4}-0\right]$$

$$= \frac{\pi}{20}$$

⑪ $\int \frac{3x-5}{x^2+16}dx$ Here we need to use a combination of integration techniques.

$$= \int \frac{3x}{x^2+16}dx - \int \frac{5}{x^2+16}dx$$

Let $u = x^2+16$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = xdx$

$$= 3\int \frac{du}{2u} - 5\int \frac{dx}{x^2+4^2}$$

$$= \frac{3}{2}\ln u - 5\left(\frac{1}{4}\right)\tan^{-1}\left(\frac{x}{4}\right)+C$$

$$= \frac{3}{2}\ln\left|x^2+16\right| - \frac{5}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$$

⑫ $\int_0^3 \frac{x-3}{x^2+9}dx$

$$= \int_0^3 \frac{x}{x^2+9}dx - \int_0^3 \frac{3}{x^2+9}dx$$

Let $u = x^2+9$ $\quad x=0 \quad x=3$

$\frac{du}{dx} = 2x \quad u=9 \quad u=18$

$\frac{du}{2} = dx$

$$= \frac{1}{2}\int_9^{18}\frac{du}{u} - 3\int_0^3 \frac{dx}{x^2+3^2}$$

$$= \frac{1}{2}\left[\ln u\right]_9^{18} - \left[3\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= \frac{1}{2}(\ln 18 - \ln 9) - \left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

Some Questions Involving Partial Fractions

Example

$$\int \frac{9x+8}{(x-2)(x^2+9)}\,dx$$

$$=\int\left(\frac{2}{x-2}+\frac{5-2x}{x^2+9}\right)dx$$

$$=\int\frac{2}{x-2}\,dx+\int\frac{5}{x^2+9}\,dx-\int\frac{2x}{x^2+9}\,dx$$

$$=2\ln|x-2|+5\int\frac{1}{x^2+9}\,dx-\ln|x^2+9|+C$$

$$=2\ln|x-2|+\frac{5}{3}\tan^{-1}\frac{x}{3}-\ln|x^2+9|+C$$

Partial fractions

$$\frac{9x+8}{(x-2)(x^2+9)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+9}$$

$$9x+8=A(x^2+9)+(Bx+C)(x-2)$$

$x=2$	$x=0$	$x=1$
$26=13A$	$8=18-2C$	$17=20-B-5$
$A=2$	$C=5$	$B=-2$

Questions

① $\displaystyle\int\frac{3x-1}{(x-2)(x^2+1)}\,dx$ ② $\displaystyle\int\frac{3x^2-4x+10}{(x+1)(x^2+16)}\,dx$ ③ $\displaystyle\int\frac{dx}{x^3-x^2+4x-4}$

Integration by Parts

Integration by parts is useful when we have to integrate the product of two functions. (It is like a product rule for integration.)

It comes from the product rule….. $\frac{d}{dx}(uv)=u'v+uv'$

If we integrate both sides….. $uv=\int u'v\,dx+\int uv'\,dx$

Rearranging this gives $\boxed{\int uv'\,dx=uv-\int u'v\,dx}$

This is the rule for integration by parts.

We denote the two functions as u and v'. We should pick the u which makes u' the simplest.

u is usually a polynomial and v' is usually a trig or exp function.

Examples

① $\int x\sin x\, dx$

Let $u = x$ $\quad v' = \sin x$

$u' = 1$ $\quad v = -\cos x$

$$= -x\cos x - \int -\cos x\, dx$$

$$= -x\cos x + \sin x + C$$

② $\int (2x+4)e^x\, dx$

Let $u = 2x+4$ $\quad v' = e^x$

$u' = 2$ $\quad v = e^x$

$$= (2x+4)e^x - \int 2e^x\, dx$$

$$= (2x+4)e^x - 2e^x + C$$

③ $\int x\sec^2 x\, dx$

Let $u = x$ $\quad v' = \sec^2 x$

$u' = 1$ $\quad v = \tan x$

$$= x\tan x - \int \tan x\, dx$$

$$= x\tan x - \int \frac{\sin x}{\cos x} dx$$

$$= x\tan x + \ln|\cos x| + C$$

④ $\int \frac{\ln x}{x^3} dx$

Let $u = \ln x$ $\quad v' = \frac{1}{x^3}$

$u' = \frac{1}{x}$ $\quad v = \frac{-1}{2x^2}$

$$= \frac{-\ln x}{2x^2} - \int \frac{-1}{2x^3} dx$$

$$= \frac{-\ln x}{2x^2} - \left(\frac{1}{4x^2}\right) + C$$

$$= \frac{-1}{2x^2}\left(\ln x + \frac{1}{2}\right) + C$$

⑤ $\int \sin x \ln|\cos x|\, dx$

Let $u = \ln|\cos x|$ $\quad v' = \sin x$

$u' = \frac{1}{\cos x}\frac{d}{dx}(\cos x)$ $\quad v = -\cos x$

$= \frac{-\sin x}{\cos x}$

$$= -\cos x \ln|\cos x| - \int \frac{\sin x}{\cos x}\cos x\, dx$$

$$= -\cos x \ln|\cos x| - \int \sin x\, dx$$

$$= -\cos x \ln|\cos x| + \cos x + C$$

$$= \cos x\left(1 - \ln|\cos x|\right) + C$$

Questions

① $\int x\cos x\ dx$ ② $\int xe^{x}\ dx$ ③ $\int x\cos\left(2x+1\right)\ dx$

④ $\int x\left(2x+3\right)^{3}\ dx$ ⑤ $\int x\sqrt{1+3x}\ dx$ ⑥ $\int\left(x+2\right)\ln x\ dx$

Definite Integrals – Integration by Parts

When we use integration by parts to find definite integrals we have

$$\int_a^b uv'\,dx=\left[uv\right]_a^b-\int_a^b u'v\ dx$$

Examples

① $\int_0^{\frac{\pi}{2}} 3x\sin x\ dx$ Let $u=3x$ $v'=\sin x$

$u'=3$ $v=-\cos x$

$$=\left[-3x\cos x\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}3\cos x\ dx$$

$$=\left(\frac{-3\pi}{2}\cos\frac{\pi}{2}-0\right)+\left[3\sin x\right]_0^{\frac{\pi}{2}}$$

$$=3\sin\frac{\pi}{2}-3\sin 0$$

$$=3$$

② $\int_1^2\frac{\ln x}{x^2}dx$ Let $u=\ln x$ $v'=\frac{1}{x^2}$

$u'=\frac{1}{x}$ $v=\frac{-1}{x}$

$$=\left[\frac{-\ln x}{x}\right]_1^2+\int_1^2\frac{1}{x^2}\,dx$$

$$=\frac{-1}{2}\ln 2-\left[\frac{1}{x}\right]_1^2$$

$$=\frac{-1}{2}\ln 2-\left(\frac{1}{2}-1\right)$$

$$=\frac{1-\ln 2}{2}$$

③ $\int_4^6 x\exp(x-4)\ dx$ $u=x$ $v'=e^{(x-4)}$

$u'=1$ $v=e^{(x-4)}$

$$=\left[xe^{(x-4)}\right]_4^6-\int_4^6 e^{(x-4)}dx$$

$$=6e^2-4e^0-\left[e^{(x-4)}\right]_4^6$$

$$=6e^2-4-\left(e^2-e^0\right)$$

$$=5e^2-3$$

④ $\int_0^{\frac{\pi}{4}} (x+4)\sin 2x\, dx$ $\qquad u = x+4 \qquad v' = \sin 2x$

$= \left[\frac{-(x+4)}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x\, dx$ $\qquad u' = 1 \qquad v = \frac{-1}{2}\cos 2x$

$= 0 - \left(\frac{-4}{2}\right) + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= 2 + \frac{1}{2}\left(\frac{1}{2}(1) - \frac{1}{2}(0)\right)$

$= 2 + \frac{1}{4}$

$= \frac{9}{4}$

⑤ $\int_0^3 x(x-2)^3 dx$ $\qquad u = x \qquad v' = (x-2)^3$

$= \left[\frac{x(x-2)^4}{4}\right]_0^3 - \frac{1}{4}\int_0^3 (x-2)^4 dx$ $\qquad u' = 1 \qquad v = \frac{1}{4}(x-2)^4$

$= \left(\frac{3}{4} - 0\right) - \frac{1}{4}\left[\frac{1}{5}(x-2)^5\right]_0^3$

$= \frac{3}{4} - \frac{1}{4}\left(\frac{1}{5} + \frac{32}{5}\right)$

$= \frac{3}{4} - \frac{33}{20}$

$= \frac{-18}{20}$

$= \frac{-9}{10}$

<u>Questions</u>

① $\int_0^{\frac{\pi}{2}} x\cos x\, dx$ ② $\int_0^1 xe^{5x} dx$ ③ $\int_0^1 x(x+1)^4\, dx$ ④ $\int_0^3 x\sqrt{x+1}\, dx$

⑤ $\int_0^{\pi} x\sin\left(\frac{x}{2}\right) dx$ ⑥ $\int_1^2 x^3 \ln x\, dx$ ⑦ $\int_0^1 xe^{-2x} dx$ ⑧ $\int_3^6 \frac{x}{\sqrt{x-2}}\, dx$

More Than One Application Of Integration By Parts

Sometimes the second integral we get also needs to be integrated using integration by parts.

Examples

① $\int x^2 e^x dx$ $\quad u = x^2 \quad v' = e^x$

$= x^2 e^x - \int 2xe^x dx$ $\quad u' = 2x \quad v = e^x$

To find $\int 2xe^x dx$, use integration by parts again.

$u = 2x \quad v' = e^x$

$= x^2 e^x - \left(2xe^x - 2\int e^x dx\right)$ $\quad u' = 2 \quad v = e^x$

$= x^2 e^x - 2xe^x + 2e^x + C$

$= e^x \left(x^2 - 2x + 2\right) + C$

② $\int x^2 \cos 3x \, dx$ $\quad u = x^2 \quad v' = \cos 3x$

$= \frac{x^2}{3} \sin 3x - \frac{2}{3} \int x \sin 3x \, dx$ $\quad u' = 2x \quad v = \frac{1}{3} \sin 3x$

To find $\int x \sin 3x \, dx$, use integration by parts again.

$u = x \quad v' = \sin 3x$

$= \frac{x^2}{3} \sin 3x - \frac{2}{3}\left(\frac{-x}{3} \cos 3x + \frac{1}{3} \int \cos 3x \, dx\right)$ $\quad u' = 1 \quad v = \frac{-1}{3} \cos 3x$

$= \frac{x^2}{3} \sin 3x + \frac{2x}{9} \cos 3x - \frac{2}{9}\left(\frac{1}{3} \sin 3x\right) + C$

$= \frac{x^2}{3} \sin 3x + \frac{2x}{9} \cos 3x - \frac{2}{27} \sin 3x + C$

③ $\int_0^{\frac{\pi}{6}} x^2 \sin 3x \, dx$ $\qquad u = x^2 \qquad v' = \sin 3x$

$= \left[\frac{-x^2}{3} \cos 3x \right]_0^{\frac{\pi}{6}} + \frac{2}{3} \int_0^{\frac{\pi}{6}} x \cos 3x \, dx$ $\qquad u' = 2x \qquad v = \frac{-1}{3} \cos 3x$

To find $\int_0^{\frac{\pi}{6}} x \cos 3x \, dx$, use integration by parts again.

$u = x \qquad v' = \cos 3x$

$= \left[\left(\frac{-\pi^2}{36 \times 3} \cos \frac{\pi}{2} \right) - 0 \right] + \frac{2}{3} \left(\left[\frac{x}{3} \sin 3x \right]_0^{\frac{\pi}{6}} - \frac{1}{3} \int_0^{\frac{\pi}{6}} \sin 3x \, dx \right)$ $\qquad u' = 1 \qquad v = \frac{1}{3} \sin 3x$

$= \frac{2}{3} \left(\left(\frac{\pi}{18} \sin \frac{\pi}{2} - 0 \right) + \frac{1}{9} [\cos 3x]_0^{\frac{\pi}{6}} \right)$

$= \frac{\pi}{27} + \frac{2}{27} \left(\cos \frac{\pi}{2} - \cos 0 \right)$

$= \frac{\pi}{27} - \frac{2}{27}$

$= \frac{\pi - 2}{27}$

Questions

Find ① $\int x^2 e^{-4x} dx$ ② $\int_0^2 x^2 e^x dx$ ③ $\int x^3 e^x dx$

Using Integration By Parts As A 'Trick' For Integrating

Some functions which at first glance look as though they cannot be integrated can be integrated by using integration by parts.

Examples

① $\int \ln x \, dx$

We write this as $\int 1 \times \ln x \, dx$ so that we can use integration by parts.

$\int 1 \times \ln x \, dx$ $\qquad u = \ln x \qquad v' = 1$

$= x \ln x - \int \frac{x}{x} dx$ $\qquad u' = \frac{1}{x} \qquad v = x$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x + C$

② $\int \tan^{-1} x \, dx$

We write this as $\int 1 \times \tan^{-1} x \, dx$ so that we can use integration by parts.

$$\int 1 \times \tan^{-1} x \, dx \qquad u = \tan^{-1} x \qquad v' = 1$$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2} dx \qquad u' = \frac{1}{1+x^2} \qquad v = x$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + C$$

③ $\int \sin^{-1} x \, dx$

$$= \int 1 \times \sin^{-1} x \, dx \qquad u = \sin^{-1} x \qquad v' = 1$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx \qquad u' = \frac{1}{\sqrt{1-x^2}} \qquad v = x$$

$$= x\sin^{-1} x + \frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

$$= x\sin^{-1} x + \frac{1}{2}\int u^{\frac{-1}{2}} du$$

$$= x\sin^{-1} x + \frac{1}{2}\left(2u^{\frac{1}{2}}\right) + C$$

$$= x\sin^{-1} x + \sqrt{1-x^2} + C$$

Use substitution

Let $u = 1 - x^2$

$$\frac{du}{dx} = -2x$$

$$x \, dx = \frac{-1}{2} du$$

Integration By Parts – Cyclical Calculations

Sometimes when we integrate by parts, we end up going round in circles.

Examples

① $\int e^x \cos x \, dx \qquad u = e^x \qquad v' = \cos x$

$$u' = e^x \qquad v = \sin x$$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx \qquad u = e^x \qquad v' = \sin x$$

$$= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx\right) \qquad u' = e^x \qquad v = -\cos x$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

The integral on the RHS is the same as the integral we began with.

If we transfer it to the LHS we can begin to approach a solution.

Add $\int e^x \cos x \, dx$ to both sides.

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$$

$$= \frac{e^x}{2}\left(\sin x + \cos x\right) + C$$

② $\int e^x \sin(1-2x) \, dx$ $\qquad u = e^x \qquad v' = \sin(1-2x)$

$$u' = e^x \qquad v = \frac{1}{2}\cos(1-2x)$$

$$\int e^x \sin(1-2x) \, dx = \frac{e^x}{2}\cos(1-2x) - \frac{1}{2}\int e^x \cos(1-2x) dx \qquad u = e^x \quad v' = \cos(1-2x)$$

$$= \frac{e^x}{2}\cos(1-2x) - \frac{1}{2}\left(\frac{-e^x}{2}\sin(1-2x) + \frac{1}{2}\int e^x \sin(1-2x) \, dx\right) \qquad u' = e^x \quad v = \frac{-1}{2}\sin(1-2x)$$

$$= \frac{e^x}{2}\cos(1-2x) + \frac{e^x}{4}\sin(1-2x) - \frac{1}{4}\int e^x \sin(1-2x) \, dx$$

$$\frac{5}{4}\int e^x \sin(1-2x) \, dx = \frac{e^x}{4}\left(2\cos(1-2x) + \sin(1-2x)\right) + C$$

$$\int e^x \sin(1-2x) \, dx = \frac{e^x}{5}\left(2\cos(1-2x) + \sin(1-2x)\right) + C$$

Questions

Find

① $\int e^{3x} \sin(1-x) \, dx$ ② $\int \frac{\sin x}{e^x} dx$ ③ $\int e^{-2x} \cos(3x-4) dx$

④ $\int e^{5x} \cos(-2x-4) dx$ ⑤ $\int \frac{\cos(1-3x)}{e^{2x}} dx$

Integration Using Reduction Formulae

This technique can be used to integrate functions involving powers which can't be integrated directly. They can be integrated by setting up a **reduction formula** to obtain an integral of the same or similar expression with a lower power and progressively simplifying the integral until it can be integrated. The reduction formula can be found by using any of the techniques already covered – by substitution, by partial fractions, by parts etc.

Examples

① (a) Evaluate $\int_0^{\pi} x\sin x\,dx.$

(b) Let $I_n = \int_0^{\pi} x^n \sin x\,dx,$ where n is a positive integer.

Show that for $n \geq 3,\ I_n = \pi^n - n\ \ n-1\ \ I_{n-2}.$ This is a **reduction formula.**

(c) Use these results to evaluate $\int_0^{\pi} x^3 \sin x\,dx.$

(a) $\int_0^{\pi} x\sin x\ dx$ Let $u = x, \quad v' = \sin x$

$= \left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} -\cos x\ dx$ $\quad u' = 1 \quad v = -\cos x$

$= \left(-\pi\cos\pi - 0\right) + \left[\sin x\right]_0^{\pi}$

$= \pi + \left(\sin\pi - \sin 0\right)$

$= \pi$

(b) $I_n = \int_0^{\pi} x^n \sin x\ dx$ Let $u = x^n, \quad v' = \sin x$

$= \left[-x^n \cos x\right]_0^{\pi} - \int_0^{\pi} -nx^{n-1}\cos x\ dx$ $\quad u' = nx^{n-1} \quad v = -\cos x$

$= \left[-x^n \cos x\right]_0^{\pi} + \int_0^{\pi} nx^{n-1}\cos x\ dx$ Let $u = nx^{n-1}, \quad v' = \cos x$

$= \left(-\pi^n \cos\pi - 0\right) + \left[nx^{n-1}\sin x\right]_0^{\pi} - \int_0^{\pi} n\left(n-1\right)x^{n-2}\sin x\ dx$ $\quad u' = n\left(n-1\right)x^{n-2}, \quad v = \sin x$

$= \pi^n + \left(\pi^{n-1}\sin\pi - 0\right) - n\left(n-1\right)\int_0^{\pi} x^{n-2}\sin x\ dx$

$= \pi^n - n\left(n-1\right)I_{n-2}$ since $I_{n-2} = \int_0^{\pi} x^{n-2}\sin x\ dx$

So $I_n = \pi^n - n\left(n-1\right)I_{n-2}$ for $n \geq 3$

(c) We can now use this reduction formula to evaluate $\int_0^{\pi} x^3 \sin x\ dx$

$\int_0^{\pi} x^3 \sin x\ dx = I_3$

$I_3 = \pi^3 - 3\left(3-1\right)I_1$

$= \pi^3 - 6I_1$

$= \pi^3 - 6\pi$

So $\int_0^{\pi} x^3 \sin x\ dx = \pi^3 - 6\pi$

② $I_n = \int_0^1 x^n e^{-x}\, dx, \;\; n \geq 1.$

Find I_1 and show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.

Hence evaluate I_3.

$$
\begin{aligned}
I_1 &= \int_0^1 xe^{-x}\, dx && \text{Let } u = x, \quad v' = e^{-x} \\
&= \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\, dx && u' = 1, \quad v = -e^{-x} \\
&= \left(-e^{-1} - 0\right) + \left[-e^{-x}\right]_0^1 \\
&= -e^{-1} + \left(-e^{-1} + 1\right) \\
&= 1 - 2e^{-1}
\end{aligned}
$$

$$
\begin{aligned}
I_n &= \int_0^1 x^n e^{-x}\, dx && \text{Let } u = x^n, \quad v' = e^{-x} \\
&= \left[-x^n e^{-x}\right]_0^1 + \int_0^1 nx^{n-1} e^{-x}\, dx && u' = nx^{n-1} \quad v = -e^{-x} \\
&= -e^{-1} + n\int_0^1 x^{n-1} e^{-x}\, dx \\
&= -e^{-1} + nI_{n-1} \\
&= nI_{n-1} - e^{-1}, \quad n \geq 2
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= 3I_2 - e^{-1} \\
&= 3\left(2I_1 - e^{-1}\right) - e^{-1} \\
&= 6I_1 - 4e^{-1} \\
&= 6\left(1 - 2e^{-1}\right) - 4e^{-1} \\
&= 6 - 16e^{-1} \simeq 0{\cdot}1139
\end{aligned}
$$

Questions

① Establish a Reduction Formula that could be used to find $\int x^n e^x\, dx$ and use it when $n = 3$.

Let *n* be a positive integer. Use Integration by Parts to derive the formula

$$\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \tfrac{n}{a}\int x^{n-1} e^{ax}\, dx, \quad (a \neq 0).$$

② Show that $\int \sin^n x\, dx = -\tfrac{1}{n}\sin^{n-1} x \cos x + \tfrac{n-1}{n}\int \sin^{n-2} x\, dx.$

Hence evaluate (i) $\int \sin^5 x\, dx$ (ii) $\int \sin^6 x\, dx.$

③ Obtain a result for $\int \cos^n x\,dx$ similar to that of Q2 above.

Use your result to evaluate (i) $\int \cos^5 x\,dx$ (ii) $\int \cos^6 x\,dx$.

④ Let $I_n = \int_0^1 \frac{1}{\left(1+x^2\right)^n}\,dx, n \geq 1.$

(a) Use integration by parts to show that $I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{\left(1+x^2\right)^{n+1}}\,dx.$

(b) Find the values of A and B for which $\frac{A}{\left(1+x^2\right)^n} + \frac{B}{\left(1+x^2\right)^{n+1}} = \frac{x^2}{\left(1+x^2\right)^{n+1}}$.

Hence show that $I_{n+1} = \frac{1}{n \times 2^{n+1}} + \left(\frac{2n-1}{2n}\right)I_n$.

(c) Hence obtain the exact value of $\int_0^1 \frac{1}{\left(1+x^2\right)^3}\,dx.$

Past Paper Questions

2001

① Find the value of $\int_0^{\pi/4} 2x\sin 4x\ dx$. **(5 marks)**

② (a) Obtain partial fractions for $\frac{x}{x^2-1}$, $x > 1$.

(b) Use the results of (a) to find $\int \frac{x^3}{x^2-1}\,dx$, $x > 1$. **(2, 4 marks)**

2002

Use integration by parts to evaluate $\int_0^1 \ln(1+x)\ dx$. **(5 marks)**

2003

Define $I_n = \int_0^1 x^n e^{-x}\ dx,\ \ n \geq 1$

(a) Use integration by parts to obtain the value of $I_1 = \int_0^1 xe^{-x}\ dx$

(b) Similarly, show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.

(c) Evaluate I_3. **(3, 4, 3 marks)**

2004

Express $\dfrac{1}{x^2 - x - 6}\ dx$ in partial fractions.

Evaluate $\int_0^1 \dfrac{1}{x^2 - x - 6}\ dx$ **(2, 4 marks)**

2005

Express $\dfrac{1}{x^3 + x}$ in partial fractions.

Obtain a formula for $I(k)$, where $I(k) = \int_1^k \dfrac{1}{x^3 + x}\ dx$, expressing it in the form ln $\dfrac{a}{b}$,

where a and b depend on k.

Write down an expression for $e^{I(k)}$ and obtain the value of $\lim\limits_{k \to \infty} e^{I(k)}$. **(4, 4, 2 marks)**

2006

① (a) Determine whether $f(x) = x^2 \sin x$ is odd, even or neither. Justify your answer.

(b) Use integration by parts to find $\int x^2 \sin x\ dx$.

(c) Hence find the area bounded by $y = x^2 \sin x$, the lines $x = \dfrac{-\pi}{4}$, $x = \dfrac{\pi}{4}$ and the x-axis.

(3, 4, 3 marks)

② (a) Show that $\int \sin^2 x \cos^2 x \, dx = \int \cos^2 x \, dx - \int \cos^4 x \, dx$.

(b) By writing $\cos^4 x = \cos x \cos^3 x$ and using integration by parts, show that

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{1}{4} + 3\int_0^{\pi/4} \sin^2 x \cos^2 x \, dx.$$

(c) Show that $\int_0^{\pi/4} \cos^2 x \, dx = \frac{\pi + 2}{8}$.

(d) Hence, using the above results, show that $\int_0^{\pi/4} \cos^4 x \, dx = \frac{3\pi + 8}{32}$. **(1, 3, 3, 3 marks)**

2007

Express $\frac{2x^2 - 9x - 6}{x\left(x^2 - x - 6\right)}$ in partial fractions.

Given that $\int_4^6 \frac{2x^2 - 9x - 6}{x\left(x^2 - x - 6\right)} \, dx = \ln \frac{m}{n}$, determine values for the integers m and n.

(3, 3 marks)

2008

① Express $\frac{12x^2 + 20}{x\left(x^2 + 5\right)}$ in partial fractions.

Hence, evaluate $\int_1^2 \frac{12x^2 + 20}{x\left(x^2 + 5\right)} \, dx$. **(3, 3 marks)**

② Use integration by parts to obtain $\int 8x^2 \sin 4x \, dx$. **(5 marks)**

2009

Use integration by parts to obtain the exact value of $\int_0^1 x \tan^{-1} x \, dx$. **(5 marks)**

2010

① (a) Use the substitution $t = x^4$ to obtain $\int \frac{x^3}{1 + x^8} \, dx$.

(b) Integrate $x^2 \ln x$ with respect to x. **(3, 4 marks)**

② Evaluate $\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx$ expressing your answer in the form $\ln\frac{a}{b}$ where a and b are integers. **(6 marks)**

2011

① Express $\frac{13-x}{x^2+4x-5}$ in partial fractions and hence obtain $\int \frac{13-x}{x^2+4x-5}\,dx$. **(5 marks)**

② (a) Obtain the exact value of $\int_0^{\pi/4} (\sec x - x)(\sec x + x)\ dx$.

(b) Find $\int \frac{x}{\sqrt{1-49x^4}}\,dx$. **(3, 4 marks)**

③ Let $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx, n \geq 1.$

(a) Use integration by parts to show that $I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx.$

(b) Find the values of A and B for which $\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$.

Hence show that $I_{n+1} = \frac{1}{n \times 2^{n+1}} + \left(\frac{2n-1}{2n}\right)I_n$.

(c) Hence obtain the exact value of $\int_0^1 \frac{1}{(1+x^2)^3}\,dx.$ **(3, 5, 3 marks)**

2012

(a) Write down the derivative of $\sin^{-1} x$.

(b) Use integration by parts to obtain $\int \sin^{-1} x.\frac{x}{\sqrt{1-x^2}}\, dx$. **(1, 4 marks)**

2013

Use integration by parts to obtain $\int x^2 \cos 3x\, dx$. **(5 marks)**

2014

(a) Use integration by parts to obtain an expression for $\int e^x \cos x\, dx$.

(b) Similarly, given $I_n = \int e^x \cos nx\, dx$ where $n \neq 0$, obtain an expression for I_n.

(c) Hence evaluate $\int_0^{\pi/2} e^x \cos 8x\, dx$. **(4, 4, 2 marks)**

2015

Obtain the exact value of $\int_0^2 x^2 e^{4x} dx$ **(5 marks)**

2016

Obtain $\int x^7 (\ln x)^2\, dx.$ **(6 marks)**