Integration

Some questions from Higher to begin:-

① $\int \left(6x^2 - 5x\right)\, dx$ ② $\int \sin\left(3x-1\right)\, dx$ ③ $\int \left(\frac{4-x}{\sqrt{x}}\right) dx$

④ $\int \left(\frac{x^4-6}{x^3}\right) dx$ ⑤ $\int \left(4x-5\right)^2\, dx$ ⑥ $\int_1^3 \left(t^2 + \frac{1}{t^2}\right) dt$

Some new results…..

In the previous outcome on differentiation we learned that

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$

It follows that

$$\boxed{\begin{aligned} &\int \sec^2 x\, dx = \tan x + C \\ &\int e^x\, dx = e^x + C \\ &\int \frac{1}{x}\, dx = \ln|x| + C, \quad |x| = \begin{cases} x, x \geq 0 \\ -x, x \leq 0 \end{cases} \end{aligned}}$$

More usefully:

$$\boxed{\begin{aligned} &\int \sec^2\left(ax+b\right) dx = \frac{1}{a}\tan\left(ax+b\right) + C \\ &\int e^{(ax+b)}\, dx = \frac{1}{a}e^{(ax+b)} + C \\ &\int \frac{1}{(ax+b)}\, dx = \frac{1}{a}\ln|ax+b| + C \end{aligned}}$$

Examples

① $\int \frac{5}{x}\,dx$ ② $\int_0^4 e^{5x}\,dx$ ③ $\int_0^1 \frac{dx}{3x+5}$ ④ $\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$ ⑤ $\int \frac{ds}{2s}$

① $= 5\int \frac{1}{x}\,dx = 5\ln|x| + C$

② $= \left[\frac{1}{5}e^{5x}\right]_0^4 = \frac{1}{5}e^{20} - \frac{1}{5} = \frac{e^{20}-1}{5}$

③ $= \left[\frac{1}{3}\ln|3x+5|\right]_0^1 = \frac{1}{3}(\ln 8 - \ln 5) = \frac{1}{3}\ln\left(\frac{8}{5}\right)$

④ $= \left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{8}} = \frac{1}{2}\left(\tan\frac{\pi}{4} - \tan 0\right) = \frac{1}{2}$

⑤ $= \frac{1}{2}\ln|2s| + C$

Questions

① $\int 8e^{-4u}\,du$ ② $\int \sec^2(4x+1)\,dx$ ③ $\int 3e^{(5x+1)}\,dx$ ④ $\int_3^7 \frac{4}{x}\,dx$

⑤ $\int_0^1 \frac{1}{3x+2}\,dx$ ⑥ $\int_0^e \frac{dx}{x+e}$ ⑦ $\int_1^2 \left(3 - e^{-x}\right)dx$ ⑧ $\int \mathrm{e}^{-5t}\,dt$

⑨ $\int (4x-3)^{-1}dx$ ⑩ $\int_0^{\frac{\pi}{4}} \sec^2(3x)\,dx$ ⑪ $\int_1^4 \frac{8}{2x+3}\,dx$ ⑫ $\int_{-1}^1 4e^{(1-2x)}\,dx$

In some cases a bit of rearranging may be required to aid the integration:-

Examples

① $\int \frac{t}{t-1}\,dt$

$= \int \frac{t-1+1}{t-1}\,dt$ Rewrite the first line- we have not changed the function.

$= \int \left(\frac{t-1}{t-1} + \frac{1}{t-1}\right)dt$ Split into proper fractions which we can integrate.

$= \int \left(1 + \frac{1}{t-1}\right)dt$

$= t + \ln|t-1| + C$

② $\int \frac{x}{x+3}\,dx$

$= \int \frac{x+3-3}{x+3}\,dx$

$= \int \left(1-\frac{3}{x+3}\right) dx$

$= x - 3\ln|x+3| + C$

③ $\int \frac{x+6}{x+1}\,dx$

$= \int \frac{x+1+5}{x+1}\,dx$

$= \int \left(1+\frac{5}{x+1}\right) dx$

$= x + 5\ln|x+1| + C$

④ $\int \frac{t}{6+2t}\,dt$

$= \frac{1}{2}\int \frac{t}{3+t}\,dt$

$= \frac{1}{2}\int \frac{t+3-3}{t+3}\,dt$

$= \frac{1}{2}\int \left(1-\frac{3}{t+3}\right) dt$

$= \frac{1}{2}\left(t - 3\ln|t+3|\right) + C$

Questions

① $\int \frac{t}{t-4}\,dt$ ② $\int \frac{5+w}{w+4}\,dw$ ③ $\int \frac{x}{6x+12}\,dx$

Remember that integration "reverses" differentiation so differentiating your answer acts as a check.

Integration By Substitution

Suppose we have the following:

$$\int x^3\left(2x^4-2\right)^3\,dx$$

How can we solve this? Expanding the function into separate terms would be a long task. We can turn it into an easier integration by substituting a different variable into the function.

Let $u = 2x^4 - 2$

$\frac{du}{dx} = 8x^3$

$\frac{1}{8}du = x^3dx$ We now replace x^3dx by $\frac{1}{8}du$ and $2x^4-2$ by u in the integrand giving

$\int x^3\left(2x^4-2\right)^3\,dx = \int \frac{1}{8}u^3du$ This is now an easy integration.

$= \frac{1}{8}\int u^3du$

$= \frac{1}{8}\left(\frac{1}{4}u^4\right) + C$ Now substitute $u = 2x^3 - 2$ back in and tidy up.

$= \frac{1}{32}\left(2x^4-2\right)^4 + C$

More examples

① $\int \sin^2 x \cos x \, dx$ Let $u = \sin x$

$= \int u^2 du$ $\frac{du}{dx} = \cos x$

$= \frac{1}{3}u^3 + C$ $du = \cos x \, dx$

$= \frac{1}{3}\sin^3 x + C$

② $\int 4x\left(x^2+7\right)^6 dx$ Let $u = x^2 + 7$

$= \int 2u^6 \, du$ $\frac{du}{dx} = 2x$

$= \frac{2u^7}{7} + C$ $du = 2x \, dx$

$= \frac{2\left(x^2+7\right)^7}{7} + C$

③ $\int 8\cos x \sin^3 x \, dx$ Let $u = \sin x$

$= \int 8u^3 \, du$ $\frac{du}{dx} = \cos x$

$= \frac{8u^4}{4} + C$ $du = \cos x \, dx$

$= 2\sin^4 x + C$

④ $\int (2x+3)\left(x^2+3x\right)^5 dx$ Let $u = x^2 + 3x$

$= \int u^5 \, du$ $\frac{du}{dx} = 2x + 3$

$= \frac{u^6}{6} + C$ $du = (2x+3)dx$

$= \frac{\left(x^2+3x\right)^6}{6} + C$

⑤ $\int 5x\sqrt{\left(1+x^2\right)}dx$ Let $u = 1 + x^2$

$= \int \frac{5}{2} u^{\frac{1}{2}} \, du$ $\frac{du}{dx} = 2x$

$= \frac{5}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$ $du = 2x \, dx$

$= \frac{5}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$

$= \frac{5}{3}\sqrt{\left(1+x^2\right)^3} + C$

⑥ $\int \cos x \, e^{\sin x} dx$ Let $u = \sin x$

$= \int e^u \, du$ $\frac{du}{dx} = \cos x$

$= e^u + C$ $du = \cos x \, dx$

$= e^{\sin x} + C$

⑦ $\int \sec^2 x \ \tan^4 x \, dx$ Let $u = \tan x$

$= \int u^4 \, du$ $\frac{du}{dx} = \sec^2 x$

$= \frac{u^5}{5} + C$ $du = \sec^2 x \, dx$

$= \frac{1}{5}\tan^5 x + C$

⑧ $\int x^2 \, \mathrm{e}^{\left(x^3+3\right)} \, dx$ Let $u = x^3 + 3$

$= \int \frac{1}{3}e^u \, du$ $\frac{du}{dx} = 3x^2$

$= \frac{1}{3}e^u + C$ $du = 3x^2 \, dx$

$= \frac{1}{3}\mathrm{e}^{\left(x^3+3\right)} + C$

⑨ $\int 3\left(3x^2+4x\right)\left(x^3+2x^2\right)^4\ dx$ Let $u=x^3+2x^2$

$=\int 3u^4\ du$ $\qquad \dfrac{du}{dx}=3x^2+4x$

$=\dfrac{3}{5}u^5+C$ $\qquad du=\left(3x^2+4x\right)dx$

$=\dfrac{3}{5}\left(x^3+2x^2\right)+C$

⑩ $\int \dfrac{4\ln x}{x}\ dx$ Let $u=\ln x$

$=\int 4u\ du$ $\qquad \dfrac{du}{dx}=\dfrac{1}{x}$

$=\dfrac{4u^2}{2}+C$

$=2\left(\ln|x|\right)^2+C$

Questions

① $\int 3x\left(2x^2-3\right)^7 dx$ ② $\int 16\cos x\sin^5 x\ dx$ ③ $\int 9x\sqrt{\left(1+3x^2\right)}\ dx$

④ $\int\left(3x^2+4\right)\left(x^3+4x\right)^4 dx$ ⑤ $\int \dfrac{3\ln|x-1|}{4x-4}\ dx$ ⑥ $\int \sin x\ e^{\cos x}dx$

⑦ $\int 4\sec^2 x\tan^6 x\ dx$ ⑧ $\int 3\left(5x^4+6x^2\right)e^{\left(x^5+2x^3\right)}\ dx$

Integration By Substitution- Definite Integrals

When we are working with a definite integral we do not have to change back to the original variable **provided we change the limits at the point of substitution**.

Examples

① Use the substitution $x=1+\cos\theta$ to evaluate $\int_0^{\frac{\pi}{2}} \dfrac{\sin\theta}{\left(1+\cos\theta\right)^3}\ d\theta$.

$\int_0^{\frac{\pi}{2}} \dfrac{\sin\theta}{\left(1+\cos\theta\right)^3}\ d\theta$ $\qquad$ Let $x=1+\cos\theta$ $\qquad$ When $\theta=0$, $x=1+\cos 0=2$

$=-\int_2^1 \dfrac{1}{x^3}\ dx$ $\qquad \dfrac{dx}{d\theta}=-\sin\theta$ $\qquad$ When $\theta=\dfrac{\pi}{2}$, $x=1+\cos\dfrac{\pi}{2}=1$

$=\left[\dfrac{1}{2x^2}\right]_2^1$ $\qquad dx=-\sin\theta\ d\theta$

$=\dfrac{1}{2}-\dfrac{1}{8}$

$=\dfrac{3}{8}$

Sometimes we also have to perform a bit of manipulation with the substitution.

② 2007 Q10 - Use the substitution $u = 1 + x^2$ to obtain $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$.

$\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$ Let $u = 1 + x^2$ When $x = 0,\ u = 1$

$= \int_1^2 \frac{x^3}{u^4}\frac{1}{2x}\,du$ $\frac{du}{dx} = 2x$ $dx = \frac{1}{2x}du$ When $x = 1,\ u = 2$

$= \frac{1}{2}\int_1^2 \frac{x^2}{u^4}\,du$ We need an expression for x^2. $x^2 = u - 1$

$$= \frac{1}{2}\int_1^2 \frac{u-1}{u^4}\,du$$

$$= \frac{1}{2}\int_1^2 \left(\frac{u}{u^4} - \frac{1}{u^4}\right) du$$

$$= \frac{1}{2}\int_1^2 \left(u^{-3} - u^{-4}\right) du$$

$$= \frac{1}{2}\left[\frac{-1}{2u^2} + \frac{1}{3u^3}\right]_1^2$$

$$= \frac{1}{2}\left(\frac{-1}{8} + \frac{1}{24} - \left(\frac{-1}{2} + \frac{1}{3}\right)\right)$$

$$= \frac{1}{24}$$

<u>Questions</u>

① $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$ Let $u = 1 - x^2$

② $\int_e^{e^2} \frac{1}{x\ln x}\,dx$ Let $t = \ln x$

③ $\int_{\pi/3}^{\pi/2} \cot x\,dx$ Let $u = \sin x$

Your answer to the last question should indicate an important result in integration which makes many integrals quicker if you can identify it.

This is ……. $\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$

From the last question $\int \cot x \; dx = \int \frac{\cos x}{\sin x} \; dx$.

Here we have the situation $\int \frac{f'(x)}{f(x)} \; dx$.

This allows us to perform the integration using the result $\int \frac{f'(x)}{f(x)} \; dx = \ln\left|f(x)\right| + C$.

$$\int \cot x \; dx = \int \frac{\cos x}{\sin x} \; dx$$
$$= \ln\left|\sin x\right| + C$$

This is definitely quicker than using the substitution method.

Examples

① $\int \frac{2e^{2x}}{1+e^{2x}} \; dx$

$= \ln\left|1+e^{2x}\right| + C$

② $\int \frac{18x^2 - 4x + 10}{6x^3 - 2x^2 + 10x + 7} \; dx$

$= \ln\left|6x^3 - 2x^2 + 10x + 7\right| + C$

Occasionally we may have to remove a common factor and do some manipulation.

③ $\int \frac{12x+8}{3x^2+4x-8} \; dx$

$= \int \frac{2(6x+4)}{3x^2+4x-8} \; dx$

$= 2\ln\left|3x^2+4x-8\right| + C$

④ $\int \frac{e^{8x}}{1+e^{8x}} \; dx$

$= \frac{1}{8}\int \frac{8e^{8x}}{1+e^{8x}} \; dx$

$= \frac{1}{8}\ln\left|1+e^{8x}\right| + C$

⑤ $\int \frac{2}{3x\ln|x|} \; dx$

$= \frac{2}{3}\int \frac{1}{x\ln|x|} \; dx$

$= \frac{2}{3}\int \frac{1/x}{\ln|x|} \; dx$

$= \frac{2}{3}\ln\left|\ln|x|\right| + C$

Questions

① $\int \frac{2x+5}{x^2+5x-3} \; dx$ ② $\int \frac{x^2}{x^3-6} \; dx$ ③ $\int \frac{4x}{1-x^2} \; dx$ ④ $\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}} \; dx$

Important Formulae Which Help Integration

Remember these! They are essential.

$$\boxed{\begin{aligned} &\sin^2 x + \cos^2 x = 1 \\ &1 + \tan^2 x = \sec^2 x \end{aligned}}$$

Also remember double angle formulae.

$$\boxed{\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \\ \sin 2x &= 2\sin x \cos x \end{aligned}}$$

Examples

① $\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx$ Let $x = \sin u$

$= \int_0^{\pi/2} \frac{\cos u}{\sqrt{1-\sin^2 u}}\ du$ $\frac{dx}{du} = \cos u$ When $x = 0,\ u = 0$

$= \int_0^{\pi/2} \frac{\cos u}{\sqrt{\cos^2 u}}\ du$ $dx = \cos u\ du$ When $x = 1,\ u = \frac{\pi}{2}$

$= \int_0^{\pi/2} \frac{\cos u}{\cos u}\ du$

$= \int_0^{\pi/2} 1\ du$

$= [u]_0^{\pi/2}$

$= \frac{\pi}{2}$

② $\int_0^{5/\sqrt{2}} \sqrt{25-x^2}\ dx$ Let $x = 5\sin\theta$

$= \int_0^{\pi/4} \sqrt{25-25\sin^2\theta}\ 5\cos\theta\ d\theta$ $\frac{dx}{d\theta} = 5\cos\theta$ When $x = 0,\ \theta = 0$

$= \int_0^{\pi/4} \sqrt{25\left(1-\sin^2\theta\right)}\ 5\cos\theta\ d\theta$ $dx = 5\cos\theta\ d\theta$ When $x = \frac{5}{\sqrt{2}},\ \theta = \frac{\pi}{4}$

$= 25\int_0^{\pi/4} \sqrt{\cos^2\theta}\ \cos\theta\ d\theta$

$= 25\int_0^{\pi/4} \cos^2\theta\ d\theta$

$= 25\int_0^{\pi/4} \frac{1}{2}\ \left(\cos 2\theta + 1\right)\ d\theta$

$= \frac{25}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi/4}$

$= \frac{25\left(2+\pi\right)}{8}$

③ $\int_0^{2/3} \frac{1}{4+9x^2}\,dx$ Let $x=\frac{2}{3}\tan u$ When $x=0,\ u=0$

$=\int_0^{\pi/4} \frac{1}{4+9\left(\frac{2}{3}\tan u\right)^2}\left(\frac{2}{3}\sec^2 u\right)du$ $\frac{dx}{du}=\frac{2}{3}\sec^2 u$ When $x=\frac{2}{3},\ u=\frac{\pi}{4}$

$=\frac{2}{3}\int_0^{\pi/4} \frac{\sec^2 u}{4+4\tan^2 u}\,du$ $dx=\frac{2}{3}\sec^2 u\ du$

$=\frac{2}{3}\int_0^{\pi/4} \frac{\sec^2 u}{4\left(1+\tan^2 u\right)}\,du$

$=\frac{2}{12}\int_0^{\pi/4} \frac{\sec^2 u}{\sec^2 u}\,du$

$=\frac{1}{6}\int_0^{\pi/4} 1\,du$

$=\frac{1}{6}\left[u\right]_0^{\pi/4}$

$=\frac{\pi}{24}$

Questions

① $\int_0^{3\sqrt{3}/2} \frac{x-2}{\sqrt{9-x^2}}\,dx$ Let $x=3\sin t$

② $\int_0^{1/4} \sqrt{1-4x^2}\,dx$ Let $x=\frac{1}{2}\sin t$

③ $\int_0^{3} \frac{1}{9+3x^2}\,dx$ Let $x=\frac{3}{\sqrt{3}}\tan u$

Integration with Partial Fractions

Remember, we have 3 different types of partial fractions- distinct linear factors, repeated linear factor and irreducible quadratic factor.

Examples

① $$\int \frac{dx}{x^2-3x+2}$$

$$= \int \frac{dx}{(x-2)(x-1)}$$

$$= \int \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx$$

$$= \ln|x-2| - \ln|x-1| + C$$

$$= \ln\left|\frac{x-2}{x-1}\right| + C$$

Partial fractions

$$\frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$1 = A(x-1) + B(x-2)$$

$$x = 2,\ A = 1$$

$$x = 1,\ B = -1$$

② $$\int \frac{x}{(x-1)(x^2-6x+9)} dx$$

$$= \int \frac{x}{(x-1)(x-3)^2} dx$$

$$= \int \left(\frac{1}{4(x-1)} - \frac{1}{4(x-3)} + \frac{3}{2(x-3)^2} \right) dx$$

$$= \frac{1}{4}\int \frac{1}{(x-1)} dx - \frac{1}{4}\int \frac{1}{(x-3)} dx + \frac{3}{2}\int \frac{1}{(x-3)^2} dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x-3| + \frac{3}{2}\left(\frac{-1}{x-3}\right) + C$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x-3}\right| - \frac{3}{2(x-3)} + C$$

Partial fractions

$$\frac{x}{(x-1)(x-3)^2} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$x = A(x-3)^2 + B(x-1)(x-3) + C(x-1)$$

$x = 1$	$x = 3$	$x = 2$
$1 = 4A$	$3 = 2C$	$2 = A - B + C$
$A = \frac{1}{4}$	$C = \frac{3}{2}$	$2 = \frac{7}{4} - B$
		$B = \frac{-1}{4}$

③ $\int \frac{x^5+2}{x^2-1}\,dx$ An improper rational function must be simplified using algebraic long division.

$$\begin{array}{r|l} & x^3 + x \\ \hline x^2-1 & x^5 \qquad\qquad + 2 \\ & \underline{x^5 \quad - x^3} \\ & \quad x^3 \qquad + 2 \\ & \quad \underline{x^3 \quad - x} \\ & \qquad\quad x + 2 \end{array}$$

$$\therefore \frac{x^5+2}{x^2-1} = x^3 + x + \frac{x+2}{(x-1)(x+1)}$$

Partial fractions

$$\frac{x+2}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

$$x+2 = A(x+1) + B(x-1)$$

$x = 1 \qquad x = -1$

$3 = 2A \qquad 1 = -2B$

$A = \frac{3}{2} \qquad B = \frac{-1}{2}$

so $\int \frac{x^5+2}{x^2-1}\,dx$

$$= \int \left(x^3 + x + \frac{x+2}{(x-1)(x+1)} \right) dx$$

$$= \int \left(x^3 + x + \frac{3}{2(x-1)} - \frac{1}{2(x+1)} \right) dx$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

Questions

① $\int \frac{6-2x}{x(x+3)}\,dx$ ② $\int \frac{3}{(x-1)(x-2)^2}\,dx$ ③ $\int \frac{5x^2-x+18}{(2x-1)(x-3)^2}\,dx$

④ $\int \frac{9}{(x-1)(x+2)(2x+1)}\,dx$ ⑤ $\int \frac{x^4+2x^3-3x^2-4x+13}{x^3+3x^2-4}\,dx$

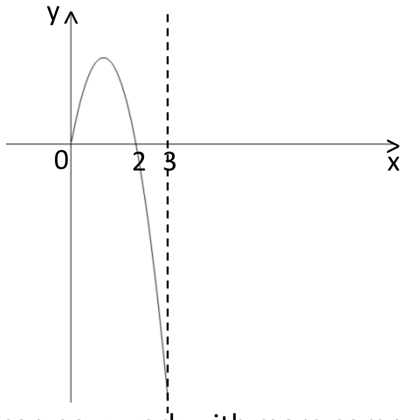
Integration and Area

In Higher we learned how to calculate the area between a curve and the x- axis.

Examples

① Calculate the area between the curve $y=4x-2x^2$, the x-axis, $x=0$ and $x=3$.

A sketch always helps. Remember that the areas above and below the x-axis must be calculated separately.



Area above

$$=\int_0^2 \left(4x-2x^2\right)\, dx$$

$$=\left[2x^2-\frac{2}{3}x^3\right]_0^2$$

$$=2\frac{2}{3}\text{ units}^2$$

Area below

$$=\int_2^3 \left(4x-2x^2\right)\, dx$$

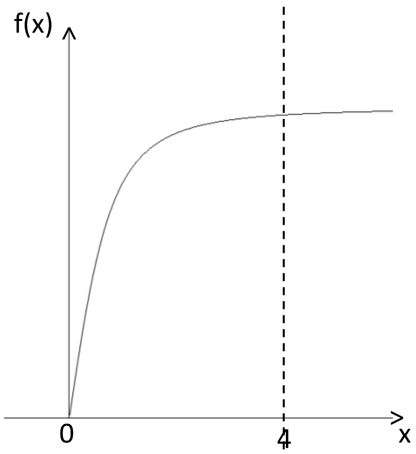
$$=\left[2x^2-\frac{2}{3}x^3\right]_2^3$$

$$=(-)2\frac{2}{3}\text{ units}^2$$

Total area $=5\frac{1}{3}$ units^2

We can now work with more complex integrals.

② Calculate the area between the curve $f(x)=\dfrac{4x}{\sqrt{x^2+2}}$, the x-axis, $x=0$ and $x=4$.



$$\text{Area } =\int_0^4 \frac{4x}{\sqrt{x^2+2}}\, dx \qquad \text{Let } u=x^2+2$$

$$=\int_2^{18} \frac{2}{\sqrt{u}}\, du \qquad \frac{du}{dx}=2x \quad \text{When } x=0,\ u=2$$

$$=2\left[2u^{1/2}\right]_2^{18} \qquad du=2x\, dx \quad \text{When } x=4,\ u=18$$

$$=2\left(2\sqrt{18}-2\sqrt{2}\right)$$

$$=8\sqrt{2}\text{ units}^2$$

Calculating the area between a curve and the y-axis

To calculate the area between a curve and the y-axis

- change the subject of the equation of the curve to x
- ensure the limits are in terms of y
- integrate x with respect to y ……… $A=\int_a^b x\, dy$

Example

Calculate the area enclosed by the curve $y=(x-4)^3$, the y-axis, $y=1$ and $y=8$.

$$y=(x-4)^3$$

$$y^{\frac{1}{3}}=x-4$$

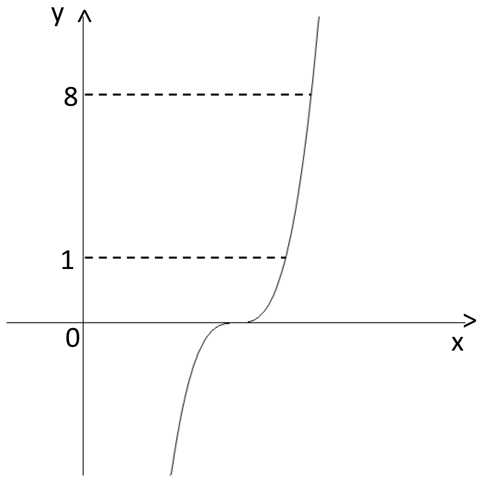
$$x=y^{\frac{1}{3}}+4$$

$$\text{Area}=\int_1^8\left(y^{\frac{1}{3}}+4\right)\,\mathrm{dy}$$

$$=\left[\frac{3}{4}y^{\frac{4}{3}}+4y\right]_1^8$$

$$=\left(\frac{3}{4}\sqrt[3]{8}^4+32\right)-\left(\frac{3}{4}+4\right)$$

$$=39\frac{1}{4}\text{ units}^2$$



Questions

① Calculate the area between the curve $y=x^2+1$, the y-axis, $y=2$ and $y=5$.

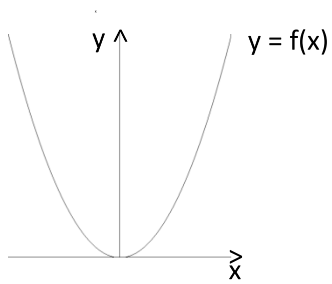
②Calculate the area between the curve $y=(x+2)^3$, the y-axis, $y=3$ and $y=8$.

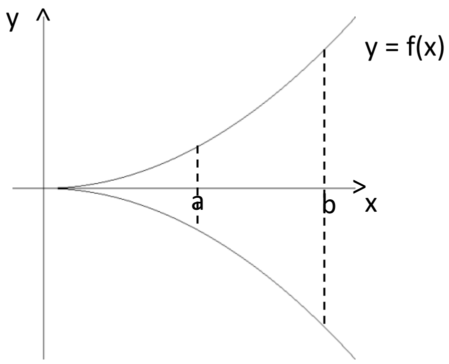
Volumes of Revolution

When any given area is rotated $360°$ about the x-axis or y-axis, the solid formed is known as a solid of revolution.

Consider the function $y=f(x)$.

We will consider the graph between $x=a$ and $x=b$ and rotate it $360°$ about the x-axis.





To understand how we can calculate this volume, think of the solid shape being made up of discs.

Just like we built up the area of rectangles when studying the area under a curve in Higher, we shall build up the volume of the solid with discs.

Consider a disc with radius y and let us say a thickness of δx. The disc is approximately cylindrical so the volume is given by $\delta V \approx \pi y^2 \delta x$ $\quad (V = \pi r^2 h$ for a cylinder).

As we make the disc thinner and thinner, $\delta x \to 0$ and the volume of the solid between $x = a$ and $x = b$ is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 dx$$

$$\boxed{V = \int_a^b \pi y^2 \; dx \quad \text{where } y = f(x)}$$

If we rotate about the y-axis we have a similar result for the volume of the solid of revolution.

$$\boxed{V = \int_a^b \pi x^2 \; dy \quad \text{where } x = f(y)}$$

Examples

① Find the volume of revolution obtained between $x = 1$ and $x = 2$ when the curve $y = x^2 + 2$ is rotated through $360°$

(a) about the x-axis

(b) about the y-axis.

(a) $V = \int_a^b \pi y^2 \; dx$

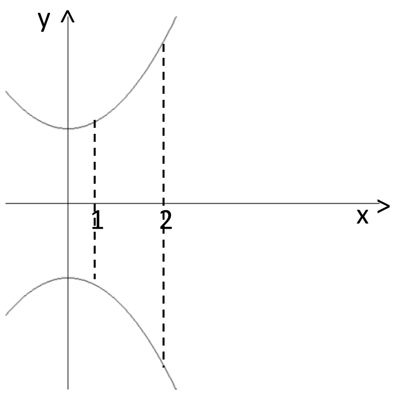
$= \int_1^2 \pi \left(x^2 + 2\right)^2 \; dx$

$= \pi \int_1^2 \left(x^4 + 4x^2 + 1\right) dx$

$= \pi \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + x\right]_1^2$

$= \pi \left(\left(\frac{32}{5} + \frac{32}{3} + 2\right) - \left(\frac{1}{5} + \frac{4}{3} + 1\right)\right)$

$= \frac{248}{15}\pi$



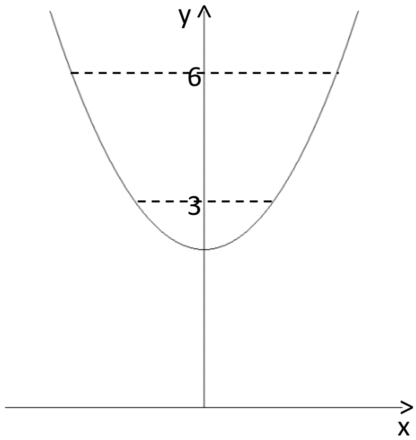
(b) When $x=1$, $y=1^2+2=3$

When $x=2$, $y=2^2+2=6$. These are our limits of integration.

We must rearrange the equation of the curve to make x the subject.

$$y=x^2+2$$
$$x^2=y-2$$
$$x=\sqrt{y-2}$$

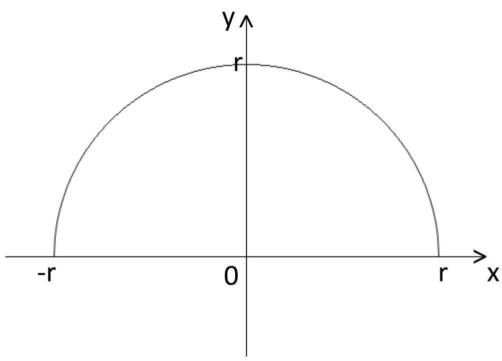
$$\begin{aligned}V&=\int_a^b \pi x^2\ dy\\&=\int_3^6 \pi\left(\sqrt{y-2}\right)^2\ dy\\&=\pi\int_1^2 (y-2)\ dy\\&=\pi\left[\frac{1}{2}y^2-2y\right]_3^6\\&=\pi\left((18-12)-\left(\frac{9}{2}-6\right)\right)\\&=\frac{15}{2}\pi\end{aligned}$$



The next example will give us a volume we already know…. the volume of a sphere.

② Calculate the volume of the solid formed when the semi-circle with equation $x^2+y^2=r^2$, $y\geq 0$ is given a full turn about the x-axis.

$$\begin{aligned}V&=\int_{-r}^r \pi y^2\ dx\\&=\pi\int_{-r}^r \left(r^2-x^2\right)\ dx\\&=\pi\left[r^2x-\frac{1}{3}x^3\right]_{-r}^r\\&=\pi\left(\left(r^3-\frac{1}{3}r^3\right)-\left(-r^3+\frac{1}{3}r^3\right)\right)\\&=\frac{4}{3}\pi r^3\end{aligned}$$



Questions

① Find the volume of the solid of revolution formed by rotating $y = \sin x$ through $360°$ about the x-axis between $x = 0$ and $x = \pi$.

② Find the volume of the solid of revolution formed by rotating $y = 2x^2 - 1$ through $360°$ about the y-axis between $y = 1$ and $y = 3$.

③ Calculate the volume of the solid formed by rotating $y = x - \frac{4}{x}$ through a complete turn about the x-axis from $x = 2$ to $x = 4$.

④ Calculate the volume of the solid formed by rotating $y = x^3$ through a complete turn about the y-axis from $y = 0$ to $y = 8$.

Rates of Change

In the last outcome, when studying rectilinear motion, we learned that

displacement → velocity → acceleration

differentiate with respect to time

It follows that

displacement ← velocity ← acceleration

integrate with respect to time

When we integrate we can use any initial conditions given to calculate the constant of integration, C.

Examples

① A particle starts from the origin at rest and at a time t seconds, its velocity is given by $V(t) = 6t^2 + 2t - 1$.

Determine the distance, velocity and acceleration after 3 seconds.

Velocity: $V(3) = 6 \times 3^2 + 2 \times 3 - 1$

$= 59$ units/second.

Distance: $s(t)=\int\left(6t^2+2t-1\right)\,dt$

$=2t^3+t^2-t+C$

When $t=0,\ s=0\ \therefore C=0$ so $s(t)=2t^3+t^2-t$

$s(3)=2\times3^3+3^2-3$

$=60$ units

Acceleration: $a(t)=\dfrac{d}{dt}\left(v(t)\right)$

$=12t+2$

$a(3)=12\times3+2$

$=38$ units/sec^2

② A particle is moving in a straight line so that at a time t, its distance from a fixed point is s. Its acceleration, a, is given by $a=k\cos^2 t$.
When $t=0$, $s=0$ units and $v=10$ units/sec.
Find formulae for $v(t)$ and $s(t)$.

$a(t)=k\cos^2 t$

$=\dfrac{k}{2}(\cos 2t+1)$

$v(t)=\int a(t)\,dt$

$=\dfrac{k}{2}\int\left(\cos 2t+1\right)\,dt$

$=\dfrac{k}{4}\sin 2t+\dfrac{kt}{2}+C$

When $t=0$, $v=10$

$\therefore 10=\dfrac{k}{4}\sin 0+0+C$

$\Rightarrow C=10$

So $v(t)=\dfrac{k}{4}\sin 2t+\dfrac{kt}{2}+10$

$s(t)=\int v(t)\,dt$

$=\int\left(\dfrac{k}{4}\sin 2t+\dfrac{kt}{2}+10\right)\,dt$

$=\dfrac{-k}{8}\cos 2t+\dfrac{kt^2}{4}+10t+C$

When $t=0$, $s=0$

$\therefore 0=\dfrac{-k}{8}\cos 0+0+0+C$

$\Rightarrow C=\dfrac{k}{8}$

$s(t)=10t-\dfrac{k}{8}\cos 2t+\dfrac{kt^2}{4}+\dfrac{k}{8}$

Questions

① A car starts from rest and travels in a straight line. Its acceleration, t seconds after its start is $\frac{1}{8}(30-t)$ m/s². Calculate its velocity after 4 seconds.

② A particle, starting from rest, proceeds in a straight line. Its acceleration after t seconds is given by $a = 4\sec^2 t$ units/s² where $0 \le t \le 1$. Calculate the velocity after $0\cdot5$ seconds and the distance travelled after $\frac{\pi}{4}$ seconds.

Past Paper Questions

2001

(a) Obtain partial fractions for

$$\frac{x}{x^2-1}, \ x>1.$$

(b) Use the result of (a) to find

$$\int \frac{x^3}{x^2-1}\,dx, \ x>1.$$

(2, 4 marks)

2002

Use the substitution $x+2=2\tan\theta$ to obtain $\int \frac{1}{x^2+4x+8}\,dx$. **(5 marks)**

2003

Use the substitution $x=1+\sin\theta$ to evaluate $\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)^3}\,d\theta$. **(5 marks)**

2004

① Express $\frac{1}{x^2-x-6}$ in partial fractions.

Evaluate $\int_0^1 \frac{1}{x^2-x-6}\,dx$. **(2, 4 marks)**

② A solid is formed by rotating the curve $y=e^{-2x}$ between $x=0$ and $x=1$ through $360°$ about the x-axis. Calculate the volume of the solid that is formed. **(5 marks)**

2005

① Use the substitution $u=1+x$ to evaluate $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$. **(5 marks)**

② Express $\frac{1}{x^3+x}$ in partial fractions.

Obtain a formula for $I(k)$, where $I(k)=\int_1^k \frac{1}{x^3+x}\,dx$, expressing it in the form $\ln\frac{a}{b}$,

where a and b depend on k.

Write down an expression for $e^{I(k)}$ and obtain the value of $\lim\limits_{k\to\infty} e^{I(k)}$. **(4, 4, 2 marks)**

③ (a) Given $f(x)=\sqrt{\sin x}$, where $0<x<\pi$, obtain $f'(x)$.

(b) If, in general, $f(x)=\sqrt{g(x)}$, where $g(x)>0$, show that $f'(x)=\frac{g'(x)}{k\sqrt{g(x)}}$,

stating the value of k.

Hence, or otherwise, find $\int \frac{x}{\sqrt{1-x^2}}\,dx$. **(1, 2, 3 marks)**

2006

Find $\int \frac{12x^3-6x}{x^4-x^2+1}\,dx$. **(3 marks)**

2007

① Express $\frac{2x^2-9x-6}{x\left(x^2-x-6\right)}$ in partial fractions.

Given that $\int_4^6 \frac{2x^2-9x-6}{x\left(x^2-x-6\right)}\,dx=\ln\frac{m}{n}$,

determine values for the integers m and n. **(3, 3 marks)**

② Use the substitution $u = 1 + x^2$ to obtain $\int_0^1 \frac{x^3}{\left(1+x^2\right)^4}\, dx$.

A solid is formed by rotating the curve $y = \frac{x^{3/2}}{\left(1+x^2\right)^2}$ between $x = 0$ and $x = 1$ through $360°$ about the x-axis. Write down the volume of this solid. **(5, 1 marks)**

2008

① Express $\frac{12x^2 + 20}{x\left(x^2+5\right)}$ in partial fractions.

Hence evaluate $\int_1^2 \frac{12x^2 + 20}{x\left(x^2+5\right)}\, dx$. **(3, 3 marks)**

② Write down the derivative of $\tan x$.

Show that $1 + \tan^2 x = \sec^2 x$.

Hence obtain $\int \tan^2 x \, dx$. **(1, 1, 2 marks)**

2009

① Show that $\int_{\ln 3/2}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx = \ln \frac{9}{5}$. **(4 marks)**

② Use the substitution $x = 2\sin\theta$ to obtain the exact value of $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\, dx$.

(Note that $\cos 2A = 1 - 2\sin^2 A$.) **(6 marks)**

2010

① Evaluate

$$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\, dx$$

expressing your answer in the form $\ln \frac{a}{b}$, where a and b are integers. **(6 marks)**

② A new board game has been invented and the symmetrical design on the board is made from 4 identical "petal" shapes. One of these petals is the region enclosed between the curves $y = x^2$ and $y^2 = 8x$ as shown shaded in diagram 1 below.

Calculate the area of the complete design, as shown in diagram 2.

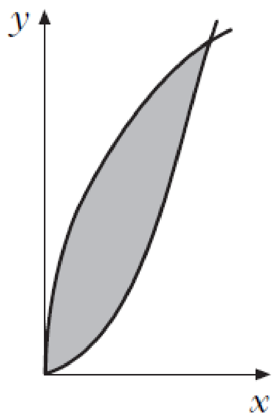


Diagram 1

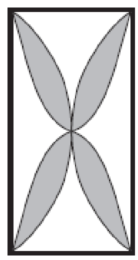


Diagram 2

The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through $360°$ about the y-axis.
Find the volume of plastic required to make one counter. **(5, 5 marks)**

2011

① Express $\frac{13-x}{x^2+4x-5}$ in partial fractions and hence obtain $\int \frac{13-x}{x^2+4x-5}\,dx$. **(5 marks)**

② Obtain the exact value of $\int_0^{\pi/4} (\sec x - x)(\sec x + x)\,dx$. **(3 marks)**

2012

Use the substitution $x = 4\sin\theta$ to evaluate $\int_0^2 \sqrt{16 - x^2}\,dx$. **(6 marks)**

2013

① The velocity, v, of a particle P at time t is given by

$$v = e^{3t} + 2e^{t}$$

(a) Find the acceleration of P at time t.

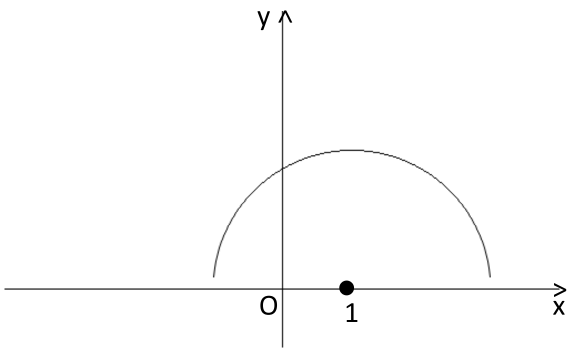
(b) Find the distance covered by P between $t = 0$ and $t = \ln 3$. **(2, 3 marks)**

② Integrate $\dfrac{\sec^2 3x}{1+\tan 3x}$ with respect to x. **(4 marks)**

2014

① A semi-circle with centre $(1,0)$ and radius 2, lies on the x-axis as shown.

Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis.



② Use the substitution $x = \tan\theta$ to determine the exact value of

$$\int_0^1 \frac{dx}{\left(1+x^2\right)^{3/2}}.$$ **(6 marks)**

2015

Find $\displaystyle\int \frac{2x^3 - x - 1}{(x-3)\left(x^2+1\right)}\,dx,\ x > 3.$ **(9 marks)**

2016

Express $\dfrac{3x+32}{(x+4)(6-x)}$ in partial fractions and hence evaluate $\displaystyle\int_3^4 \frac{3x+32}{(x+4)(6-x)}\,dx.$

Give your answer in the form $\ln\left(\dfrac{p}{q}\right)$. **(9 marks)**