Further Sequences and Series (Maclaurin Series)

A power series is an expression which takes the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots\ldots$$

where $a_0, a_1, a_2 \ldots$ are constants and x is a variable.

Power series are used to solve many differential equations which occur in Physics and are used to determine how many decimal places are required in a computation to guarantee a specific accuracy.

Functions such as $\sin x,\ \cos x,\ e^x,\ \tan^{-1} x,\ \ln(1+x)$ and $(1+x)^n$ can be expressed in terms of power series.

Maclaurin Series

There are some functions which can be continuously differentiated and which can be differentiated when $x = 0$. Functions which satisfy these conditions can be expressed as a **Maclaurin series** which is a special type of power series.

The Maclaurin series for the function $f(x)$ is

$$\sum_{r=0}^{\infty} f^r(0)\frac{x^r}{r!} = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + \ldots\ldots + f^n(0)\frac{x^n}{n!} + \ldots\ldots$$

Examples

① Find the first 5 terms of the Maclaurin series generated by $f(x) = e^x$.

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$f'''(x) = e^x \qquad f'''(0) = 1$

$f^{iv}(x) = e^x \qquad f^{iv}(0) = 1$

So the Maclaurin series generated by $f(x) = e^x$ becomes

$$\sum_{r=0}^{\infty} f^r(0)\frac{x^r}{r!} = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!} + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots$$

② Find the Maclaurin series generated by $f(x) = \sin x$ as far as the x^5 term.

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \\ f^{iv}(x) = \sin x & f^{iv}(0) = 0 \\ f^{v}(x) = \cos x & f^{v}(0) = 1 \end{array}$$

So the Maclaurin series generated by $f(x) = \sin x$ becomes

$$\begin{aligned} \sum_{r=0}^{\infty} f^{r}(0)\frac{x^r}{r!} &= f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!} + f^{v}(0)\frac{x^5}{5!} + \ldots\ldots \\ &= 0 + 1\frac{x}{1!} + 0\frac{x^2}{2!} - 1\frac{x^3}{3!} + 0\frac{x^4}{4!} + 1\frac{x^5}{5!} + \ldots \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots\ldots \end{aligned}$$

③ Find the Maclaurin series generated by $f(x) = \cos x$.

$$\begin{array}{ll} f(x) = \cos x & f(0) = 1 \\ f'(x) = -\sin x & f'(0) = 0 \\ f''(x) = -\cos x & f''(0) = -1 \\ f'''(x) = \sin x & f'''(0) = 0 \\ f^{iv}(x) = \cos x & f^{iv}(0) = 1 \\ f^{v}(x) = -\sin x & f^{v}(0) = 0 \end{array}$$

So the Maclaurin series generated by $f(x) = \cos x$ becomes

$$\begin{aligned} \sum_{r=0}^{\infty} f^{r}(0)\frac{x^r}{r!} &= f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!} + f^{v}(0)\frac{x^5}{5!} + \ldots\ldots \\ &= 1 + 0\frac{x}{1!} - 1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 1\frac{x^4}{4!} + 0\frac{x^5}{5!} + \ldots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots\ldots \end{aligned}$$

④ Find the Maclaurin series generated by $f(x)=(1+x)^n$.

$$f(x)=(1+x)^n \qquad f(0)=1$$
$$f'(x)=n(1+x)^{n-1} \qquad f'(0)=n$$
$$f''(x)=n(n-1)(1+x)^{n-2} \qquad f''(0)=n(n-1)$$
$$f'''(x)=n(n-1)(n-2)(1+x)^{n-3} \qquad f'''(0)=n(n-1)(n-2)$$
$$f^{iv}(x)=n(n-1)(n-2)(n-3)(1+x)^{n-4} \qquad f^{iv}(0)=n(n-1)(n-2)(n-3)$$
$$f^{v}(x)=n(n-1)(n-2)(n-3)(n-4)(1+x)^{n-5} \qquad f^{v}(0)=n(n-1)(n-2)(n-3)(n-4)$$

So the Maclaurin series generated by $f(x)=(1+x)^n$ becomes

$$\sum_{r=0}^{\infty} f^{r}(0)\frac{x^r}{r!}=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+f^{iv}(0)\frac{x^4}{4!}+f^{v}(0)\frac{x^5}{5!}+\ldots\ldots$$
$$=1+n\frac{x}{1!}+n(n-1)\frac{x^2}{2!}+n(n-1)(n-2)\frac{x^3}{3!}+n(n-1)(n-2)(n-3)\frac{x^4}{4!}+n(n-1)(n-2)(n-3)(n-4)\frac{x^5}{5!}+\ldots$$
$$=1+nx+n(n-1)\frac{x^2}{2}+n(n-1)(n-2)\frac{x^3}{6}+n(n-1)(n-2)(n-3)\frac{x^4}{24}+n(n-1)(n-2)(n-3)(n-4)\frac{x^5}{120}+\ldots$$

⑤ Find the Maclaurin series generated by $f(x)=\ln(1+x)$.

$$f(x)=\ln(1+x) \qquad f(0)=0$$
$$f'(x)=\frac{1}{1+x} \qquad f'(0)=1$$
$$f''(x)=\frac{-1}{(1+x)^2} \qquad f''(0)=-1$$
$$f'''(x)=\frac{2}{(1+x)^3} \qquad f'''(0)=2$$
$$f^{iv}(x)=\frac{-6}{(1+x)^4} \qquad f^{iv}(0)=-6$$
$$f^{v}(x)=\frac{24}{(1+x)^5} \qquad f^{v}(0)=24$$

So the Maclaurin series generated by $f(x)=\ln(1+x)$ becomes

$$\sum_{r=0}^{\infty} f^{r}(0)\frac{x^r}{r!}=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+f^{iv}(0)\frac{x^4}{4!}+f^{v}(0)\frac{x^5}{5!}+\ldots\ldots$$
$$=0+1\frac{x}{1!}-1\frac{x^2}{2!}+2\frac{x^3}{3!}-6\frac{x^4}{4!}+24\frac{x^5}{5!}+\ldots$$
$$=x-\frac{x^2}{2}+2\frac{x^3}{6}-6\frac{x^4}{24}+24\frac{x^5}{120}+\ldots$$
$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\ldots$$

The above expansions can be termed Maclaurin Series but are only valid within specific intervals of convergence, similar to when we looked at S_∞.

In that case S_∞ existed when $|r|<1$.

A ratio test is required to find the interval of convergence. This is beyond AH level. The basics are contained in the handout and are all that is really required at this stage.

Composite Maclaurin Series

We have looked at functions which are straightforward to differentiate but more complex ones may now be introduced.
We can also differentiate and integrate these series expansions provided we are within the range of validity.

Example 2001 QB4

Find the first four terms in the Maclaurin Series for $(2+x)\ln(2+x)$.

$f(x)=(2+x)\ln(2+x)$ $\qquad f(0)=2\ln 2$

$f'(x)=\ln(2+x)+\frac{2+x}{2+x}=\ln(2+x)+1$ $\qquad f'(0)=\ln 2+1$

$f''(x)=\frac{1}{2+x}$ $\qquad f''(0)=\frac{1}{2}$

$f'''(x)=\frac{-1}{(2+x)^2}$ $\qquad f'''(0)=\frac{-1}{4}$

So the Maclaurin Series generated by $f(x)=(2+x)\ln(2+x)$ becomes

$$\sum_{r=0}^{\infty} f^r(0)\frac{x^r}{r!}=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+f^{iv}(0)\frac{x^4}{4!}+f^{v}(0)\frac{x^5}{5!}+\ldots\ldots$$

$$=2\ln 2+(\ln 2+1)\frac{x}{1!}+\frac{\frac{1}{2}x^2}{2!}-\frac{\frac{1}{4}x^3}{3!}+\ldots$$

$$=2\ln 2+(\ln 2+1)x+\frac{x^2}{4}-\frac{x^3}{24}+\ldots$$

Example - 2002 QB3

Find the Maclaurin expansion of $f(x)=\ln(\cos x)$, $0\le x\le\frac{\pi}{2}$, as far as the term in x^4.

$f(x)=\ln(\cos x)$ $\qquad f(0)=0$

$f'(x)=\frac{1}{\cos x}\cdot(-\sin x)=-\tan x$ $\qquad f'(0)=0$

$f''(x)=-\sec^2 x=\frac{-1}{\cos^2 x}$ $\qquad f''(0)=-1$

$f'''(x)=-2\tan x\sec x(\sec x)=-2\sec^2 x\tan x$ $\qquad f'''(0)=0$

$f^{iv}(x)=-4\sec^2 x\tan^2 x-2\sec^4 x$ $\qquad f^{iv}(0)=-2$

So the Maclaurin series is

$$f(x) = (-1)\frac{x^2}{2!} + (-2)\frac{x^4}{4!} + \ldots\ldots$$

$$f(x) = \frac{-x^2}{2} - \frac{x^4}{12} + \ldots\ldots$$

<u>Example - 2003 QB4</u>

Obtain the Maclaurin Series for $f(x) = \sin^2 x$ up to the term in x^4.

Hence write down a series for $\cos^2 x$ up to the term in x^4.

$$f(x) = \sin^2 x \qquad f(0) = 0$$
$$f'(x) = 2\sin x\cos x = \sin 2x \qquad f'(0) = 0$$
$$f''(x) = 2\cos 2x \qquad f''(0) = 2$$
$$f'''(x) = -4\sin 2x \qquad f'''(0) = 0$$
$$f^{iv}(x) = -8\cos 2x \qquad f^{iv}(0) = -8$$

$$\therefore \sin^2 x = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{8x^4}{4!} + \ldots$$
$$= x^2 - \frac{1}{3}x^4 + \ldots$$

Now $\sin^2 x + \cos^2 x = 1$

$$\cos^2 x = 1 - \sin^2 x$$
$$\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \ldots$$

<u>Example - 2004 Q7</u>

Obtain the first three non-zero terms in the Maclaurin Series of $f(x) = e^x \sin x$.

$$f(x) = e^x \sin x \qquad f(0) = 0$$
$$f'(x) = e^x \sin x + e^x \cos x \qquad f'(0) = 1$$
$$f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x \qquad f''(0) = 2$$
$$f'''(x) = 2e^x \cos x - 2e^x \sin x \qquad f'''(0) = 2$$

$$\therefore f(x) = e^x \sin x = 0 + \frac{1x}{1!} + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \ldots$$
$$= x + x^2 + \frac{1}{3}x^3 + \ldots$$

Example - 2005 Q2

Write down the Maclaurin expansion of e^x as far as the term in x^4.

Deduce the Maclaurin expansion of e^{x^2} as far as the term in x^4.

Hence or otherwise, find the Maclaurin expansion of e^{x+x^2} as far as the term in x^4.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$e^{x^2} = 1 + \left(x^2\right) + \frac{\left(x^2\right)^2}{2} + \frac{\left(x^2\right)^3}{6} + \frac{\left(x^2\right)^4}{24} + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2} + \cdots$$

$$e^{x+x^2} = e^x e^{x^2}$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(1 + x^2 + \frac{x^4}{2} + \cdots\right)$$

$$= 1 + x^2 + \frac{x^4}{2} + x + x^3 + \frac{x^2}{2} + \frac{x^4}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

$$= 1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \frac{25x^4}{24} + \cdots$$

Example - 2007 Q6

Find the Maclaurin Series for $\cos x$ as far as the term in x^4.

Deduce the Maclaurin Series for $f\left(x\right) = \frac{1}{2}\cos 2x$ as far as the term in x^4.

Hence write down the first three non-zero terms of the series for $f\left(3x\right)$.

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$f\left(x\right) = \frac{1}{2}\cos 2x = \frac{1}{2}\left(1 - \frac{1}{2}\left(2x\right)^2 + \frac{1}{24}\left(2x\right)^4\right)$$

$$= \frac{1}{2}\left(1 - 2x^2 + \frac{2}{3}x^4\right)$$

$$= \frac{1}{2} - x^2 + \frac{1}{3}x^4$$

$$f\left(3x\right) = \frac{1}{2} - \left(3x\right)^2 + \frac{1}{3}\left(3x\right)^4$$

$$= \frac{1}{2} - 9x^2 + 27x^4$$

Example - 2008 Q12

Obtain the first three non-zero terms in the Maclaurin expansion of $x\ln(2+x)$.

Hence, or otherwise, deduce the first three non-zero terms in the Maclaurin expansion of $x\ln(2-x)$.

Hence obtain the first two non-zero terms in the Maclaurin expansion of $x\ln(4-x^2)$.

$f(x)=x\ln(2+x)$ $\qquad f(0)=0$

$f'(x)=\ln(2+x)+\dfrac{x}{2+x}$ $\qquad f'(0)=\ln 2$

$f''(x)=\dfrac{1}{2+x}+\dfrac{(2+x)-x}{(2+x)^2}=\dfrac{1}{2+x}+\dfrac{2}{(2+x)^2}$ $\qquad f''(0)=\dfrac{1}{2}+\dfrac{2}{4}=1$

$f'''(x)=\dfrac{-1}{(2+x)^2}-\dfrac{4}{(2+x)^3}$ $\qquad f'''(0)=\dfrac{-1}{4}-\dfrac{4}{8}=\dfrac{-3}{4}$

$$\therefore f(x)=x\ln(2+x)=(\ln 2)x+\frac{x^2}{2}-\frac{\frac{3}{4}x^3}{6}+\cdots$$
$$=(\ln 2)x+\frac{x^2}{2}-\frac{x^3}{8}+\cdots$$

$$g(x)=x\ln(2-x)$$
$$=-(-x)\ln(2+(-x))$$
$$=-f(-x)$$

$\therefore$ Maclaurin expansion of $g(x)=-\left((\ln 2)(-x)+\dfrac{1}{2}(-x)^2-\dfrac{(-x)^3}{8}\right)$

$$=(\ln 2)x-\frac{x^2}{2}-\frac{x^3}{8}+\cdots$$

Expansion of $x\ln(4-x^2)=x(\ln(2-x)(2+x))$

$$=x(\ln(2-x)+\ln(2+x))$$
$$=x\ln(2-x)+x\ln(2+x)$$
$$=(\ln 2)x-\frac{x^2}{2}-\frac{x^3}{8}+(\ln 2)x+\frac{x^2}{2}-\frac{x^3}{8}$$
$$=(2\ln 2)x-\frac{x^3}{4}+\cdots$$

Example - 2009 Q14

Express $\frac{x^2+6x-4}{(x+2)^2(x-4)}$ in partial fractions. Hence, or otherwise, obtain the first 3 non-zero terms in the Maclaurin expansion of $\frac{x^2+6x-4}{(x+2)^2(x-4)}$.

Let $\frac{x^2+6x-4}{(x+2)^2(x-4)}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}+\frac{C}{(x-4)}$

$$x^2+6x-4=A(x+2)(x-4)+B(x-4)+C(x+2)^2$$

$x=4$	$x=-2$	$x=0$
$36=36C$	$-12=-6B$	$-4=-8A-4B+4C$
$C=1$	$B=2$	$0=-8A$
		$A=0$

So $\frac{x^2+6x-4}{(x+2)^2(x-4)}=\frac{2}{(x+2)^2}+\frac{1}{(x-4)}$

$$f(x)=\frac{2}{(x+2)^2}+\frac{1}{(x-4)} \qquad f(0)=\frac{1}{4}$$

$$f'(x)=\frac{-4}{(x+2)^3}-\frac{1}{(x-4)^2} \qquad f'(0)=\frac{-9}{16}$$

$$f''(x)=\frac{12}{(x+2)^4}+\frac{2}{(x-4)^3} \qquad f''(0)=\frac{23}{32}$$

$$\therefore f(x)=\frac{1}{4}-\frac{9}{16}x+\frac{23}{32}\left(\frac{x^2}{2!}\right)$$

$$=\frac{1}{4}-\frac{9}{16}x+\frac{23}{64}x^2$$

<u>Questions</u>

Find the Maclaurin series for each term up to and including the term in x^4.

(a) $f(x) = e^x$ (b) $f(x) = e^{2x}$ (c) $f(x) = e^{3x}$ (d) $f(x) = e^{-2x}$

(e) $f(x) = \sin x$ (f) $f(x) = \sin 2x$ (g) $f(x) = \cos x$ (h) $f(x) = \cos 3x$

(i) $f(x) = \ln(1-x)$ (j) $f(x) = \ln(1-2x)$ (k) $f(x) = \ln(1+3x)$ (l) $f(x) = (1-x)^{-1}$

(m) $f(x) = (1-4x)^{-1}$ (n) $f(x) = (1+x)^{1/2}$ (o) $f(x) = (1+x)^{-2}$ (p) $f(x) = (1-x)^{-3}$

(q) $f(x) = (1-x)^{-1/2}$ (r) $f(x) = e^{\frac{1}{2}x}$ (s) $f(x) = \sqrt{1+2x}$ (t) $f(x) = \cos^2 x$

(u) $f(x) = \tan^{-1} x$ (v) $f(x) = e^x \sin x$ (w) $f(x) = e^{x^2}$ (x) $f(x) = x\ln(2-x)$

(y) $f(x) = (3+x)\ln(3+x)$ (z) $f(x) = \exp(\sin x)$

Past Paper Questions

2010 – Q9

Obtain the first three non-zero terms in the Maclaurin expansion of $\left(1+\sin^2 x\right)$.

(4 marks)

2011 – Q5

Obtain the first four terms in the Maclaurin series of $\sqrt{1+x}$ and hence write down the first four terms in the Maclaurin series of $\sqrt{1+x^2}$.

Hence obtain the first four terms in the Maclaurin series of $\sqrt{(1+x)\left(1+x^2\right)}$.

(4, 2 marks)

2012 – Q6

Write down the Maclaurin expansion of e^x as far as the term in x^3.
Hence, or otherwise, obtain the Maclaurin expansion of $\left(1+e^x\right)^2$ as far as the term in x^3.

(1, 4 marks)

2013 – Q17

Write down the sums to infinity of the geometric series

$$1+x+x^2+x^3+\ldots$$

and

$$1-x+x^2-x^3+\ldots$$

valid for $|x|<1$.

Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x|<1$,

$$\ln\left(\frac{1+x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right).$$

Show how this series can be used to evaluate $\ln 2$.
Hence determine the value of $\ln 2$ correct to 3 decimal places. **(7, 3 marks)**

2014 – Q9

Give the first three non-zero terms of the Maclaurin series for $\cos 3x$.
Write down the first four terms of the Maclaurin series for e^{2x}.
Hence, or otherwise, determine the Maclaurin series for $e^{2x}\cos 3x$ up to, and including, the term in x^3. **(2, 1 ,3 marks)**

2016 – Q6

Find Maclaurin expansions for $\sin 3x$ and e^{4x} up to and including the term in x^3.
Hence obtain an expansion for $e^{2x}\cos 3x$ up to, and including, the term in x^3. **(6 marks)**