Matrices

A matrix is a rectangular array of numbers. Each number in the matrix is called an element of the matrix. The order of the matrix is described by writing the number of rows followed by the number of columns.

$A=\begin{pmatrix} 5 & 0 \\ 1 & -3 \end{pmatrix}$ has order 2×2 $\quad B=\begin{pmatrix} 9 \\ -3 \\ 5 \end{pmatrix}$ has order 3×1 $\quad C=\begin{pmatrix} 2 & 5 & 3 \\ 1 & 9 & 0 \end{pmatrix}$ has order 2×3

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A is a matrix of order $m\times n$.

a_{mn} is an element of matrix A.

If $m=n$ then A is a square matrix.

The **zero matrix**, $\mathbf{O}$, is the matrix consisting entirely of zeros.

Addition, Subtraction, Multiplication by a Scalar

Matrices can be added or subtracted if they have the same order.

e.g. $\begin{pmatrix} 4 & 1 & 3 \\ 2 & 0 & -3 \end{pmatrix}+\begin{pmatrix} 1 & -4 & 1 \\ 0 & -2 & -9 \end{pmatrix}$
$=\begin{pmatrix} 5 & -3 & 4 \\ 2 & -2 & -12 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}-\begin{pmatrix} -2 & 9 \\ 3 & -4 \end{pmatrix}$
$=\begin{pmatrix} 3 & -9 \\ -4 & 8 \end{pmatrix}$

Matrices can also be multiplied by a scalar.

e.g. $6\begin{pmatrix} 1 & -3 \\ 0 & 7 \end{pmatrix}=\begin{pmatrix} 6 & -18 \\ 0 & 42 \end{pmatrix}$

Multiplying by -1 gives the negative of the matrix.

Transpose of a Matrix

To obtain the transpose of a matrix we switch the rows and columns.

The transpose of matrix A is denoted either A^T or A'.

e.g. $A=\begin{pmatrix}4 & 1 & 3\\ 2 & 4 & 0\end{pmatrix}$ $B=\begin{pmatrix}1 & 4 & -5\end{pmatrix}$ $C=\begin{pmatrix}x & y\\ z & 0\end{pmatrix}$

$A^T=\begin{pmatrix}4 & 2\\ 1 & 4\\ 3 & 0\end{pmatrix}$ $B^T=\begin{pmatrix}1\\ 4\\ -5\end{pmatrix}$ $C^T=\begin{pmatrix}x & z\\ y & 0\end{pmatrix}$

If $A^T=A$ then the matrix is **symmetric**.

e.g. $A=\begin{pmatrix}1 & 3 & 7\\ 3 & -2 & 0\\ 7 & 0 & 4\end{pmatrix}$ $A^T=\begin{pmatrix}1 & 3 & 7\\ 3 & -2 & 0\\ 7 & 0 & 4\end{pmatrix}$ $A^T=A$

The matrix is symmetric about the lead diagonal.

If $A^T=-A$ then the matrix is **skew-symmetric**.

e.g. $A=\begin{pmatrix}0 & -1 & 2\\ 1 & 0 & \frac{1}{5}\\ -2 & -\frac{1}{5} & 0\end{pmatrix}$ $A^T=\begin{pmatrix}0 & 1 & -2\\ -1 & 0 & -\frac{1}{5}\\ 2 & \frac{1}{5} & 0\end{pmatrix}$ $A^T=-A$

The matrix is skew-symmetric.

Skew-symmetric matrices will always have 0's in the lead diagonal.

Questions

① $\begin{pmatrix}3 & 1\\ 4 & 7\\ 9 & -3\end{pmatrix}+\begin{pmatrix}-3 & 2\\ -7 & 9\\ 6 & -4\end{pmatrix}$

② $\begin{pmatrix}3 & 0\\ 0 & 5\\ 4 & 1\end{pmatrix}+\begin{pmatrix}2 & 3\\ 1 & 4\\ 4 & 5\end{pmatrix}-\begin{pmatrix}4 & -2\\ -3 & 3\\ -4 & 9\end{pmatrix}$

③ $\begin{pmatrix}3 & 7 & 2\\ -2 & 1 & 6\end{pmatrix}+\begin{pmatrix}-2 & 0 & -2\\ 1 & 4 & 3\end{pmatrix}$

④ $\begin{pmatrix}\frac{1}{2} & 1\\ 4 & \frac{3}{2}\end{pmatrix}-2\begin{pmatrix}\frac{1}{4} & 0\\ 3 & \frac{7}{2}\end{pmatrix}$

⑤ Calculate matrix B if $\begin{pmatrix}3 & 1\\ 2 & 5\end{pmatrix}+B=\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$.

⑥ Solve these matrix equations.

(a) $2B+\begin{pmatrix}1 & 4\\ 5 & 7\end{pmatrix}=\begin{pmatrix}5 & 6\\ 1 & 9\end{pmatrix}$

(b) $2C-3\begin{pmatrix}1 & 1 & 2\\ 2 & -1 & 0\end{pmatrix}=\begin{pmatrix}7 & 5 & 1\\ 6 & -2 & 3\end{pmatrix}$

⑦ Write down the transpose of each matrix. Comment where appropriate.

(a) $\begin{pmatrix} 17 & 1 \\ 4 & 3 \\ -2 & 8 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -1 & -6 \\ -1 & 3 & 2 \\ -6 & 2 & \frac{1}{2} \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -3 & 4 \\ 3 & 0 & 1 \\ -4 & -1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 3 & 4 & 5 \\ -2 & 3 & 0 & -6 \\ -2 & 0 & 3 & 4 \end{pmatrix}$

The Identity Matrix

The **identity matrix** has ones on the lead diagonal and zeros for every other element.

I is used to represent the identity matrix.

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In the world of matrices I has the same properties as 1 has in the world of real numbers under multiplication.

Hence we have $AI = IA = A$ for a matrix A.

Matrix Multiplication

A matrix A can multiply a matrix B if A has order m x n and B has order n x p.

We can multiply a 2×1 matrix by a 1×2 matrix to get a 2×2 matrix

3×2 matrix by a 2×3 matrix to get a 3×3 matrix

1×4 matrix by a 4×3 matrix to get a 1×3 matrix

2×3 matrix by a 3×3 matrix to get a 2×3 matrix and so on.

Examples

① $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \end{pmatrix}\begin{matrix} r_1 \\ r_2 \end{matrix}$ $B = \begin{array}{c} \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 1 & 0 \end{pmatrix} \\ \begin{matrix} c_1 & c_2 \end{matrix} \end{array}$ A is 2×3 and B is 3×2 so AB will be 2×2.

$$AB = \begin{pmatrix} r_1c_1 & r_1c_2 \\ r_2c_1 & r_2c_2 \end{pmatrix}$$

The elements in this matrix are obtained by multiplying and adding the elements in the rows and columns of A and B as follows.

$$AB=\begin{pmatrix} 2\times1+1\times3+3\times1 & 2\times(-1)+1\times2+3\times0 \\ 0\times1+(-1)\times3+4\times1 & 0\times(-1)+(-1)\times2+4\times0 \end{pmatrix}$$

$$AB=\begin{pmatrix} 2+3+3 & -2+2+0 \\ 0-3+4 & 0-2+0 \end{pmatrix}$$

$$AB=\begin{pmatrix} 8 & 0 \\ 1 & -2 \end{pmatrix}$$

② $X=\begin{pmatrix} 1 & 3 \\ 7 & 6 \\ 4 & -1 \end{pmatrix}$ $Y=\begin{pmatrix} 1 & -1 & 3 \\ 4 & -2 & 0 \end{pmatrix}$ X is 3×2 and Y is 2×3 so AB will be 3×3.

$$XY=\begin{pmatrix} 1\times1+3\times4 & 1\times(-1)+3\times(-2) & 1\times3+3\times0 \\ 7\times1+6\times4 & 7\times(-1)+6\times(-2) & 7\times3+6\times0 \\ 4\times1+(-1)\times4 & 4\times(-1)+(-1)\times(-2) & 4\times3+(-1)\times0 \end{pmatrix}$$

$$XY=\begin{pmatrix} 13 & -7 & 3 \\ 31 & -19 & 21 \\ 0 & -2 & 12 \end{pmatrix}$$

③ $G=\begin{pmatrix} -1 & 2 & -4 \end{pmatrix}^T$ $H=\begin{pmatrix} 1 & 2 & -2 \end{pmatrix}$ G^T is 3×1 and H is 1×3 so G^TH will be 3×3.

$$G^TH=\begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \end{pmatrix}$$

$$G^TH=\begin{pmatrix} -1\times1 & -1\times2 & -1\times(-2) \\ 2\times1 & 2\times2 & 2\times(-2) \\ -4\times1 & -4\times2 & -4\times(-2) \end{pmatrix}$$

$$G^TH=\begin{pmatrix} -1 & -2 & 2 \\ 2 & 4 & -4 \\ -4 & -8 & 8 \end{pmatrix}$$

We can also calculate HG^T here as H is 1×3 and G^T is 3×1 resulting in HG^T being 1×1, a scalar.

$$HG^T=\begin{pmatrix} 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$$

$$HG^T = 1\times(-1)+2\times2+(-2)\times(-4)$$

$$HG^T = 11$$

Questions

Given

$$A=\begin{pmatrix}2 & 1 & -1\\ 3 & -2 & 1\end{pmatrix} \quad B=\begin{pmatrix}1 & 2\\ -2 & 1\\ 0 & 3\end{pmatrix} \quad C=\begin{pmatrix}1 & 0 & 3\\ 2 & -1 & -1\\ -3 & 2 & -2\end{pmatrix} \quad D=\begin{pmatrix}3 & 1 & 1 & -2\\ 1 & 1 & 0 & -1\\ 2 & -2 & 1 & 2\end{pmatrix}$$

calculate the following products.

(a) AB (b) AC (c) BA (d) CB (e) CD (f) AD

Orthogonal Matrices

A is an **orthogonal** matrix if $A^TA = AA^T = I$.

Example

Which of these matrices are orthogonal?

$$A=\begin{pmatrix}3/5 & -4/5 & 0\\ 4/5 & 3/5 & 0\\ 0 & 0 & 1\end{pmatrix} \quad B=\begin{pmatrix}2 & 0 & 1\\ 1 & 2 & 2\\ 2 & 1 & 0\end{pmatrix} \quad C=\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$$

$$A^TA=\begin{pmatrix}3/5 & 4/5 & 0\\ -4/5 & 3/5 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}3/5 & -4/5 & 0\\ 4/5 & 3/5 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A^TA=\begin{pmatrix}9/25+16/25 & -12/25+12/25 & 0\\ -12/25+12/25 & 16/25+9/25 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A^TA=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$A^TA = I$ so A is orthogonal

$$B^TB=\begin{pmatrix}2 & 1 & 2\\ 0 & 2 & 1\\ 1 & 2 & 0\end{pmatrix}\begin{pmatrix}2 & 0 & 1\\ 1 & 2 & 2\\ 2 & 1 & 0\end{pmatrix}$$

$$B^TB=\begin{pmatrix}9 & 4 & 4\\ 4 & 5 & 4\\ 4 & 4 & 5\end{pmatrix}$$

$B^TB \neq I$ so B is not orthogonal.

$$C^TC=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$C^TC=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$C^TC=I$ so C is orthogonal.

The Determinant of a 2×2 Matrix

If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det A=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$.

Examples

① $X=\begin{pmatrix} 3 & 1 \\ 2 & \frac{1}{2} \end{pmatrix}$

$$\det X=\begin{vmatrix} 3 & 1 \\ 2 & \frac{1}{2} \end{vmatrix}$$

$$\det X=\frac{3}{2}-2$$

$$\det X=\frac{-1}{2}$$

② $Y=\begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$

$$\det Y=\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$$

$$\det Y=4-0$$

$$\det Y=4$$

③ $Z=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{3} & \frac{1}{4} \end{pmatrix}$

$$\det Z=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{3} & \frac{1}{4} \end{vmatrix}$$

$$\det Z=\frac{1}{8}+\frac{1}{6}$$

$$\det Z=\frac{7}{24}$$

Questions

Find the determinant of the following matrices.

① $A=\begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}$ ② $B=\begin{pmatrix} -2 & 3 \\ 0 & 7 \end{pmatrix}$ ③ $C=\begin{pmatrix} 6 & \frac{1}{2} \\ -1 & 0 \end{pmatrix}$ ④ $D=\begin{pmatrix} -1 & 3 \\ -4 & \frac{1}{3} \end{pmatrix}$

The Determinant of a 3×3 Matrix

A 3×3 determinant is calculated slightly differently.

If
$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then $\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$\det A=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Examples

① Calculate $|A|$ when $A = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 1 & 1 \\ -1 & 2 & -2 \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 1 & 1 \\ -1 & 2 & -2 \end{vmatrix}$$

$$\det A = 1\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}$$

$$\det A = (-2-2) - 0 + 4(6+1)$$

$$\det A = 24$$

② Calculate $|B|$ when $B = \begin{pmatrix} 4 & 1 & 3 \\ 0 & 2 & -1 \\ 3 & 0 & 7 \end{pmatrix}$

$$\det B = \begin{vmatrix} 4 & 1 & 3 \\ 0 & 2 & -1 \\ 3 & 0 & 7 \end{vmatrix}$$

$$\det B = 4\begin{vmatrix} 2 & -1 \\ 0 & 7 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 3 & 7 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix}$$

$$\det B = 4(14-0) - 1(0-(-3)) + 3(0-6)$$

$$\det B = 35$$

Questions

Calculate the determinant of the following matrices.

① $A = \begin{pmatrix} 4 & 1 & 2 \\ 3 & -1 & 0 \\ 1 & 4 & -2 \end{pmatrix}$ ② $B = \begin{pmatrix} 1 & -1 & 4 \\ 0 & 3 & 2 \\ -3 & 6 & 1 \end{pmatrix}$ ③ $C = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 4 \\ 0 & 2 & -1 \end{pmatrix}$ ④ $D = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$

Singular/Non-singular Matrices

A matrix which has a determinant of 0 is known as a **singular** matrix.

All other matrices are known as **non-singular**.

Inverse Matrices

Any number n has an inverse $\frac{1}{n}$ such that $n \times \frac{1}{n} = 1$.

In a similar way, a matrix A has an inverse denoted A^{-1} such that $AA^{-1} = I$ provided that

- the matrix is square
- the matrix is non-singular.

Inverse of a 2×2 Matrix

The inverse of a matrix A is denoted A^{-1}.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is known as the adjugate of A.

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}A$$

Here we can clearly see why singular matrices have no inverse.

Examples

① $X = \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$ $|X| = 3 - 0 = 3$

$X^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$

$X^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

② $Y = \begin{pmatrix} 2 & 3 \\ 1 & -\frac{1}{2} \end{pmatrix}$ $|Y| = -1 - 3 = -4$

$Y^{-1} = \frac{-1}{4}\begin{pmatrix} -\frac{1}{2} & -3 \\ -1 & 2 \end{pmatrix}$

$Y^{-1} = \begin{pmatrix} \frac{1}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$

Questions

Find the inverse of the following matrices.

① $A = \begin{pmatrix} -3 & 2 \\ 1 & 4 \end{pmatrix}$ ② $B = \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix}$ ③ $C = \begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}$

Inverse of a 3×3 Matrix

Finding $\text{adj}A$ and $\det A$ for a 3×3 matrix can be a lengthy process so we use another method which involves the identity matrix.
The identity matrix is also useful for checking we have calculated our inverse matrix correctly as $AA^{-1}=I$.
The method we use is called Gauss-Jordan elimination.
This involves using **Elementary Row Operations (ERO's)**.

- The order of the rows can be changed.
- A row can be multiplied by a constant.
- One row can be added to or subtracted from another.

Example

$A=\begin{pmatrix}1&1&1\\2&-3&-1\\5&2&3\end{pmatrix}$ is a non-singular matrix. Calculate its inverse.

We start with matrix A and write it down with the identity matrix next to it.

$$\begin{array}{c}\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\2&-3&-1&0&1&0\\5&2&3&0&0&1\end{array}\right)\begin{matrix}R_1\\R_2\\R_3\end{matrix}\\ A \qquad\qquad\quad I\end{array}$$

This is called the augmented matrix.

We now have to turn matrix A on the left into the identity matrix using elementary row operations.

$$\begin{matrix}\\2R_1-R_2\\5R_1-R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&5&3&2&-1&0\\0&3&2&5&0&-1\end{array}\right)\begin{matrix}R_1\\R_2\\R_3\end{matrix}$$

$$\begin{matrix}\\R_2\div5\\5R_3-3R_2\end{matrix}\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&\tfrac{3}{5}&\tfrac{2}{5}&\tfrac{-1}{5}&0\\0&0&1&19&3&-5\end{array}\right)\begin{matrix}R_1\\R_2\\R_3\end{matrix}$$

$$\begin{matrix}R_1-R_3\\R_2-\tfrac{3}{5}R_3\\ \\\end{matrix}\left(\begin{array}{ccc|ccc}1&1&0&-18&-3&5\\0&1&0&-11&-2&3\\0&0&1&19&3&-5\end{array}\right)\begin{matrix}R_1\\R_2\\R_3\end{matrix}$$

$$\begin{array}{c}\begin{matrix}R_1-R_2\\ \\ \\\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&-7&-1&2\\0&1&0&-11&-2&3\\0&0&1&19&3&-5\end{array}\right)\begin{matrix}R_1\\R_2\\R_3\end{matrix}\\ I \qquad\qquad\quad A^{-1}\end{array} \quad\Rightarrow\quad A^{-1}=\begin{pmatrix}-7&-1&2\\-11&-2&3\\19&3&-5\end{pmatrix}$$

So matrix A has been turned into the identity matrix and the matrix on the right is A^{-1}.

How does this work?

Think of it like this -

$\left(\, 5 \mid 1 \,\right)$ turn 5 into 1 by ÷ by 5 and do the same to the 1 to turn it into $\frac{1}{5}$.

$\left(\, 1 \mid \tfrac{1}{5} \,\right)$ and $\frac{1}{5}$ is the (multiplicative) inverse of 5.

In terms of matrix A -

$\left(\, A \mid I \,\right)$

$\left(\, A^{-1}A \mid A^{-1}I \,\right)$ The effect of all the elementary row operations is the same as multiplying by A^{-1}.

$\left(\, I \mid A^{-1} \,\right)$ since $A^{-1}A = I$ and $A^{-1}I = A^{-1}$.

Questions

① Use Gauss-Jordan elimination to find the inverse of the following matrices.

(a) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ (b) $B = \begin{pmatrix} 1 & 5 & 2 \\ 4 & 3 & -1 \\ 5 & 10 & 2 \end{pmatrix}$ (c) $C = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$

② Show that these matrices are invertible and find the inverse in each case.

(a) $A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 6 & 4 \\ 7 & 2 & 3 \end{pmatrix}$ (b) $B = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

Solving a System of Linear Equations using the Inverse Matrix

A system of linear equations can be represented by the equation $AX = B$ where A is a matrix consisting of the coefficients of the unknowns, X is the column matrix of the unknowns and B is the column matrix of the constants.

e.g. The system

$$\begin{aligned} 5x+3y-2z &= -4 \\ 2x+2y+2z &= 0 \\ 3x+2y+z &= 1 \end{aligned}$$

can be represented by

$$\underset{A}{\begin{pmatrix} 5 & 3 & -2 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} \underset{=}{=} \underset{B}{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}}$$

We can set up the augmented matrix $\left(A|I\right)$ and convert it to $\left(I\middle|A^{-1}\right)$ using elementary row operations to find the inverse matrix A^{-1}.

The system of equations is then solved as follows.

$$AX = B$$

Multiply both sides by A^{-1}

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

So the column matrix of unknowns is found by multiplying the column matrix of constants by the inverse of the matrix of coefficients.

We will solve the example above by this method.

The augmented matrix is
$$\left(\begin{array}{ccc|ccc} 5 & 3 & -2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ 2R_1-5R_2 \\ 3R_1-5R_3 \end{array}\left(\begin{array}{ccc|ccc} 5 & 3 & -2 & 1 & 0 & 0 \\ 0 & -4 & -14 & 2 & -5 & 0 \\ 0 & -1 & -11 & 3 & 0 & -5 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ \\ R_2-4R_3 \end{array}\left(\begin{array}{ccc|ccc} 5 & 3 & -2 & 1 & 0 & 0 \\ 0 & -4 & -14 & 2 & -5 & 0 \\ 0 & 0 & 30 & -10 & -5 & 20 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} 15R_1+R_3 \\ 30R_2+14R_3 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 75 & 45 & 0 & 5 & -5 & 20 \\ 0 & -120 & 0 & -80 & -220 & 280 \\ 0 & 0 & 30 & -10 & -5 & 20 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} 8R_1+3R_2 \\ \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 600 & 0 & 0 & -200 & -700 & 1000 \\ 0 & -120 & 0 & -80 & -220 & 280 \\ 0 & 0 & 30 & -10 & -5 & 20 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} R_1 \div 600 \\ R_2 \div -120 \\ R_3 \div 30 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{7}{6} & \frac{5}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{11}{6} & -\frac{7}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}\right)$$

So the inverse matrix is $A^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{7}{6} & \frac{5}{3} \\ \frac{2}{3} & \frac{11}{6} & -\frac{7}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}$.

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

so $X = \begin{pmatrix} \frac{-1}{3} & \frac{-7}{6} & \frac{5}{3} \\ \frac{2}{3} & \frac{11}{6} & \frac{-7}{3} \\ \frac{-1}{3} & \frac{-1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}+0+\frac{5}{3} \\ \frac{-8}{3}+0-\frac{7}{3} \\ \frac{4}{3}+0+\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$

The solution of the system of equations is $x = 3,\ y = -5,\ z = 2$.

An alternative and usually quicker method for solving a system of 3 equations is Gaussian elimination which we will meet later on.

However, the determinant can tell us whether a system of equations has a unique solution or not. If the matrix of coefficients is square and non-singular then an inverse exists and so there is a unique solution to the system of equations. In geometric terms, the 3 equations are equations of planes. If there is a unique solution this means that the intersection of the 3 planes is at a unique point.

Matrix Properties

There are several properties we need to learn and be able to use.

① $A+B=B+A$

② $AB \neq BA$ (in general)

③ $(AB)C = A(BC)$ (the associative law)

④ $A(B+C) = AB + AC$ (the distributive law)

⑤ $\left(A^T\right)^T = A$

⑥ $(A+B)^T = A^T + B^T$

⑦ $(AB)^T = B^T A^T$

⑧ $(AB)^{-1} = B^{-1}A^{-1}$

⑨ $|AB| = |A||B|$

Examples

① Given that $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}$ show that

(a) $A^{-1}B^{-1} = (BA)^{-1}$ (b) $(AB)^{-1} = B^{-1}A^{-1}$ (c) $|A||B| = |AB|$

(a) $A^{-1} = \dfrac{1}{6-5}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ $B^{-1} = \dfrac{1}{7-8}\begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} -1 & 2 \\ 4 & -7 \end{pmatrix}$

$$A^{-1}B^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 4 & -7 \end{pmatrix} = \begin{pmatrix} -23 & 41 \\ 9 & -16 \end{pmatrix}$$

$$BA=\begin{pmatrix}7&2\\4&1\end{pmatrix}\begin{pmatrix}2&5\\1&3\end{pmatrix}=\begin{pmatrix}16&41\\9&23\end{pmatrix} \qquad (BA)^{-1}=\frac{1}{(16\times23-41\times9)}\begin{pmatrix}23&-41\\-9&16\end{pmatrix}=\begin{pmatrix}-23&41\\9&-16\end{pmatrix}$$

$$\therefore \quad A^{-1}B^{-1}=(BA)^{-1}$$

(b) $AB=\begin{pmatrix}2&5\\1&3\end{pmatrix}\begin{pmatrix}7&2\\4&1\end{pmatrix}=\begin{pmatrix}34&9\\19&5\end{pmatrix} \qquad (AB)^{-1}=\frac{1}{(34\times5-9\times19)}\begin{pmatrix}5&-9\\-19&34\end{pmatrix}=\begin{pmatrix}-5&9\\19&-34\end{pmatrix}$

$$B^{-1}A^{-1}=\begin{pmatrix}-1&2\\4&-7\end{pmatrix}\begin{pmatrix}3&-5\\-1&2\end{pmatrix}=\begin{pmatrix}-5&9\\19&-34\end{pmatrix}$$

$$\therefore \quad (AB)^{-1}=B^{-1}A^{-1}$$

(c) $|A|=2\times3-5\times1=1 \qquad |B|=7\times1-2\times4=-1 \qquad |AB|=34\times5-9\times19=-1$

$$\therefore \quad |A||B|=|AB|$$

② For what values of k is the matrix $A=\begin{pmatrix}2+k&-6\\4&3-k\end{pmatrix}$ singular?

A matrix is singular if its determinant = 0.

$$\begin{vmatrix}2+k&-6\\4&3-k\end{vmatrix}=(2+k)(3-k)+24$$

$$0=30+k-k^2$$

$$0=(6-k)(5+k)$$

$k=6 \quad \text{or} \quad k=-5 \Rightarrow$ Matrix is singular when $k=6$ or $k=-5$.

③ (a) $A=\begin{pmatrix}2&3\\7&11\end{pmatrix}$ Show that $A^2=13A-I$.

(b) Hence show (without evaluating A^3 or A^{-1}) that $A^3=168A-13I$ and $A^{-1}=13I-A$.

(a) $A^2=\begin{pmatrix}2&3\\7&11\end{pmatrix}\begin{pmatrix}2&3\\7&11\end{pmatrix}=\begin{pmatrix}25&39\\91&142\end{pmatrix} \qquad 13A-I=13\begin{pmatrix}2&3\\7&11\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}25&39\\91&142\end{pmatrix}$

$$\therefore \quad A^2=13A-I$$

b)

$A^3 = A^2A$	$A^2 = 13A - I$
$= \left(13A - I\right)A$	$AA = 13A - I$
$= 13A^2 - IA$	$AA - 13A = -I$
$= 13\left(13A - I\right) - A$	$13A - AA = I$
$= 169A - 13I - A$	$A\left(13I - A\right) = I$
$= 168A - 13I$	$AA^{-1} = I$
	$\therefore A^{-1} = 13I - A$

④ Given that A and B are square matrices, simplify and comment

(a) $A^{-1}B^{-1}BA$ (b) $BAA^{-1}B^{-1}$ (c) $B^{-1}A^{-1}AB$ (d) $ABB^{-1}A^{-1}$

(a) $A^{-1}B^{-1}BA$	(b) $BAA^{-1}B^{-1}$	(c) $B^{-1}A^{-1}AB$	(d) $ABB^{-1}A^{-1}$
$= A^{-1}IA$	$= BIB^{-1}$	$= B^{-1}IB$	$= AIA^{-1}$
$= A^{-1}A$	$= BB^{-1}$	$= B^{-1}B$	$= AA^{-1}$
$= I$	$= I$	$= I$	$= I$

In general $\left(AB\right)^{-1} = B^{-1}A^{-1}$ and $\left(BA\right)^{-1} = A^{-1}B^{-1}$.

Matrices and Transformations

We can use a 2×2 matrix to map any point $\left(x, y\right)$ to another point $\left(x', y'\right)$ with this result…

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Transformations can be rotations, reflections or dilatations.

Reflection in the x-axis

Under reflection in the x-axis the image of point $P\left(x, y\right)$ is $P'\left(x, -y\right)$.

This can be represented by the matrix equation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

So the matrix associated with reflection in the x-axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Reflection in the y-axis

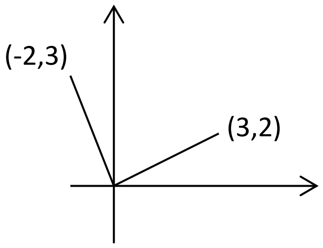
Under reflection in the y-axis the image of point $P(x,y)$ is $P'(-x,y)$.

This can be represented by the matrix equation $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-x\\ y\end{pmatrix}$.

So the matrix associated with reflection in the y-axis is $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$.

Rotation

Take the point $(3,2)$. If we rotate it 90° anticlockwise about the origin it becomes $(-2,3)$.



$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}3\\ 2\end{pmatrix}=\begin{pmatrix}-2\\ 3\end{pmatrix}$$

$$\begin{pmatrix}3a+2b\\ 3c+2d\end{pmatrix}=\begin{pmatrix}-2\\ 3\end{pmatrix}$$

Pick simple values of $a,\ b,\ c$ and d.
$a=0,\ b=-1,\ c=1,\ d=0$

The transformation matrix is $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

This is a specific case but there is a general matrix of rotation $R=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ where θ is the angle of rotation anticlockwise about the origin.

In the example above, $\theta=90^\circ$ so the rotation matrix is $\begin{pmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

Example

Find the coordinates of the point $(6,4)$ under an anticlockwise rotation of 60° about the origin.

$$\begin{pmatrix}\cos 60^\circ & -\sin 60^\circ\\ \sin 60^\circ & \cos 60^\circ\end{pmatrix}\begin{pmatrix}6\\ 4\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2} & \frac{-\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}6\\ 4\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3-2\sqrt{3}\\ 3\sqrt{3}+2\end{pmatrix}$$

After rotation the coordinates of the point are $\left(3-2\sqrt{3},\ 3\sqrt{3}+2\right)$

Questions

① Find the coordinates of the point $(8,1)$ after a rotation of $30°$ anticlockwise about the origin.

② Find the coordinates of the point $\left(1,-\sqrt{3}\right)$ rotated through an angle of $60°$ clockwise about the origin.

③ The point $(2,1)$ maps to the point $\left(\tfrac{3}{2},\tfrac{11}{6}\right)$ under a rotation about the origin. Find to the nearest degree the angle through which the point has been rotated.

④ A point is rotated about the origin by $100°$ clockwise and finishes at the point with coordinates $(-1.796,\ 1.332)$. Find the original coordinates to the nearest whole number.

Reflection in a Line Passing Through the Origin

In a similar way, there is another 2×2 matrix which represents a reflection of any point in a line passing through the origin.

This matrix is $\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$ where $\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$ (or $-90°\le\theta\le 90°$) and θ is the angle between the line of reflection and the x-axis.

Example

Find the coordinates of the point $(-3,1)$ when reflected in the line $y=3x$.

$$m=\tan\theta$$
$$3=\tan\theta$$
$$\theta=71\cdot 6° \qquad \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$
$$\begin{pmatrix}\cos 143.13 & \sin 143.13\\ \sin 143.13 & -\cos 143.13\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$
$$\begin{pmatrix}-3\cos 143.13+\sin 143.13\\ -3\sin 143.13-\cos 143.13\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$
$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3\\ -1\end{pmatrix}$$

After reflection the image point is $(3,-1)$.

Questions

① Find the coordinates, correct to 1 d.p. of the point $(3,1)$, reflected in a line through the origin at an angle of $-30°$.

② The point $(-1,-2)$ maps to the point $(-2,-1)$ under a reflection in a line at an angle of θ to the x-axis. Find the angle θ to the nearest degree.

③ A point is reflected in the line $y=-4x$ and has its image at the point with coordinates $(-1,0)$. Find the original coordinates to 1 d.p.

Dilatation (Scaling)

The third transformation matrix we need to learn is $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$.

When this matrix multiplies the column vector $\begin{pmatrix}x\\ y\end{pmatrix}$ it scales the x-coordinate by λ and the y-coordinate by μ.

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\lambda x\\ \mu y\end{pmatrix}$$

Examples

① Find the effect of scaling the point $(2,3)$ by 4 in the x direction.

$\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix}=\begin{pmatrix}8\\ 3\end{pmatrix}$ The image point is $(8,3)$.

② Find the scaling matrix and coordinates of the image of the point $(2,3)$ after scaling by 3 in the x direction and scaling by 5 in the y direction.

Scaling matrix is $\begin{pmatrix}3 & 0\\ 0 & 5\end{pmatrix}$.

$\begin{pmatrix}3 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix}=\begin{pmatrix}6\\ 15\end{pmatrix}$ The image point is $(6,15)$.

Questions

① Find the scaling matrix and coordinates of the image of the point $(3,1)$ scaled in the x direction by -4.

② The point $(-2,-3)$ maps to the point $(-4,{}^{-3}\!/_{2})$ under a scaling in both directions. Find the scaling matrix.

③ A point $(2,5)$ is scaled to give an image at the point with coordinates $(-1,0)$. Find the scaling matrix.

Composition of Transformations

If matrix M_1 is associated with transformation T_1 and matrix M_2 is associated with transformation T_2 then transformation T_1 followed by transformation T_2 is associated with the matrix M_2M_1. Notice the order in which the matrices are multiplied. This is very important.

Example

Find the image of $(3,4)$ after an anti-clockwise rotation of $\dfrac{\pi}{2}$ radians about the origin followed by reflection in the x-axis.

Matrix associated with an anti-clockwise rotation of $\dfrac{\pi}{2}$ radians about the origin = $\begin{pmatrix} \cos {}^{\pi}\!/_{2} & -\sin {}^{\pi}\!/_{2} \\ \sin {}^{\pi}\!/_{2} & \cos {}^{\pi}\!/_{2} \end{pmatrix}$

Matrix associated with reflection in the x-axis = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Matrix associated with an anti-clockwise rotation of $\dfrac{\pi}{2}$ radians about the origin followed by reflection in the x-axis = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos {}^{\pi}\!/_{2} & -\sin {}^{\pi}\!/_{2} \\ \sin {}^{\pi}\!/_{2} & \cos {}^{\pi}\!/_{2} \end{pmatrix} = \begin{pmatrix} \cos {}^{\pi}\!/_{2} & -\sin {}^{\pi}\!/_{2} \\ -\sin {}^{\pi}\!/_{2} & -\cos {}^{\pi}\!/_{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ The image point is $(-4,-3)$.

This combination of transformations is equivalent to a reflection in the line $y=-x$.

Past Paper Questions

2001 – B3

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$

Show that $AB = kI$ for some constant k, where I is the identity matrix .

Hence obtain (i) the inverse matrix A^{-1} and (ii) the matrix A^2B. **(4 marks)**

2002 – Q14

Write down the 2×2 matrix A representing a reflection in the x-axis and the 2×2 matrix B representing an anti-clockwise rotation of $30°$ about the origin.

Hence show that the image of a point (x, y) under the transformation A followed by the transformation B is $\left(\frac{kx+y}{2}, \frac{x-ky}{2}\right)$, stating the value of k . **(4 marks)**

2003 – Q13

The matrix A is such that $A^2 = 4A - 3I$ where I is the corresponding identity matrix.

Find integers p and q such that $A^4 = pA + qI$. **(4 marks)**

2004 – Q6

Write down the 2×2 matrix M_1 associated with an anti-clockwise rotation of $\frac{\pi}{2}$ radians about the origin.

Write down the matrix M_2 associated with reflection in the x-axis.

Evaluate M_2M_1 and describe geometrically the effect of the transformation represented by M_2M_1.

(2, 1, 2 marks)

2005 – Q7

Given the matrix $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$, show that $A^2 + A = kI$ for some constant k, where I is the 3×3 unit matrix.

Obtain the values of p and q for which $A^{-1} = pA + qI$. **(4, 2 marks)**

2006 – Q13

Calculate the inverse of the matrix $\begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}$.

For what value of x is this matrix singular? **(4 marks)**

2007 – Q5

Matrices A and B are defined by $A=\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 1 & 2\end{pmatrix}$, $B=\begin{pmatrix}x+2 & x-2 & x+3\\ -4 & 4 & 2\\ 2 & -2 & 3\end{pmatrix}$.

(a) Find the product AB.

(b) Obtain the determinants of A and AB.

Hence or otherwise, obtain an expression for $\det B$. **(2, 2, 1 marks)**

2008 – Q6

Let the matrix $A=\begin{pmatrix}1 & x\\ x & 4\end{pmatrix}$.

(a) Obtain the value(s) of x for which A is singular.

(b) When $x=2$, show that $A^2=pA$ for some constant p.

Determine the value of q such that $A^4=qA$. **(2, 3 marks)**

2009 – Q2

Given the matrix $A=\begin{pmatrix}t+4 & 3t\\ 3 & 5\end{pmatrix}$.

(a) Find A^{-1} in terms of t when A is non-singular.

(b) Write down the value of t such that A is singular.

(c) Given that the transpose of A is $\begin{pmatrix}6 & 3\\ 6 & 5\end{pmatrix}$, find t. **(3, 1, 1 marks)**

2010

Q4 Obtain the 2×2 matrix M associated with an enlargement, scale factor 2, followed by a clockwise rotation of $60°$ about the origin.

(4 marks)

Q14b Use Gaussian elimination to show that the set of equations

$$x-y+z=1$$
$$x+y+2z=0$$
$$2x-y+az=2$$

has a unique solution when $a\neq2\cdot5$.

Explain what happens when $a=2\cdot5$.

Obtain the solution when $a=3$.

Given $A=\begin{pmatrix}5 & 2 & -3\\ 1 & 1 & -1\\ -3 & -1 & 2\end{pmatrix}$ and $B=\begin{pmatrix}1\\ 0\\ 2\end{pmatrix}$, calculate AB.

Hence or otherwise state the relationship between A and the matrix $C=\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & 2\\ 2 & -1 & 3\end{pmatrix}$.

(5, 1, 1, 1, 2 marks)

2011 – Q4

(a) For what value of λ is $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}$ singular?

(b) For $A=\begin{pmatrix} 2 & 2\alpha+\beta & -1 \\ 3\alpha+2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$ obtain values of α and β such that $A'=\begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$.

(3, 3 marks)

2012 – Q9

A non-singular $n\times n$ matrix A satisfies the equation $A+A^{-1}=I$ where I is the $n\times n$ identity matrix. Show that $A^3=kI$ and state the value of k. **(4 marks)**

2013 – Q3

Matrices A and B are defined by $A=\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}$

(a) Find A^2.

(b) Find the value of p for which A^2 is singular.

(c) Find the values of p and x if $B=3A'$. **(1, 2, 2 marks)**

2015

Q5 Obtain the value(s) of p for which the matrix $A=\begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$ is singular. **(4 marks)**

Q11 Write down the 2×2 matrix, M_1, associated with a reflection in the y-axis.

Write down a second 2×2 matrix, M_2, associated with an anticlockwise rotation through an angle of $\frac{\pi}{2}$ radians about the origin.

Find the 2×2 matrix, M_3, associated with an anticlockwise rotation through $\frac{\pi}{2}$ radians about the origin followed by a reflection in the y-axis.

What single transformation is associated with M_3? **(4 marks)**

2016 – Q7

(a) A is the matrix $\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix}$.

Find the determinant of the matrix A.

(b) Show that A^2 can be expressed in the form $pA+qI$, stating the values of p and q.

(c) Obtain a similar expression for A^4.

(1, 3, 2 marks)

2017 – Q7

Matrices P and Q are defined by $P = \begin{pmatrix} x & 2 \\ -5 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & -3 \\ 4 & y \end{pmatrix}$, where $x, y \in \boldsymbol{R}$

(a) Given the determinant of P is 2, obtain:

(i) The value of x

(ii) P^{-1}

(iii) $P^{-1}Q'$, where Q' is the transpose of Q

(b) The matrix R is defined by $R = \begin{pmatrix} 5 & -2 \\ z & -6 \end{pmatrix}$, where $z \in \boldsymbol{R}$.

Determine the value of z such that R is singular. **(1,1,2,2 marks)**