Proof & Elementary Number Theory – Teacher Version

There are 5 main methods of proof to be considered; these are,

- Direct proof
- Proof by counterexample
- Proof by contradiction
- Proof by contrapositive
- Proof by induction

Notation

$x \Rightarrow y$	x implies that y
$x \Leftarrow y$	x is implied by y
$\Leftrightarrow$ or *iff*	if & only if
$\forall x$	for all values of x
$\exists x$	there exists a value of x.

Direct Proof

While Proof, in many respects is a new topic for us at Advanced Higher, we have considered some of the ideas previously, including the following questions which you have already answered in homework.

Questions

① Given that $f(n) = n^2 + n$, where *n* is a positive integer, prove $f(n)$ is always even.

② Prove that for all natural numbers n, $\frac{n}{3}+\frac{n^2}{2}+\frac{n^3}{6}$ is a natural number.

③ By considering $(a-b)^2$, show that $a^2+b^2 \geq 2ab$ for real numbers *a* and *b*.

④ By considering $(a-b)^2+(b-c)^2+(c-a)^2$, show that $a^2+b^2+c^2 \geq ab+bc+ca$ for real numbers *a*, *b* and *c*.

Direct proof usually just involved you using your mathematical skills to take the LHS of a statement and prove you can write it in the form of the RHS (or vice versa). You should **never** be working with both sides at once.

Direct proof **does not** involve substituting in some numbers and proving it works for those numbers…this does not prove the general case. You should always be working algebraically.

Sometimes to aid our in working out our proof it is useful to remember that numbers are either odd $2k \pm 1$ or even $2k$. That may be useful in proving the following statements.

Questions

① Prove that the product of two even numbers is always even.
② Prove that the product of an odd and an even number is always even.
③ Prove that if n is odd, n^2 is odd.
④ Prove that the product of three consecutive numbers is always divisible by six.

Proof by Counterexample

Find a counterexample to disprove each of the following statements.

① $\forall x \in W$, $5x > 4x$
② $\forall x \in \mathbb{R}, 2x \neq 2^x$
③ $\forall x \in \mathbb{R}, ax = bx \Rightarrow a = b$
④ $\forall x \in \mathbb{Z}, \sqrt{x^2} = x$
⑤ $\forall x \in \mathbb{R}, \sqrt{1-\sin^2 x} = \cos x$
⑥ $\forall a, b \in \mathbb{R}, |a| + |b| \leq |a+b|$
⑦ $\forall x \in W, e^{\ln x} = x$

Solutions

① When x = 0 — $5x$ = 0, $4x$ = 0 — $5x = 4x$ — ∴ Statement is false.

② When x = 1 — $2x$ = 2, $2^1 = 2$ — $2x = 2^x$ — ∴ Statement is false.

③ When x = 0, a = 1, b = 2 — $2 \times 0 = 1 \times 0$ but $2 \neq 1, a \neq b$ — ∴ Statement is false.

④ When x = -1 — $\sqrt{(-1)^2} = 1 \neq x$ — ∴ Statement is false.

⑤ i.e. $\sqrt{\cos^2 x} = \cos x$ When $x = \pi$

$\sqrt{\cos^2 \pi} = \sqrt{(-1)^2} = 1$ — $\cos \pi = -1$ — $1 \neq -1$ ∴ Statement is false.

⑥ When a = 2, b = -3 — $|2| + |-3| = 2 + 3 = 5$ — $|2-3| = |-1| = 1$ — $5 > 1$ — ∴ Statement is false.

⑦ When x = 0 — $e^{\ln 0}$
ln0 is undefined — ∴ Statement is false.

Questions

By finding a counterexample, prove each of the following conjectures is false.

① $u_n = n^3 - 6n^2 + 13n - 7, n \in Z^+$ gives the sequence of odd numbers 1, 3, 5, ...

② $P|(a+b)$ and $P|(b+c) \Rightarrow P|(a+c)$ ($a|b$ means '*a* is a factor of *b*' or '*a divides b*')

③ $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$, $\forall n \in \mathbb{Z}$.

④ $S(n)$ is defined as the sum of the divisors of *n* other than *n* itself.
Conjecture is $S(n) < n, \forall n \in \mathbb{N}$.

⑤ All odd numbers between 2 and 14 are prime.

Solutions

① If true, when *n* = 4, $u_n = 7$ $13 \neq 7$, $\therefore$ Conjecture is False.

$u_4 = 4^3 - 6\times 4^2 + 13\times 4 - 7$

= 64 – 96 + 52 – 7

= 13

② Let *P* = 5, *a* = 3, *b* = 2 and *c* = 8

Now $5|(3+2)$ and $5|(2+8)$, but $a+c=11$ which is <u>not</u> divisible by 5, $\therefore$ Conjecture is False.

③ When *n* = -1

$\int \frac{1}{x} dx = \ln|x| + c$, $\therefore$ Conjecture is False.

④ *S*(6) = 1 + 2 + 3 = 6

6 = 6 $\therefore$ Conjecture is False.

⑤ $2 < 9 < 14$ but $3|9$, $\therefore$ 9 is not prime; Conjecture is False.

Proof by Contradiction

When we prove a conjecture by contradiction we initially assume the conjecture is false. We then must show this leads to the deduction of a false conclusion & so the original conjecture must therefore have been true.

Examples

① '$\sqrt{2}$ is not a rational number.'

Assume that $\sqrt{2}$ is rational

$\therefore \sqrt{2} = \dfrac{p}{q}, \quad p \,\&\, q \in \mathbb{Z}^+$ (with no common factors – this is termed relatively prime)

$\Rightarrow 2 = \dfrac{p^2}{q^2}$

$\Rightarrow 2q^2 = p^2$

$\Rightarrow p^2$ is even, $\therefore p$ is even.

Then $p = 2k$ for $k \in \mathbb{Z}$.

Thus $\quad 2q^2 = (2k)^2$

$2q^2 = 4k^2$

$q^2 = 2k^2 \Rightarrow q^2$ is even, $\therefore q$ is even.

$\therefore$ If assumption is true *p* and *q* are both even.

HOWEVER $p \,\&\, q \in \mathbb{Z}^+$, with no common factors $\therefore$ by contradiction assumption was false.

$\Rightarrow$ Original conjecture is true, $\sqrt{2}$ is not a rational number.

② 'If m^2 is even then *m* is even.'

Assume that is m^2 is even then *m* is odd.

$\Rightarrow m = 2k - 1, \quad k \in \mathbb{Z}$

$\Rightarrow (2k-1)^2$ is even.

$= (2k)^2 - 4k + 1$

$= 4k^2 - 4k + 1$

$= 2(2k^2 - 2k) + 1 \qquad$ Let $2k^2 - 2k = r$

$= 2r + 1$

$= odd$

This is a contradiction $\therefore$ assumption is false.

$\Rightarrow$ Original conjecture is true, 'If m^2 is even then *m* is even.'

③ Prove, by contradiction, that $\frac{a+b}{2} \geq \sqrt{ab} \quad \forall a,b \in \mathbb{N}$.

Assume $\frac{a+b}{2} < \sqrt{ab}$ for some $\forall a,b \in \mathbb{N}$

$$\Rightarrow a+b < 2\sqrt{ab}$$
$$\Rightarrow (a+b)^2 < 4ab$$
$$\Rightarrow a^2 + 2ab + b^2 < 4ab$$
$$\Rightarrow a^2 - 2ab + b^2 < 0$$
$$(a-b)^2 < 0, \ \Rightarrow a,b \in \mathbb{N}$$

This is a contradiction $\therefore$ assumption is false.

$\Rightarrow$ Original conjecture is true, $\frac{a+b}{2} \geq \sqrt{ab} \quad \forall a,b \in \mathbb{N}$.

④ Prove that the number of primes in infinite.

Assume the number of prime numbers is finite: P_n is the greatest.

The prime numbers are $p_1, p_2, p_3, \ldots\ldots, p_n$
There is a $q \in \mathbb{N}$ s.t. $q = p_1 \times p_2 \times p_3 \times \ldots\ldots \times p_n + 1$

Clearly $q > 1$, now $q \div p_1 = p_2 p_3 \ldots\ldots p_n$ remainder 1. $\Rightarrow q$ is not divisible by any p_n.

Now all numbers $\in \mathbb{N}$ can be divided by 1 & a prime number.

Thus $\exists$ a prime number <u>not</u> between p_1 & p_n which divides *q*.
$\therefore P_n$ is not the greatest $\therefore$ assumption is false.

$\Rightarrow$ Original conjecture is true, the number of primes is infinite.

<u>Further Questions</u>

Prove each of these conjectures by contradiction.

① '$\sqrt{7}$ is irrational.'
② ' If n^3 is odd, n is odd.'
③ 'If $m^2 = 10$ then m is not a rational number.'
④ 'If the sum of 2 real numbers is irrational, at least one of the numbers is irrational.'

<u>Solutions</u>

① Assume that $\sqrt{7}$ is rational

$\therefore \sqrt{7} = \dfrac{p}{q}, \quad p \,\&\, q \in \mathbb{Z}^{+}$ and are relatively prime

$\Rightarrow 7 = \dfrac{p^2}{q^2}$

$\Rightarrow 7q^2 = p^2$

$\Rightarrow p^2$ is divisible by 7.

Then $p = 7k$ for $k \in \mathbb{Z}$.

Thus $7q^2 = (7k)^2$

$7q^2 = 49k^2$

$q^2 = 7k^2 \Rightarrow q^2$ is divisible by 7

$\therefore 7 | p^2 \,\&\, 7 | q^2 \Rightarrow p^2 \,\&\, q^2$ are not relatively prime

$\therefore$ by contradiction assumption was false.

$\Rightarrow$ Original conjecture is true, $\sqrt{7}$ is irrational.

② Assume that is n^3 is odd then *n* is even.

$\Rightarrow n = 2k, \quad k \in \mathbb{Z}$

$\Rightarrow (2k)^3$ is even.

$= 8k^3$

$= 2(4k^3)$ Let $4k^2 = r$

$= 2r$

$= even$

This is a contradiction $\therefore$ assumption is false.

$\Rightarrow$ Original conjecture is true, If n^2 is odd then *n* is odd.’

③ Assume that $m \in \mathbb{Q}$

$\therefore m = \frac{p}{q}, \quad p \,\&\, q \in \mathbb{Z}$ and are relatively prime

$\Rightarrow m^2 = \frac{p^2}{q^2}$

$\Rightarrow 10 = \frac{p^2}{q^2}$

$\Rightarrow 10q^2 = p^2$

$\Rightarrow p^2$ is even $\therefore p$ is even.

Then $p = 2k$ for $k \in \mathbb{Z}$.

Thus $10 = \frac{4k^2}{q^2}$

$10q^2 = 4k^2$

$5q^2 = 2k^2$

$2k^2$ is even so $5q^2$ is even

$\Rightarrow q^2$ is even $\therefore q$ is even.

$\therefore$ If assumption is true *p* and *q* are both even.

HOWEVER $p \,\&\, q \in \mathbb{Z}^+$, with no common factors $\therefore$ by contradiction assumption was false.

$\Rightarrow$ Original conjecture is true, 'If $m^2 = 10$ then m is not a rational number.'

④ Assume $a + b$ is irrational, $a, b \in \mathbb{Q}$

Let $a = \frac{p}{q}, \quad b = \frac{r}{s}$ where $p, q, r, s \in \mathbb{Z}^+$ & *p,q*:*r,s* are relatively prime.

$a + b = \frac{p}{q} + \frac{r}{s}$

$a + b = \frac{ps + rq}{qs}$

$$\left.\begin{array}{l} ps + rq \in \mathbb{R} \\ qs \in \mathbb{R} \end{array}\right\} \therefore \frac{ps + rq}{qs} \in \mathbb{Q}$$

Assumption is False $\Rightarrow$ Original conjecture is true.

Proof by Contrapositive

Pythagoras' Theorem states "If triangle ABC is right-angled at C then $c^2 = a^2 + b^2$".
This is known as an implication as it is an "If.....then" statement.
The contrapositive of this statement is "If $c^2 \neq a^2 + b^2$ then the triangle ABC is not right-angled at C."

An implication and its contrapositive are logically equivalent i.e. if the implication is true then the contrapositive is true and vice versa.

This means that if you can prove the contrapositive of a statement is true then the original statement must also be true.

Examples

① Prove that for $x \in \mathbb{Z}$, if $5x+9$ is even, then x is odd.

First we need to write down the contrapositive of the statement:
For $x \in \mathbb{Z}$, if x is not odd then $5x+9$ is not even.
This can also be written as for $x \in \mathbb{Z}$, if x is even then $5x+9$ is odd.
We now prove the contrapositive directly.

Proof : x is even so $x = 2k$ for some integer k.
So $5x+9 = 5(2k)+9 = 10k+9 = 2(5k+4)+1$ which is odd as $5k+4$ is an integer.
$\therefore$ If x is even then $5x+9$ is odd, for $x \in \mathbb{Z}$.
As the contrapositive has been proven true this means the original statement is also true i.e. for $x \in \mathbb{Z}$, if $5x+9$ is even then x is odd.

② Prove that for $n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

Contrapositive of the statement: For $n \in \mathbb{Z}$, if n is not odd then n^2 is not odd.
This can also be written as for $n \in \mathbb{Z}$, if n is even then n^2 is even.

Proof: n is even so $n = 2k$ for some integer k.
So $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ which is even as $2k^2$ is an integer.
$\therefore$ If n is even then n^2 is even for $n \in \mathbb{Z}$.
As the contrapositive has been proven true this means the original statement is also true i.e. for $n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

Questions

Prove the following by using the method of proof by contrapositive.

① For $n \in \mathbb{Z}$, if n^2 is even, then n is even.

② For $a \in \mathbb{Z}$, if a^2 is not divisible by 4, then a is odd

③ For $n \in \mathbb{Z}$, if $3 \nmid n^2$, then $3 \nmid n$.

④ For $x, y \in \mathbb{R}$, if $y^3 + yx^2 \le x^3 + xy^2$, then $y \le x$.

⑤ For $x \in \mathbb{R}$, if $x^5 + 7x^3 + 5x \ge x^4 + x^2 + 8$, then $x \ge 0$.

Solutions

① Contrapositive of the statement: For $n \in \mathbb{Z}$, if n is odd, then n^2 is odd.
Proof: n is odd so $n = 2k+1$ for some integer k.
So $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ which is odd.
$\therefore$ If n is odd then n^2 is odd for $n \in \mathbb{Z}$.
As the contrapositive has been proven true this means the original statement is also true i.e. for $n \in \mathbb{Z}$, if n^2 is even, then n is even.

② Contrapositive of the statement: For $a \in \mathbb{Z}$, if a is even, then a^2 is divisible by 4.
Proof: a is even so $a = 2k$ for some integer k.
So $a^2 = (2k)^2 = 4k^2$ which is divisible by 4.
$\therefore$ For $a \in \mathbb{Z}$, if a is even, then a^2 is divisible by 4.
As the contrapositive has been proven true this means the original statement is also true i.e. for $a \in \mathbb{Z}$, if a^2 is not divisible by 4, then a is odd.

③ Contrapositive of the statement: For $n \in \mathbb{Z}$, if $3/n$ then $3/n^2$.
Proof: $3/n$ so $n = 3k$ for some integer k.
So $n^2 = (3k)^2 = 9k^2 = 3(3k^2)$ which is divisible by 3.
$\therefore$ For $n \in \mathbb{Z}$, if $3/n$ then $3/n^2$.
As the contrapositive has been proven true this means the original statement is also true i.e. for $n \in \mathbb{Z}$, if $3 \nmid n^2$, then $3 \nmid n$.

④ Contrapositive of the statement: If $y > x$ then $y^3 + yx^2 > x^3 + xy^2$.

Proof : $y > x$ so $y - x > 0$.

Multiply both sides by $x^2 + y^2$.

$(y-x)(x^2+y^2) > (y-x)0$

$yx^2 + y^3 - x^3 - xy^2 > 0$

$y^3 + yx^2 > x^3 + xy^2$

As the contrapositive has been proven true this means the original statement is also true i.e. for $x, y \in \mathbb{R}$, if $y^3 + yx^2 \le x^3 + xy^2$, then $y \le x$.

⑤ Contrapositive of the statement: If $x < 0$ then $x^5 + 7x^3 + 5x < x^4 + x^2 + 8$.

Proof : $x < 0$ so x is negative. This means that x^5, $7x^3$ and $5x$ are all negative.

$\Rightarrow x^5 + 7x^3 + 5x < 0$

As x is negative, x^4, x^2 and 8 are all positive.

$\Rightarrow x^4 + x^2 + 8 > 0$

So $x^5 + 7x^3 + 5x < x^4 + x^2 + 8$.

As the contrapositive has been proven true this means the original statement is also true i.e. for $x \in \mathbb{R}$, if $x^5 + 7x^3 + 5x \ge x^4 + x^2 + 8$, then $x \ge 0$.

Proof by Induction

Proof by induction involves a set process and is a mechanism to prove a conjecture.

STEP 1: Show conjecture is true for $n=1$ (or the first value *n* can take.)

STEP 2: Assume statement is true for $n=k$

STEP 3: Show conjecture is true for $n=k+1$

STEP 4: Closing Statement (this is crucial in gaining all the marks).

Examples

① Prove that $2^n > n, \quad \forall n \in \mathbb{N}$.

When $n=1$ $\quad 2^1 = 2 \quad 2>1,\ \therefore$ true for $n=1$.

Assume true for $n=k \Rightarrow 2^k > k$

Consider when $n=k+1$ $\quad 2^{k+1} = 2.2^k \quad \therefore 2^{k+1} > 2^k$
$> k+1$ $\therefore$ Statement is true for $k+1$.

As statement is true for $n=1$ & true for $k \Rightarrow$ true for $k+1$, $\therefore$ by induction is true $\forall n \geq 1, n \in \mathbb{N}$.

② Prove that $n^3 + 2n$ is divisible by 3 $\forall n \geq 1, n \in \mathbb{N}$.

When $n=1$ $\quad 1^3 + 2\times 1 = 3$ divisible by 3 $\therefore$ true for $n=1$.

Assume true for $n=k \Rightarrow k^3 + 2k$ is divisible by 3.

Consider when $n=k+1$

$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$
$$= (k^3 + 2k) + 3k^2 + 3k + 3$$
$$= (k^3 + 2k) + 3(k^2 + k + 1)$$

$(k^3 + 2k)$ and $3(k^2 + k + 1)$ are both divisible by 3 $\therefore$ Statement is true for $k+1$.

As statement is true for $n=1$ & true for $k \Rightarrow$ true for $k+1$, $\therefore$ by induction is true $\forall n \geq 1, n \in \mathbb{N}$.

③ Show that the triangular numbers are generated by the recurrence relation $u_{n+1} = u_n + n + 1$ where $u_n = \frac{1}{2}n(n+1), n \in \mathbb{N}$. (The triangular numbers are 1, 3, 6, 10….)

When $n = 1$ $\quad u_1 = \frac{1}{2} \times 1 \times (1+1) = 1$ $\therefore$ true for $n = 1$.

Assume true for $n = k \Rightarrow u_k = \frac{1}{2}k(k+1)$

Consider when $n = k+1$

$$\begin{aligned} u_{k+1} &= u_k + k + 1 \\ &= \frac{1}{2}k(k+1) + k + 1 \\ &= \frac{1}{2}(k(k+1) + 2k + 2) \\ &= \frac{1}{2}(k^2 + 3k + 2) \\ &= \frac{1}{2}(k+1)(k+2) \\ &= \frac{1}{2}(k+1)((k+1)+1) \end{aligned}$$

$\therefore$ Statement is true for $k+1$.

As statement is true for $n = 1$ & true for $k \Rightarrow$ true for $k+1$, $\therefore$ by induction is true $\forall n \geq 1, n \in \mathbb{N}$.

④ Examine the number pattern generated below.

$$1 = 1^2$$
$$1+3 = 2^2$$
$$1+3+5 = 3^2$$
$$1+3+5+7 = 4^2$$

i) Make a conjecture about the sum of the first n odd numbers.
ii) Prove your conjecture by induction.

i) $S_n = 1+3+5+7+\ldots+(2n-1) = n^2 \qquad \Rightarrow \sum_{r=1}^{n}(2r-1) = n^2$

ii) When $n=1$ $\quad (2n-1) = 2\times1-1$
$= 1$
$= 1^2$
$= n^2$
$\therefore$ true for $n=1$.

Assume true for $n = k \Rightarrow \sum_{r=1}^{k}(2r-1) = k^2$

Consider when $n = k+1$

$$\sum_{r=1}^{k+1}(2r-1) = \sum_{r=1}^{k}(2r-1) + (2(k+1)-1)$$
$$= k^2 + (2k+1)$$
$$= (k+1)^2$$

$\therefore$ Statement is true for $k+1$.

As statement is true for $n=1$ & true for $k \Rightarrow$ true for $k+1$, $\therefore$ by induction is true $\forall n \geq 1, n \in \mathbb{N}$.

⑤ Prove that the recurrence relation defined by $u_{n+1} = u_n + 4n + 4$ has an n^{th} term $u_n = 2n(n+1)$.

When $n = 1$ $\quad u_1 = 2(1+1) = 4$ $\therefore$ true for $n = 1$.

Assume true for $n = k \Rightarrow u_k = 2k(k+1)$

Consider when $n = k+1$

$$\begin{aligned} u_{k+1} &= u_k + 4k + 4 \\ &= 2k(k+1) + 4k + 4 \\ &= 2k^2 + 2k + 4k + 4 \\ &= 2k^2 + 6k + 4 \\ &= 2(k^2 + 3k + 2) \\ &= 2(k+1)(k+2) \\ &= 2(k+1)((k+1)+1) \end{aligned}$$

$\therefore$ Statement is true for $k+1$.

As statement is true for $n = 1$ & true for $k \Rightarrow$ true for $k+1$, $\therefore$ by induction is true $\forall n \geq 1, n \in \mathbb{N}$.

⑥ Prove by induction that $5 + 7 + 9 + ... + (2n-1) = n^2 - 4, \forall n \geq 3, n \in \mathbb{N}$. i.e. $\sum_{r=3}^{n} (2r-1) = n^2 - 4$.

When $n = 3$ $\quad$ RHS: $3^2 - 4 = 5$

LHS: $2 \times 3 - 1 = 5$ $\quad \therefore$ true for $n = 3$.

Assume true for $n = k \Rightarrow \sum_{r=3}^{k} (2r-1) = k^2 - 4$

Consider when $n = k+1$

$$\begin{aligned} \sum_{r=3}^{k+1} (2r-1) &= \sum_{r=3}^{k} (2r-1) + (2(k+1)-1) \\ &= k^2 - 4 + 2k + 1 \\ &= k^2 + 2k - 4 + 1 \\ &= (k+1)^2 - 4 \end{aligned}$$

$\therefore$ Statement is true for $k+1$.

As statement is true for $n = 3$ & true for $k \Rightarrow$ true for $k+1$, $\therefore$ by induction is true $\forall n \geq 3, n \in \mathbb{N}$.

⑦ Prove by induction that $2^n > n^2, \quad \forall n > 4, n \in \mathbb{N}.$

When $n = 5$ LHS: $2^5 = 32$ $32 > 25, \therefore$ true for $n = 5$.
RHS: $5^2 = 25$

Assume true for $n = k \Rightarrow 2^k > k^2$

Consider when $n = k+1$ LHS: $2^{k+1} = 2.2^k$ $\Rightarrow 2.2^k > 2k^2$

$$(k+1)^2 = k^2 + 2k + 1$$
$$= k^2 + 2\left(k + \frac{1}{2}\right)$$

So if true for $n = k+1$ we must show $2k^2 > k^2 + 2\left(k + \frac{1}{2}\right)$

$$k^2 > 2(k + \frac{1}{2}) \quad \forall k \geq 3$$

$\therefore$ when n = k , $2^{k+1} > (k+1)^2$ $\therefore$ Statement is true for $k+1$.
As statement is true for $n = 5$ & true for $k \Rightarrow$ true for $k+1$, $\therefore$ by induction is true $\forall n > 4, n \in \mathbb{N}.$

Past Paper Questions (Excluding Proof by Induction)

2003 – Q8

Given that $p(n) = n^2 + n,$ where n is a positive integer, consider the statements:

A $p(n)$ is always even

B $p(n)$ is always a multiple of 3.

For each statement, prove if it is true or, otherwise, disprove it.

(4 marks)

2006 – Q7

For all natural numbers n, prove whether the following results are true or false.

(a) $n^3 - n$ is always divisible by 6.

(b) $n^3 + n + 5$ is always prime. **(5 marks)**

2008 – Q11

For each of the following statements, decide whether it is true or false and prove your conclusion.

A For all natural numbers $m,$ if m^2 is divisible by 4 then m is divisible by 4.

B The cube of any odd integer p plus the square of any even integer q is always odd.

(5 marks)

2010 – Q8

(a) Prove that the product of two odd integers is odd.

(b) Let p be an odd integer. Use the result of (a) to prove by induction that p^n is odd for all positive integers n.

(2, 4 marks)

2010 – Q12

Prove by contradiction that if x is an irrational number, then $2 + x$ is irrational.

(4 marks)

2013 – Q12

Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A If n is a multiple of 9, then so is n^2.

B If n^2 is a multiple of 9, then so is n.

(4 marks)

2015 – Q12

Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8.

(3 marks)

2016 – Q10

For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A If a positive integer p is prime, then so is $2p+1$.

B If a positive integer n has remainder 1 when divided by 3, then n^3 also has remainder 1 when divided by 3.

(4 marks)

2017 – Q13

Let *n* be an integer.

Using proof by contrapositive, show that if n^2 is even, then *n* is even.

(4 marks)

Past Paper Questions (Proof by Induction only)

2001 – QA4

Prove by induction , that for all integers $n \geq 1$,

$$2+5+8+\ldots+(3n-1)=\frac{1}{2}n(3n+1).$$

(5 marks)

2002 – Q7

Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers n.

(5 marks)

2002 – Q12

A matrix $A=\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. Prove by induction that

$$A^n=\begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix},$$

Where n is any positive integer. **(6 marks)**

2003 – Q16

(a) Prove by induction that for all natural numbers $n \geq 1$

$$\sum_{r=1}^{n} 3(r^2-r)=(n-1)n(n+1).$$

(b) Hence evaluate $\sum_{r=11}^{40} 3(r^2-r)$. **(4, 2 marks)**

2004 – Q12

Prove by induction that $\frac{d^n}{dx^n}(xe^x)=(x+n)e^x$ for all integers $n \geq 1$. **(5 marks)**

2005 – Q10

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

State the value of $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$. **(5, 1 marks)**

2006 – Q13

The square matrices A and B are such that $AB = BA$. Prove by induction that $A^nB = BA^n$ for all integers $n \geq 1$.

(5 marks)

<u>2007 - Q12</u>
Prove by induction that for $a > 0$,

$$(1+a)^n \geq 1+na$$

For all positive integers n. **(5 marks)**

<u>2009 – Q4</u>
Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.$$

(5 marks)

<u>2011 – Q12</u>
Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 **for all integers** $n \geq 2$. **(5 marks)**

<u>2012 – Q16</u>
Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$. **(6 marks)**

<u>2013 – Q9</u>
Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} (4r^3 + 3r^2 + r) = n(n+1)^3$$

(6 marks)

<u>2014 – Q7</u>
Given A is the matrix $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$,
Prove by induction that

$$A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, \; n \geq 1.$$

(4 marks)

<u>2016 – Q5</u>
Prove by induction that

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1). \qquad \forall n \in \mathbb{N}$$

(4 marks)