Properties of Functions

Some Definitions

Number sets –

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$ is the set of natural numbers.

$\mathbb{W} = \{0, 1, 2, 3, 4, \dots\}$ is the set of whole numbers.

$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, , \dots\}$ is the set of integers.

$\mathbb{Q}$ = the set of rational numbers (all numbers which can be written as fractions).

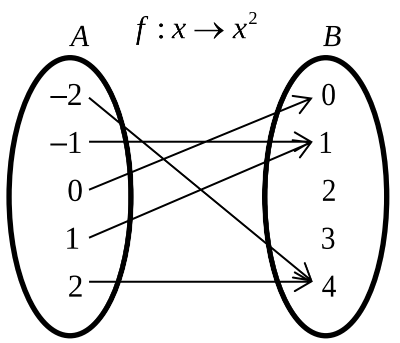
$\mathbb{R}$ = the set of real numbers (all rational and irrational numbers).

These sets have subsets e.g. $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$.

A **function** is a rule which maps each element in a set A to an element in set B.

Set A is called the **domain** and set B is called the **codomain**.

The subset of B which are the solutions of the function is called the **range**.



Domain $= \{-2, -1, 0, 1, 2\}$

Codomain $= \{0, 1, 2, 3, 4\}$

Range $= \{0, 1, 4\}$

You can see that it is possible for $f(a) = f(b)$ and $a \neq b$.

Range and domain of functions

① $f(x) = x^2$ — Domain $x \in \mathbb{R}$ — Range $f(x) \geq 0$

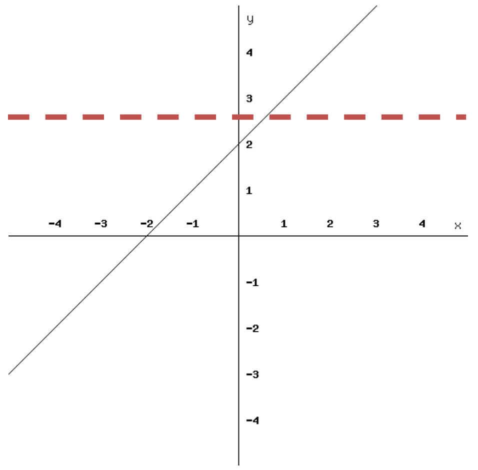
② $f(x) = \cos x$ — Domain $x \in \mathbb{R}$ — Range $-1 \leq f(x) \leq 1$ or $f(x) \in [-1, 1]$

③ $f(x) = \sqrt{x - 2}$ — Domain $x \geq 2$ — Range $f(x) \geq 0$

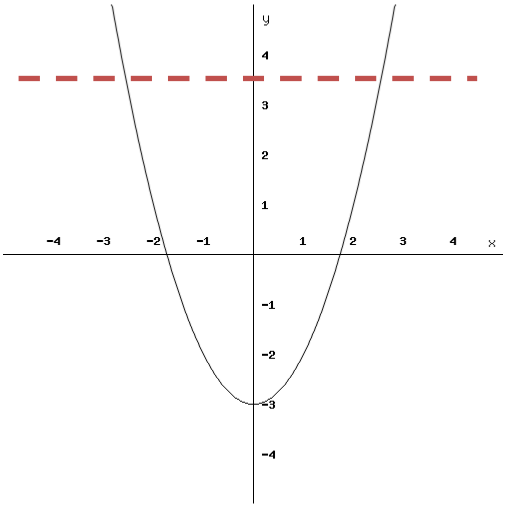
④ $f(x) = \dfrac{1}{\sin x}$ Domain $x \in \mathbb{R} - k\pi$, Range $f(x) \geq 1$ *or* $f(x) \leq -1$

One to One Functions and Inverse Functions

A function is described as one to one when $f(a) = f(b) \Leftrightarrow a = b$.



This function is one to one as only one value of x will give each value of $f(x)$.



This function is not one to one as more than one value of x will result in a single value of $f(x)$.

As each output from a one to one function comes from only one input we can "reverse" the function. The function which performs this reversal is known as the inverse function.

The inverse of $f(x)$ is denoted by $f^{-1}(x)$.

To find the inverse function, we write $f(x) = y$ and change the subject to x.

We then replace x by $f^{-1}(x)$ and y by x in the solution.

Examples

① $f(x) = 2x+1$

$y = 2x+1$

$y-1 = 2x$

$x = \dfrac{y-1}{2}$

$f^{-1}(x) = \dfrac{x-1}{2}$

② $f(x) = 8x^3$

$y = 8x^3$

$x = \sqrt[3]{\dfrac{y}{8}}$

$f^{-1}(x) = \dfrac{\sqrt[3]{x}}{2}$

③ $f(x) = \dfrac{1}{x+1}$

$y = \dfrac{1}{x+1}$

$x+1 = \dfrac{1}{y}$

$x = \dfrac{1}{y} - 1$

$f^{-1}(x) = \dfrac{1}{x} - 1$

Questions

Find the inverse of each of these functions.

① $f(x) = 4 - 2x$ ② $f(x) = 4x^3 - 9$ ③ $f(x) = \dfrac{3x}{x+2}$

Restricting the Domain

With a function which is not one to one we can **restrict the domain** to make it one to one and then find the inverse. Sometimes completing the square can help.

Examples

① $f(x) = x^2 - 4, \quad x \geq 0$

This is only a one to one function with the domain restricted to $x \geq 0$.

$$y = x^2 - 4$$
$$x^2 = y + 4$$
$$x = \sqrt{y+4}$$
$$f^{-1}(x) = \sqrt{x+4}, \quad x \geq -4$$

② $f(x) = x^2 + 6x - 1, \quad x \geq -3$

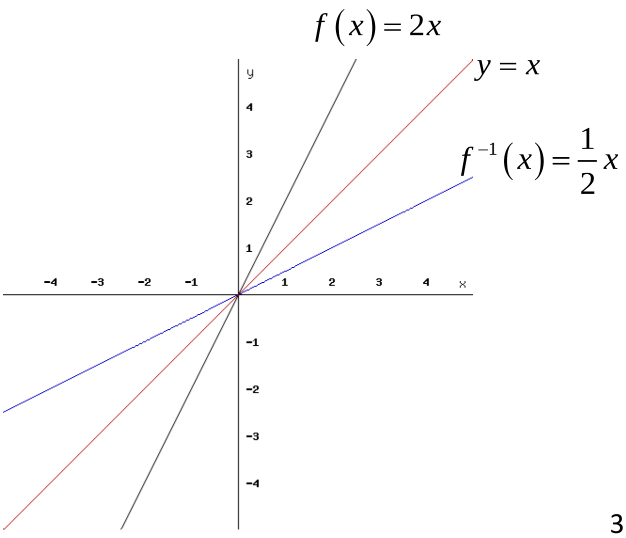
This is only a one to one function with the domain restricted to $x \geq -3$.

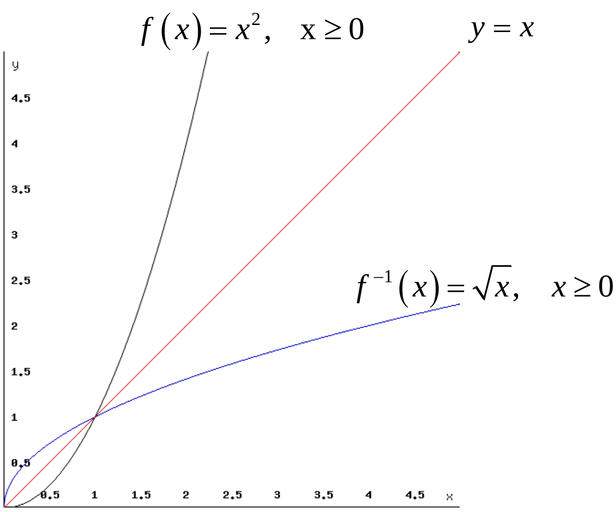
Complete the square to get a single x term.

$$y = x^2 + 6x - 1$$
$$y = (x+3)^2 - 10$$
$$y + 10 = (x+3)^2$$
$$x = \sqrt{y+10} - 3$$
$$f^{-1}(x) = \sqrt{x+10} - 3, \quad x \geq -10$$

The domain of $f(x)$ is the range of $f^{-1}(x)$ and the range of $f(x)$ is the domain of $f^{-1}(x)$.

Remember from Higher, the graph of $f^{-1}(x)$ can be obtained by reflecting the graph of $f(x)$ in the line $y = x$.





The Modulus Function

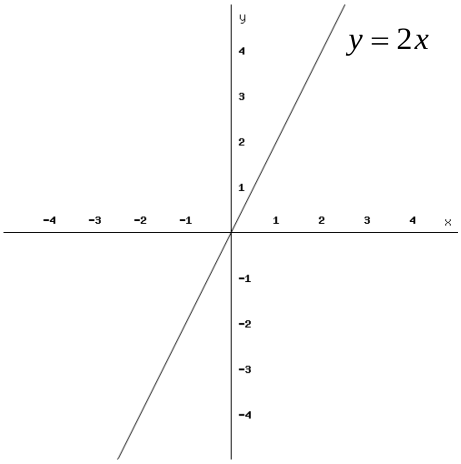
The modulus of x is sometimes known as the absolute value. It is denoted by $|x|$.

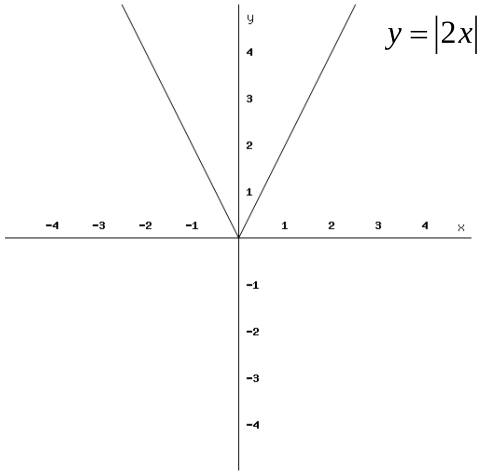
$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

To draw the graph of the modulus function, reflect any part below the x-axis in the x axis so that the whole graph lies above or on the x-axis.

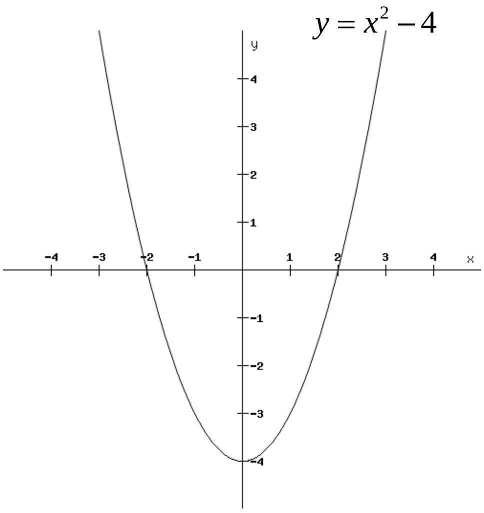
Examples

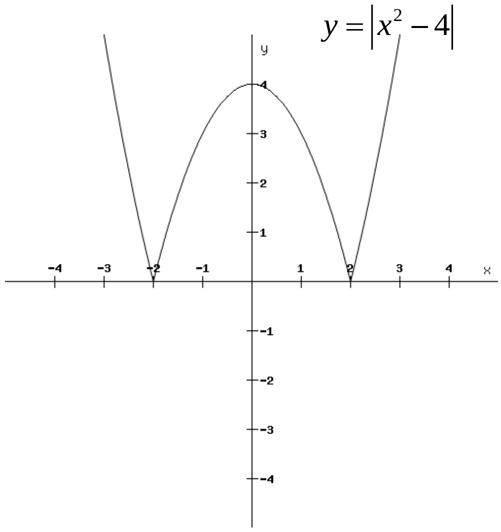
①





②





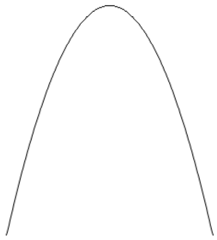
Concavity

The concavity of a graph relates to whether $f'(x)$ is increasing or decreasing.

Here $f(x)$ is concave up.

$f'(x)$ is increasing.

$f''(x) > 0$

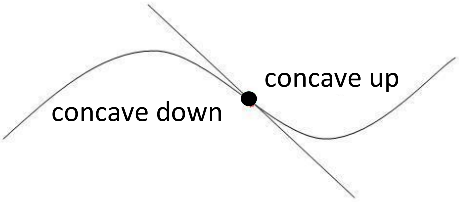


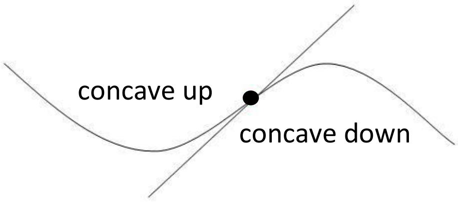
Here $f(x)$ is concave down.

$f'(x)$ is decreasing.

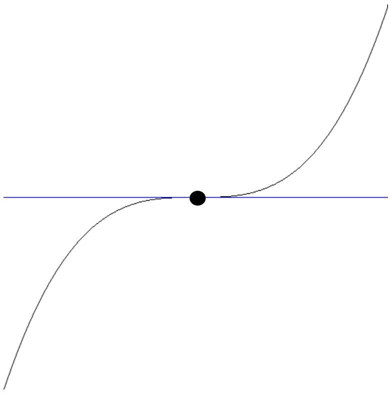
$f''(x) < 0$

We can use this to determine the nature of stationary points instead of using a nature table.
A point where concavity changes is known as a point of inflexion.

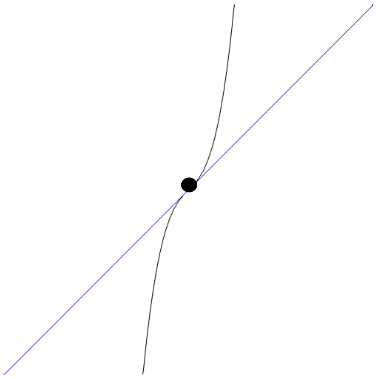




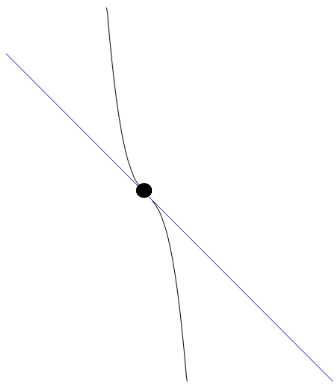
In Higher, horizontal points of inflexion were considered.
At these points $f'(x) = 0$ so horizontal points of inflexion are always stationary points.



This is not always the case however.



Here there is a point of inflexion as the graph changes from concave down to concave up.
$f'(x) > 0$ at all points so there are no stationary points.



Here there is a point of inflexion as the graph changes from concave up to concave down.
$f'(x)<0$ at all points so there are no stationary points.

At a point of inflexion $f''(x)=0$ **or** concavity changes at that point.

Examples

① $f(x)=x^3+x$.

$f'(x)=3x^2+1$ $\quad$ $f'(x)>0$ for all values of x so there are no stationary points.

$f''(x)=6x$ $\quad$ $f''(x)=0$ at $x=0$ so there could be a point of inflexion here.

We need to check if the concavity changes at this point to see if it is an inflexion point.

x	<0	0	>0
$f''(x)$	$-$	0	$+$
$f(x)$	concave down	0	concave up

At $x=0$, $f''(x)=0$ and concavity changes so $(0,0)$ is a point of inflexion.

② $f(x)=x^4$

$f'(x)=4x^3$ $\quad$ $f'(x)=0$ at $x=0$ so there is a stationary point here.

$f''(x)=12x^2$ $\quad$ $f''(x)=0$ at $x=0$ so there could be a point of inflexion here.

We need to check if the concavity changes at this point to see if it is an inflexion point.

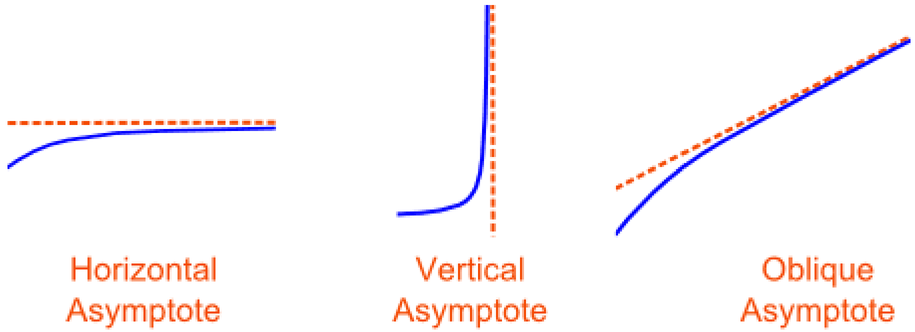
x	<0	0	>0
$f''(x)$	$+$	0	$+$
$f(x)$	concave up	0	concave up

At $x=0$, $f''(x)=0$ but concavity does not change so $(0,0)$ is not a point of inflexion.
$(0,0)$ is a minimum turning point here.

Asymptotes

Asymptotes are “invisible” lines on a graph which a curve approaches as it heads towards infinity.

There are 3 types of asymptotes – horizontal, vertical and oblique.



Example

Consider $y = \dfrac{1}{x}, \quad x \neq 0$.

The function is defined for all $x \in \mathbb{R}$ except $x = 0$. This means that $x = 0$ (the y-axis) is a vertical asymptote.

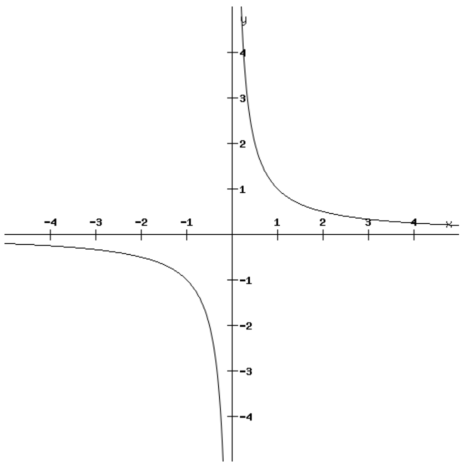
As $x \to 0^+$, $\dfrac{1}{x} \to +\infty$. As $x \to 0^-$, $\dfrac{1}{x} \to -\infty$

As $x \to \infty$, $\dfrac{1}{x} \to 0$ and $\dfrac{1}{x}$ is +ve.

As $x \to -\infty$, $\dfrac{1}{x} \to 0$ and $\dfrac{1}{x}$ is –ve.

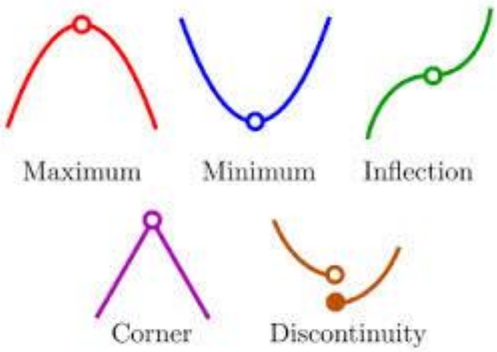
This means that $y = 0$ (the x-axis) is a horizontal asymptote.

Sketch of graph:-



Critical Points

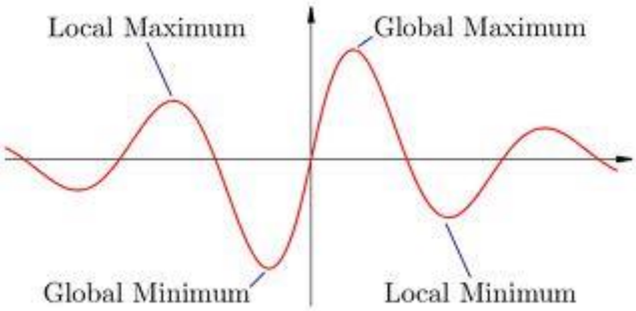
Critical points of a function are points where $f'(x)=0$ or where $f'(x)$ is undefined. These are maximum and minimum turning points, horizontal points of inflexion, points where the function is discontinuous or points where the graph of the function is a "corner".



Local and Global Maximum and Minimum Turning Points

A **local maximum** turning point occurs where a function has a greater value at that point than at any points close to it but it is not necessarily the greatest value of the function.
A **local minimum** turning point occurs where a function has a lesser value at that point than at any points close to it but it is not necessarily the least value of the function.

A **global maximum** turning point occurs where a function has the greatest value over a defined domain.
A **global minimum** turning point occurs where a function has the least value over a defined domain.



End-point Maximum and Minimum

An **endpoint maximum** is a point at the end of a given domain where a function has a greater value at that point than at any points close to it but it is not necessarily the greatest value of the function.
An **endpoint minimum** is a point at the end of a given domain where a function has a lesser value at that point than at any points close to it but it is not necessarily the least value of the function.

Curve Sketching

When sketching the graph of a function we must find

- the equations of the asymptotes
- the points where the graph crosses the x and y axes
- any stationary points and their nature using the second derivative test for concavity
- any points of inflexion
- the shape of the graph- where it is concave up and concave down using the second derivative
- how the graph behaves as it approaches the asymptotes
- how the graph behaves as $x \to \pm\infty$.

Examples

① Sketch the graph of $y = \dfrac{1}{x-1}$, $x \neq 1$

- Function is undefined at $x = 1$ so $x = 1$ is a vertical asymptote.
- Graph crosses the x-axis when $y = 0$

  $\dfrac{1}{x-1} = 0$ This has no solution so the graph does not cross the x-axis.

  The x-axis is a horizontal asymptote.
- Graph crosses the y-axis when $x = 0$, $y = -1$.
- Stationary points occur when $\dfrac{dy}{dx} = 0$

  $\dfrac{dy}{dx} = \dfrac{-1}{(x-1)^2}$ so $\dfrac{dy}{dx} < 0 \ \forall x \in \mathbb{R}$, $x \neq 1$ - so the graph has no stationary points.
- Points of inflexion/concavity

  $\dfrac{d^2y}{dx^2} = \dfrac{2}{(x-1)^3}$

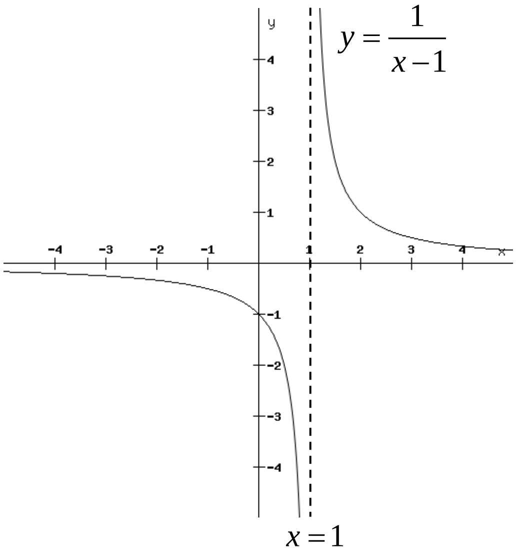
  $\dfrac{d^2y}{dx^2} = 0$ for points of inflexion. $\dfrac{2}{(x-1)^3} = 0$ has no solutions so there are no points of inflexion.

  $\dfrac{d^2y}{dx^2} > 0 \ \forall\, x > 1$ so concave up for $x > 1$

  $\dfrac{d^2y}{dx^2} < 0 \ \forall\, x < 1$ so concave down for $x < 1$

- As $x \to \infty, y \to 0$ and y is +ve. As $x \to -\infty, y \to 0$ and y is –ve

Sketch:



This graph is the translation of the graph of $y = \dfrac{1}{x}$ by 1 place to the right which should be recalled from Higher.

② Sketch the graph of $y = \dfrac{-12}{2x+4}$, $x \neq -2$

- The function is undefined at $x = -2$ so $x = -2$ is a vertical asymptote.
- Graph crosses the x-axis when $y = 0$

  $\dfrac{-12}{2x+4} = 0$ This has no solution so the graph does not cross the x-axis.

  The x-axis is a horizontal asymptote.
- Graph crosses the y-axis when $x = 0$, $y = -3$.
- Stationary points occur when $\dfrac{dy}{dx} = 0$

  $\dfrac{dy}{dx} = \dfrac{24}{(2x+4)^2}$ so $\dfrac{dy}{dx} > 0$ $\forall x \in \mathbb{R}$, $x \neq -2$ - so the graph has no stationary points.
- Points of inflexion/concavity

  $$\frac{d^2y}{dx^2} = \frac{-96}{(2x+4)^3}$$

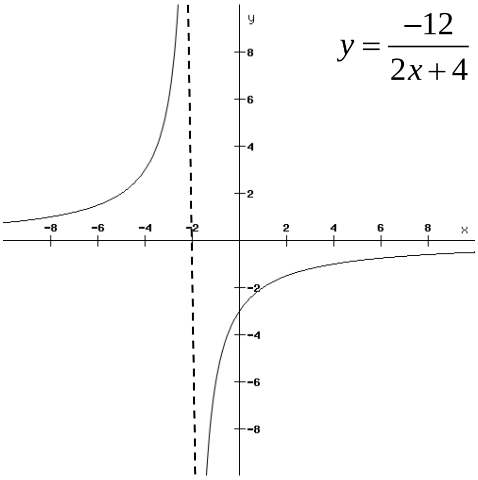
  $\dfrac{d^2y}{dx^2} = 0$ for points of inflexion. $\dfrac{-96}{(2x+4)^3} = 0$ has no solutions so there are no points of inflexion.

  $\dfrac{d^2y}{dx^2} > 0$ when $2x+4 < 0$, $x < -2$ so concave up for $x < -2$.

  $\dfrac{d^2y}{dx^2} < 0$ when $2x+4 > 0$, $x > -2$ so concave down for $x > -2$.

- As $x \to \infty, y \to 0$ and y is -ve. As $x \to -\infty, y \to 0$ and y is +ve

Sketch:



③ Sketch the graph of $y = \frac{-8}{x^2-4}$

- $y = \frac{-8}{(x-2)(x+2)}$ so the function is undefined at $x = 2$ and $x = -2$.

  $x = 2$ and $x = -2$ are vertical asymptotes.
- Graph crosses the x-axis when $y = 0$

  $\frac{-8}{x^2-4} = 0$ This has no solution so the graph does not cross the x-axis.

  The x-axis is a horizontal asymptote.
- Graph crosses the y-axis when $x = 0$, $y = 2$.
- Stationary points occur when $\frac{dy}{dx} = 0$

  $\frac{dy}{dx} = 8\left(x^2-4\right)^{-2}(2x)$

  $\frac{dy}{dx} = \frac{16x}{\left(x^2-4\right)^2}$

  $\frac{16x}{\left(x^2-4\right)^2} = 0$

  $x = 0$, $y = 2$ so there is a stationary point at $(0,2)$.

  $$\frac{d^2y}{dx^2} = \frac{16\left(x^2-4\right)^2 - 16x\left(2\left(x^2-4\right)2x\right)}{\left(x^2-4\right)^4} = \frac{16\left(x^2-4\right) - 64x^2}{\left(x^2-4\right)^3} = \frac{-48x^2-64}{\left(x^2-4\right)^3}$$

  At $x = 0$, $\frac{d^2y}{dx^2} = \frac{-64}{-64} = 1$ so concave up. $(0,2)$ is a local minimum.

- Points of inflexion/concavity

  $\frac{d^2y}{dx^2}=0$ for points of inflexion.

  $$\frac{-48x^2-64}{\left(x^2-4\right)^3}=0$$

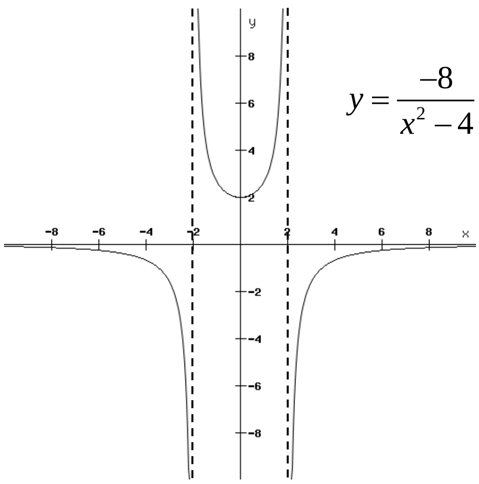
  $48x^2=-64$ - no solutions so there are no points of inflexion.

  $\frac{d^2y}{dx^2}>0$ when $x^2-4<0$, $-2<x<2$ so concave up for $-2<x<2$.

  $\frac{d^2y}{dx^2}<0$ when $x^2-4>0$, $x<-2$ and $x>2$ so concave down for $x<-2$ and $x>2$.

- As $x\to\infty, y\to 0$ and y is -ve. As $x\to-\infty, y\to 0$ and y is –ve.

Sketch:



$$y=\frac{-8}{x^2-4}$$

$x=-2 \quad x=2$

④ Sketch the graph of $f\left(x\right)=\frac{1}{x\left(2x+1\right)} \qquad x\neq 0, \quad x\neq\frac{-1}{2}$

- The function is undefined at $x=0$ and $x=\frac{-1}{2}$ so $x=0$ and $x=\frac{-1}{2}$ are vertical asymptotes.
- Graph crosses the x-axis when $y=0$

  $\frac{1}{x\left(2x+1\right)}=0$ This has no solution so the graph does not cross the x-axis.

  The x-axis is a horizontal asymptote.
- Graph crosses the y-axis when $x=0$. Undefined at $x=0$, does not cross y-axis.

  y-axis is a vertical asymptote.

- Stationary points occur when $f'(x)=0$

$$f'(x)=\frac{-(4x+1)}{x^2(2x+1)^2}=\frac{-(4x+1)}{(2x^2+x)^2}$$

$$\frac{-(4x+1)}{(2x^2+x)^2}=0$$

$x=\frac{-1}{4},\ y=-8$ so $\left(\frac{-1}{4},-8\right)$ is a stationary point.

$$f''(x)=\frac{-4(2x^2+x)^2+(4x+1)2(2x^2+x)(4x+1)}{(2x^2+x)^4}$$

$$f''(x)=\frac{-4(2x^2+x)+2(4x+1)^2}{(2x^2+x)^3}$$

$$f''(x)=\frac{-8x^2-4x+32x^2+16x+2}{(2x^2+x)^3}$$

$$f''(x)=\frac{24x^2+12x+2}{(2x^2+x)^3}=\frac{2(12x^2+6x+1)}{(2x^2+x)^3}$$

$$f''\left(\tfrac{-1}{4}\right)=\frac{2\left(12\left(\frac{-1}{4}\right)^2+6\left(\frac{-1}{4}\right)+1\right)}{\left(2\left(\frac{-1}{4}\right)^2+\left(\frac{-1}{4}\right)\right)^3}=-256$$

$f''\left(\frac{-1}{4}\right)<0$ so $\left(\frac{-1}{4},-8\right)$ is a local maximum.

- Points of inflexion/concavity

$f''(x)=0$ for points of inflexion.

$12x^2+6x+1=0$

$b^2-4ac=6^2-4\times12\times1=-12$

No solutions so no points of inflexion.

$f''(x)>0$ when $(2x^2+x)^3>0$

$$2x^2+x>0$$
$$x(2x+1)>0$$
$$x<\frac{-1}{2},\quad x>0$$

concave up

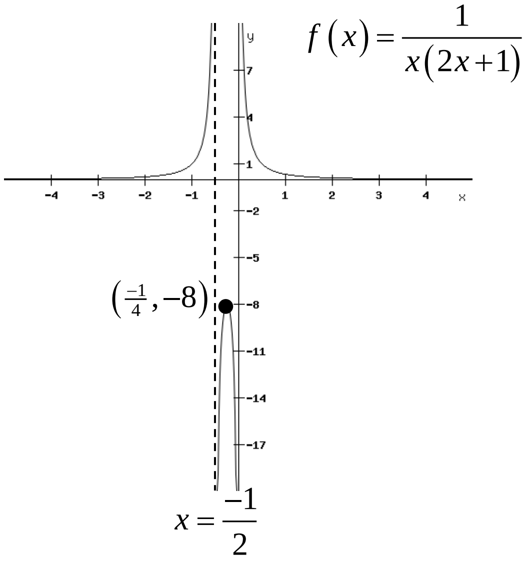
$f''(x)<0$ when $(2x^2+x)^3<0$

$$2x^2+x<0$$
$$x(2x+1)<0$$
$$\frac{-1}{2}<x<0$$

concave down

- As $x\to\infty$, $f(x)\to0$ and $f(x)$ is +ve. As $x\to-\infty$, $f(x)\to0$ and $f(x)$ is +ve.

Sketch:



Questions

Sketch the graphs of the following curves showing the asymptotes, stationary points, points of inflexion and points where the graphs cross the axes.

① $y=\dfrac{1}{(x+2)}$ ② $y=\dfrac{4}{2x-1}$ ③ $y=\dfrac{1}{(x-2)(x+4)}$ ④ $y=\dfrac{5}{x^2+3x-4}$

Oblique and Horizontal Asymptotes

Oblique and horizontal asymptotes can be identified in rational functions where the numerator is a polynomial of a degree greater or equal to the degree of the denominator. If the degrees of the numerator and denominator are equal we have a horizontal asymptote. If the degree of the numerator is greater than the degree of the denominator, we have an oblique asymptote.

To obtain the equation of the oblique or horizontal asymptote we must perform algebraic long division. The resulting quotient is the equation of the oblique or horizontal asymptote.

Examples

① Draw the graph of the function $f(x)=\dfrac{2x^3-20x^2+64x-37}{(x-4)^2}, \quad x\neq 4$

showing where it crosses the y-axis, any stationary points and the equations of the asymptotes.

Degree of numerator > degree of denominator so perform algebraic long division first.

$$
\begin{array}{r}
2x \;-\; 4 \\
x^2-8x+16 \overline{\smash{\big)}\, 2x^3-20x^2+64x-37} \\
\underline{2x^3-16x^2+32x \qquad} \\
-4x^2+32x-37 \\
\underline{-4x^2+32x-64} \\
27
\end{array}
$$

so $f\left(x\right)=2x-4+\dfrac{27}{\left(x-4\right)^2}$

- The equation of the oblique asymptote is $y=2x-4$.
- $f\left(x\right)$ is undefined at $x=4$ so the equation of the vertical asymptote is $x=4$.
- Crosses y-axis when $x=0$, $y=\dfrac{-37}{16}$

- Stationary points occur when $f'\left(x\right)=0$

  $f'\left(x\right)=2-\dfrac{54}{\left(x-4\right)^3}$

  $2-\dfrac{54}{\left(x-4\right)^3}=0$

  $\left(x-4\right)^3=27$

  $x-4=3$

  $x=7$

  $y=13$ $\qquad$ $\left(7,13\right)$ is a stationary point.

  $f''\left(x\right)=\dfrac{162}{\left(x-4\right)^4}$

  $f''\left(7\right)=\dfrac{162}{\left(7-4\right)^4}=2$ $\qquad$ $f''\left(x\right)>0$ so $\left(7,13\right)$ is a local minimum.

- Points of inflexion/concavity

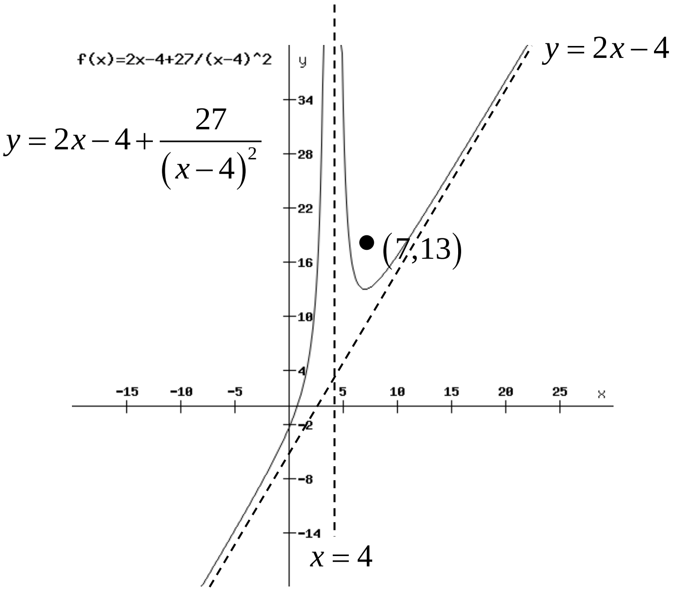
  $f''\left(x\right)=0$ for points of inflexion.

  $\dfrac{162}{\left(x-4\right)^4}=0$ - no solutions so there are no points of inflexion.

  $f''\left(x\right)>0 \;\; \forall\, x\in\mathbb{R}$ so the graph is concave up

- As $x\to\infty, f\left(x\right)\to 2x-4$. As $x\to-\infty, f\left(x\right)\to 2x-4$

Sketch:



② Sketch the graph of $y = \frac{x^2 + 3x + 5}{x + 4}$, $x \neq -4$

Degree of numerator> degree of denominator so perform algebraic long division first.

$$\begin{array}{r|l} & x - 1 \\ \hline x + 4 & x^2 + 3x + 5 \\ & \underline{x^2 + 4x} \\ & \quad -x + 5 \\ & \quad \underline{-x - 4} \\ & \qquad 9 \end{array}$$

$$y = x - 1 + \frac{9}{x + 4}, \quad x \neq -4$$

- The equation of the oblique asymptote is $y = x - 1$.
- $f(x)$ is undefined at $x = -4$ so the equation of the vertical asymptote is $x = -4$.
- Crosses y-axis when $x = 0$, $y = \frac{5}{4}$
- Crosses x-axis when $y = 0$

$$\frac{x^2 + 3x + 5}{x + 4} = 0$$

$$x^2 + 3x + 5 = 0$$

$b^2 - 4ac = 3^2 - 4 \times 1 \times 5 = -11$ No solutions so does not cross the x-axis.

- Stationary points occur when $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 1 - \frac{9}{(x+4)^2}$$

$$1-\frac{9}{(x+4)^2}=0$$

$$(x+4)^2=9$$

$x=-1$ or $x=-7$

$y=1$ $\qquad y=-11$

Stationary points at $(-1,1)$ and $(-7,-11)$

$$\frac{d^2y}{dx^2}=\frac{18}{(x+4)^3}$$

At $(-1,1)$, $\dfrac{d^2y}{dx^2}=\dfrac{18}{(-1+4)^3}=\dfrac{2}{3}$, $\quad \dfrac{d^2y}{dx^2}>0$ so $(-1,1)$ is a local minimum.

At $(-7,-11)$, $\dfrac{d^2y}{dx^2}=\dfrac{18}{(-7+4)^3}=\dfrac{-2}{3}$, $\quad \dfrac{d^2y}{dx^2}<0$ so $(-7,-11)$ is a local maximum.

- Points of inflexion/concavity

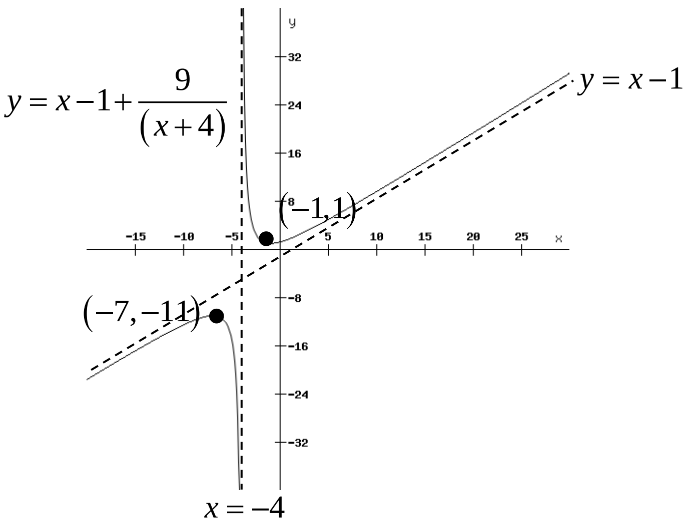
  $\dfrac{d^2y}{dx^2}=0$ for points of inflexion.

  $\dfrac{18}{(x+4)^3}=0$ - no solutions so no points of inflexion.

  $\dfrac{d^2y}{dx^2}>0$ when $x+4>0$, $x>-4$ so concave up for $x>-4$.

  $\dfrac{d^2y}{dx^2}<0$ when $x+4<0$, $x<-4$ so concave down for $x<-4$
- As $x\to\infty, y\to x-1$. As $x\to-\infty, y\to x-1$.

Sketch:



③ Sketch the graph of $y=\dfrac{3x^2-4x+2}{(x-2)^2}$, $x\neq 2$

Degree of numerator = degree of denominator so perform algebraic long division first.

$$\begin{array}{r|l} & \quad\quad\quad 3 \\ x^2-4x+4 & 3x^2-4x+2 \\ & \underline{3x^2-12x+12} \\ & \quad\quad 8x-10 \end{array}$$

So $y=3+\dfrac{8x-10}{(x-2)^2}$, $x\neq 2$

- The equation of the horizontal asymptote is $y=3$.
- y is undefined at $x=2$ so the equation of the vertical asymptote is $x=2$.
- Crosses y-axis when $x=0$, $y=\dfrac{1}{2}$.
- Crosses x-axis when $y=0$

  $3x^2-4x+2=0$

  $(-4)^2-4(3)(2)<0$ so no solutions. Does not cross the x-axis.
- Stationary points occur when $\dfrac{dy}{dx}=0$

  $$\frac{dy}{dx}=\frac{8(x-2)^2-(8x-10).2(x-2)}{(x-2)^4}=\frac{-8x+4}{(x-2)^3}$$

  $$\frac{-8x+4}{(x-2)^3}=0$$

  $$x=\frac{1}{2}$$

  A stationary point occurs at $\left(\dfrac{1}{2},\dfrac{1}{3}\right)$.

  $$\frac{d^2y}{dx^2}=\frac{-8(x-2)^3-(-8x+4)3(x-2)^2}{(x-2)^6}$$

  $$\frac{d^2y}{dx^2}=\frac{16x+4}{(x-2)^4}$$

  At $x=\dfrac{1}{2}$, $\dfrac{d^2y}{dx^2}>0$ so $\left(\dfrac{1}{2},\dfrac{1}{3}\right)$ is a local minimum.
- Points of inflexion/concavity

$\frac{d^2y}{dx^2}=0$ for points of inflexion.

$\frac{16x+4}{(x-2)^4}=0$

$x=\frac{-1}{4}$ There could be an inflexion point at $x=\frac{-1}{4}$. Check concavity.

x	$<\frac{-1}{4}$	$\frac{-1}{4}$	$>\frac{-1}{4}$
$\frac{d^2y}{dx^2}$	$-$	0	$+$
y	concave down		concave up

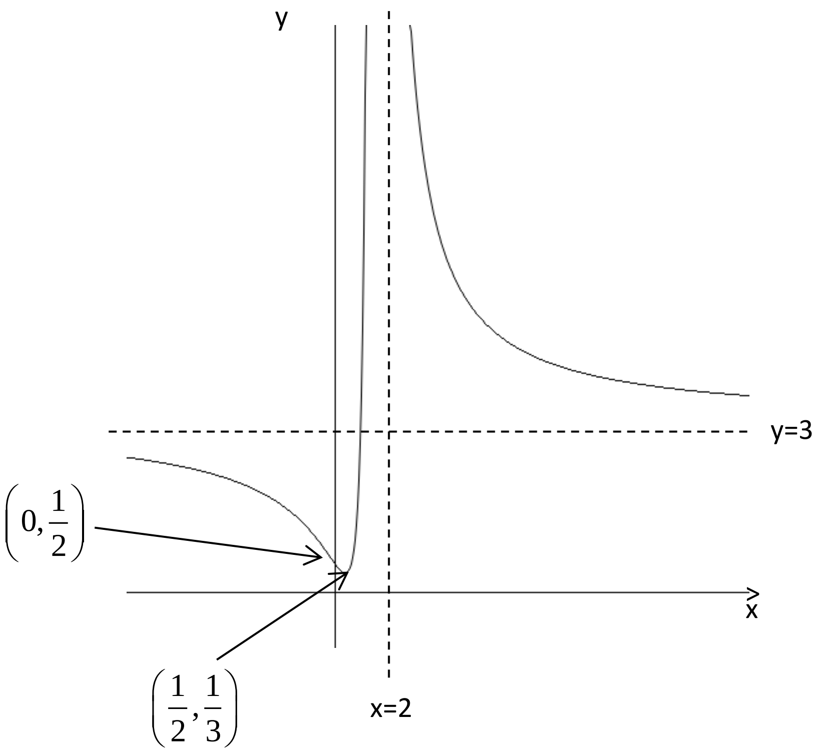
Concavity changes so there is an inflexion point at $\left(\frac{-1}{4},\frac{17}{27}\right)$.

$\frac{d^2y}{dx^2}>0$ when $16x+4>0$, $x>\frac{-1}{4}$ so concave up for $x>\frac{-1}{4}$

$\frac{d^2y}{dx^2}<0$ when $16x+4<0$, $x<\frac{-1}{4}$ so concave down for $x<\frac{-1}{4}$

- As $x\to\infty, y\to 3$. As $x\to-\infty, y\to 3$.

Sketch:



Questions

Sketch the graphs of the following curves showing the asymptotes, stationary points, points of inflexion and points where the graphs cross the axes.

① $y=\dfrac{x^2-3x-10}{x-2}$ ② $y=\dfrac{(x-1)(x+2)}{x-2}$ ③ $y=\dfrac{x^2+2x+1}{(x+2)}$ ④ $y=\dfrac{(x+2)^2}{(x-3)}$

Odd and Even Functions

An **even** function is a function for which $f(-x)=f(x)$.
The graph of an even function has line symmetry about the y-axis.
For example $f(x)=\cos x$ is an even function as $f(-x)=\cos(-x)=\cos x=f(x)$.
The graph of $y=\cos x$ is symmetrical about the y-axis.

An **odd** function is a function for which $f(-x)=-f(x)$.
The graph of an odd function has half turn symmetry about the origin.
For example $f(x)=\sin x$ is an odd function as $f(-x)=\sin(-x)=-\sin x=-f(x)$.
The graph of $y=\sin x$ has half turn symmetry about the origin.

Examples

Determine whether the following functions are odd, even or neither.

① $f(x)=x+x^5$

$$\begin{aligned} f(-x)&=-x+(-x)^5\\ &=-x-x^5\\ &=-(x+x^5)\\ &=-f(x) \end{aligned}$$

function is odd

② $f(x)=x^2+x$

$$\begin{aligned} f(-x)&=(-x)^2+(-x)\\ &=x^2-x \end{aligned}$$

neither odd nor even

③ $f(x)=x^4-1$

$$\begin{aligned} f(-x)&=(-x)^4-1\\ &=x^4-1\\ &=f(x) \end{aligned}$$

function is even

Questions

Determine whether the following functions are odd, even or neither.

① $f(x) = x^3 + x$ ② $f(x) = \frac{x^2+1}{x^2}$ ③ $f(x) = x + \frac{1}{x}$ ④ $f(x) = x^3 + x^2$

⑤ $f(x) = e^{x^2}$ ⑥ $f(x) = e^x + e^{-x}$ ⑦ $f(x) = e^x - e^{-x}$ ⑧ $f(x) = \cos x \sin x$

Past Paper Questions

2001

A function is defined by $f(x) = \frac{x^2+6x+12}{x+2}$, $x \neq -2$.

(a) Express $f(x)$ in the form $ax + b + \frac{b}{x+2}$ stating the values of a and b.

(b) Write down an equation for each of the two asymptotes.

(c) Show that $f(x)$ has two stationary points.

Determine the coordinates and the nature of the stationary points.

(d) Sketch the graph of f.

(e) State the range of values of k such that the equation $f(x) = k$ has no solution.

(2, 2, 4, 1, 1 marks)

2002

Express $\frac{x^2}{(x+1)^2}$ in the form $A + \frac{B}{x+1} + \frac{C}{(x+1)^2}$, $(x \neq -1)$, stating the values of the constants A, B and C.

A curve is defined by $y = \frac{x^2}{(x+1)^2}, (x \neq -1)$.

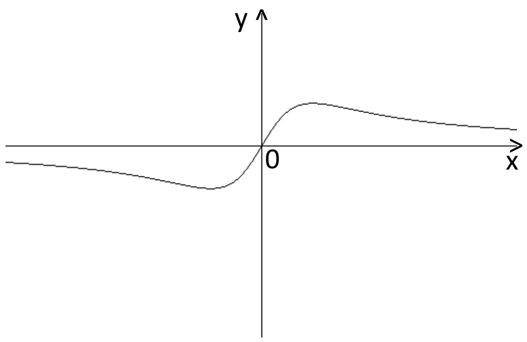
(i) Write down equations for its asymptotes.

(ii) Find the stationary point and justify its nature.

(iii) Sketch the curve showing clearly the features found in (i) and (ii).

(3, 2, 4, 2 marks)

2003



The diagram shows the shape of $y = \frac{x}{1+x^2}$.

Obtain the stationary points of the graph.

Sketch the graph of $y = \left|\frac{x}{1+x^2}\right|$ and identify its three critical points.

(4, 3 marks)

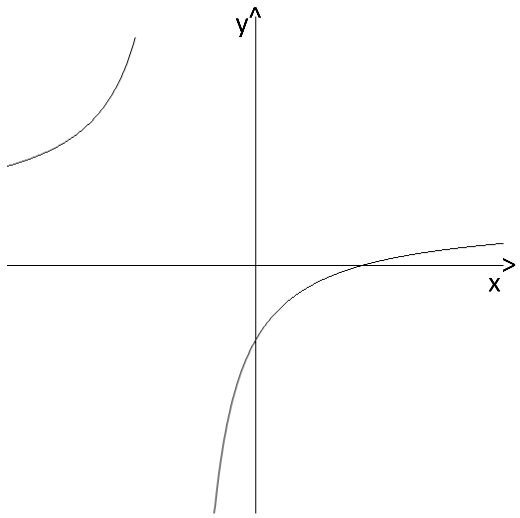
2004

Determine whether the function $f(x) = x^4 \sin 2x$ is odd, even or neither.

Justify your answer. **(3 marks)**

2004

The function f is defined by $f(x) = \frac{x-3}{x+2}$, $x \neq -2$, and the diagram shows part of its graph.



(a) Obtain algebraically the asymptotes of the graph of f.

(b) Prove that f has no stationary values.

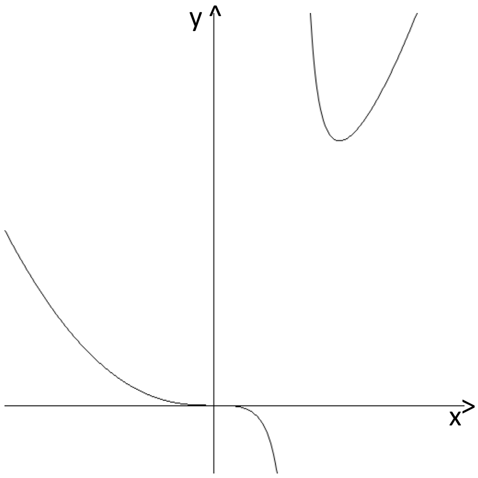
(c) Does the graph of f have any points of inflexion? Justify your answer.

(d) Sketch the graph of the inverse function f^{-1}.

State the asymptotes and domain of f^{-1}. **(3, 2, 2, 3 marks)**

2005

The diagram shows part of the graph of $y = \frac{x^3}{x-2}$, $x \neq 2$.



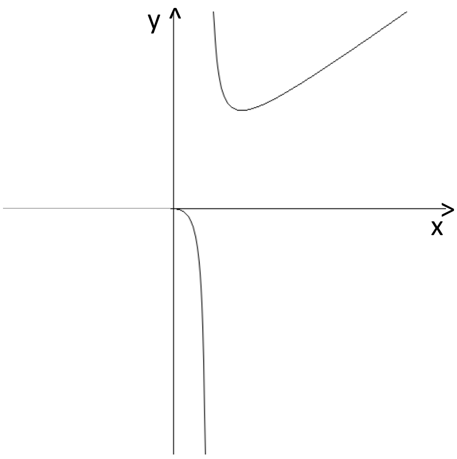
(a) Write down the equation of the vertical asymptote.

(b) Find the coordinates of the stationary points of the graph of $y = \frac{x^3}{x-2}$.

(c) Write down the coordinates of the stationary points of the graph of $y = \left|\frac{x^3}{x-2}\right| + 1$.

(1, 4, 2 marks)

2006



The diagram shows part of the graph of a function f which satisfies the following conditions:

(i) f is an even function;

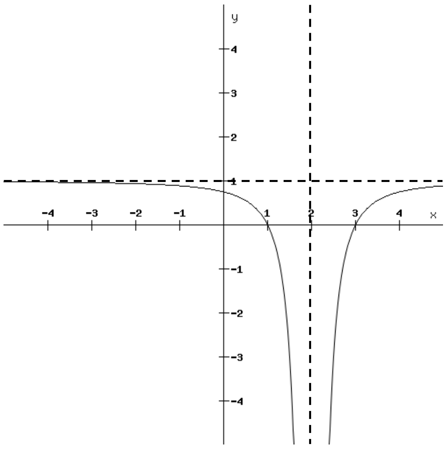
(ii) two of the asymptotes of the graph $y = f(x)$ are $y = x$ and $x = 1$.

Copy the diagram and complete the graph. Write down equations for the other two asymptotes. **(3 marks)**

2008

Part of the graph $y = f(x)$ is shown below, where the dotted lines indicate asymptotes.

Sketch the graph of $y = -f(x+1)$ showing its asymptotes. Write down the equations of the asymptotes.



(4 marks)

2009

The function $f(x)$ is defined by $f(x) = \dfrac{x^2 + 2x}{x^2 - 1}$ $(x \neq \pm 1)$.

Obtain equations for the asymptotes of the graph of $f(x)$.

Show that $f(x)$ is a strictly decreasing function.

Find the coordinates of the points where the graph of $f(x)$ crosses

(i) the x-axis and

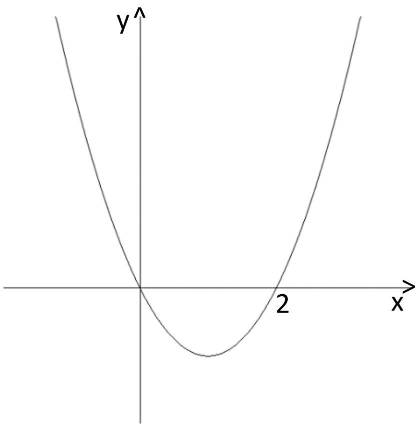
(ii) the horizontal asymptote.

Sketch the graph of $f(x)$, showing clearly all relevant features. **(3, 3, 2, 2 marks)**

2010

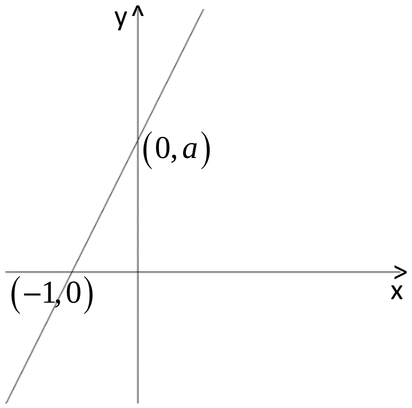
The diagram below shows part of the graph of a function $f(x)$.

State whether $f(x)$ is odd, even or neither. Fully justify your answer.



(3 marks)

2011



The diagram shows part of the graph of a function $f(x)$.

Sketch the graph of $\left|f^{-1}(x)\right|$ showing the points of intersection with the axes. **(4 marks)**

2012

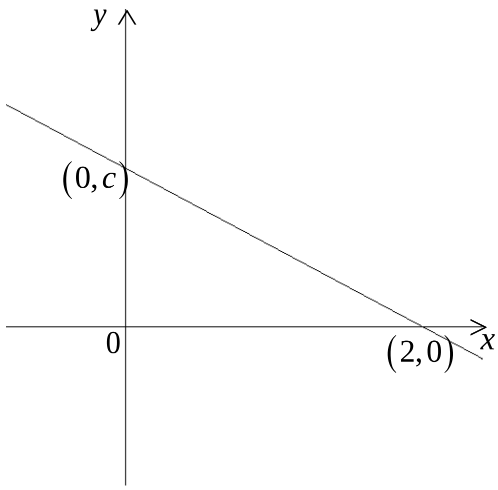
A function is defined by $f(x)=|x+2|$ for all x.

(a) Sketch the graph of the function for $-3 \leq x \leq 3$.

(b) On a separate diagram, sketch the graph of $f'(x)$. **(2, 2 marks)**

2013

Part of the straight line graph of a function $f(x)$ is shown.



(a) Sketch the graph of $f^{-1}(x)$, showing points of intersection with the axes.

(b) State the value of k for which $f(x)+k$ is an odd function.

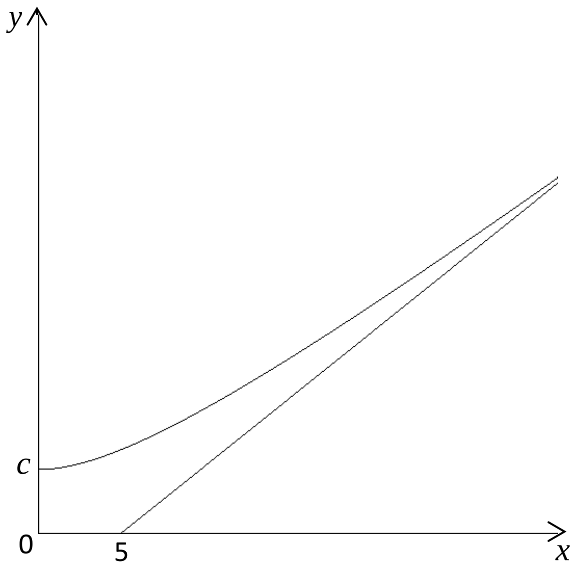
(c) Find the value of h for which $\left|f(x+h)\right|$ is an even function. **(2, 1, 2 marks)**

2014

The function $f(x)$ is defined for all $x \geq 0$.

The graph of $y = f(x)$ intersects the y-axis at $(0, c)$ where $0 < c < 5$.

The graph of the function and its asymptote, $y = x - 5$, are shown below.



(a) Copy the diagram above.

On the same diagram, sketch the graph of $y = f^{-1}(x)$.

Clearly show any points of intersection and any asymptotes.

(b) What is the equation of the asymptote of the graph $y = f(x+2)$?

(c) Why does your diagram show that the equation $x = f(f(x))$ has at least one solution?

(4, 1, 1 marks)

2015

For some function, f, define $g(x) = f(x) + f(-x)$ and

$$h(x) = f(x) - f(-x).$$

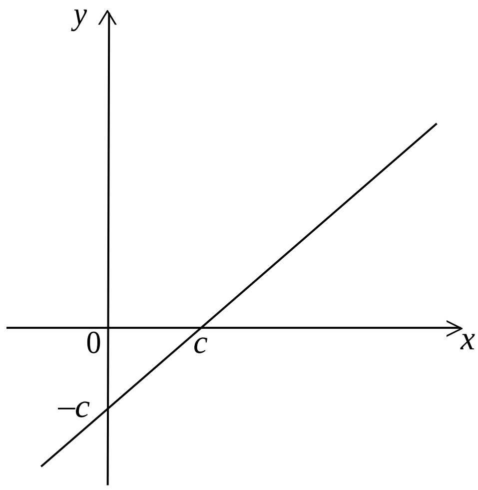
Show that $g(x)$ is an even function and that $h(x)$ is an odd function.

Hence show that $f(x)$ can be expressed as the sum of an even and an odd function.

(4 marks)

<u>2016</u>

Below is a diagram showing the graph of a linear function $y = f\left(x\right)$.



On separate diagrams show:

(a) $y = \left| f(x) - c \right|$

(b) $y = \left| 2f(x) \right|$ **(2, 2 marks)**