# Sequences and Series

A **sequence** takes the form $1,4,7,10,13,\ldots$ while $1+4+7+10+13+\ldots$ is a **series**.
There are two types of sequence/series – arithmetic and geometric.

## Arithmetic

An **arithmetic sequence** is one where we establish the next number in the sequence by adding (or subtracting) a given number. This number is known as the **common difference**.

Generally:

Let $a$ be the first term in the sequence and $d$ be the common difference.

$u_1 = a$
$u_2 = a + d$
$u_3 = u_2 + d = a + 2d$
$u_4 = u_3 + d = a + 3d$
$\vdots$

$$\boxed{u_n = a + (n-1)d}$$

This formula is important as it allows us to find the $n^{th}$ term in an arithmetic sequence provided we know the first term and the common difference.

If we sum the terms of this sequence we have an **arithmetic series**. The sum to $n$ terms of this arithmetic series is given by the formula

$$\boxed{S_n = \frac{n}{2}\left(2a + (n-1)d\right)}$$

**Proof**

Let $S_n = u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n$

Rewriting this backwards $S_n = u_n + u_{n-1} + u_{n-2} + \ldots + u_2 + u_1$

Adding these together $2S_n = (u_1 + u_n) + (u_2 + u_{n-1}) + (u_3 + u_{n-2}) + \ldots + (u_{n-1} + u_2) + (u_n + u_1)$

$$2S_n = \left(a + a + (n-1)d\right) + \left(a + d + a + (n-2)d\right) + \left(a + 2d + a + (n-3)d\right) + \ldots$$
$$\ldots + \left(a + (n-2)d + a + d\right) + \left(a + (n-1)d + a\right)$$

$$2S_n = \left(2a + (n-1)d\right) + \left(2a + (n-1)d\right) + \left(2a + (n-1)d\right) + \ldots\ldots + \left(2a + (n-1)d\right)$$

$$2S_n = n\left(2a + (n-1)d\right)$$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

**Examples**

① (a) Find the $20th$ term of the arithmetic sequence $4,11,18,25,32,\ldots$
(b) Find the sum of the first eight terms.

(a) $a=4, d=7$

$u_n = a+(n-1)d$

$u_n = 4+7(n-1)$

$u_{20} = 4+7(20-1)$

$=137$

(b) $S_n = \frac{n}{2}(2a+(n-1)d)$

$S_8 = \frac{8}{2}(2\times4+7\times7)$

$=4(8+49)$

$=228$

② (a) Find the $40th$ term of the arithmetic sequence $2,\frac{3}{2},1,\frac{1}{2},\ldots$
(b) Find the sum of the first fifteen terms.

(a) $a=2, d=\frac{-1}{2}$

$u_n = a+(n-1)d$

$u_n = 2-\frac{1}{2}(n-1)$

$u_{40} = 2-\frac{1}{2}(40-1)$

$=\frac{-35}{2}$

(b) $S_n = \frac{n}{2}(2a+(n-1)d)$

$S_{15} = \frac{15}{2}\left(2\times2+(15-1)\left(\frac{-1}{2}\right)\right)$

$=\frac{15}{2}(4-7)$

$=\frac{-45}{2}$

③ (a) Find the $75th$ term of the arithmetic sequence $40,35,30,25,\ldots$
(b) Find the sum of the first $100$ terms of the series.

(a) $a=40, d=-5$

$u_n = a+(n-1)d$

$u_n = 40-5(n-1)$

$u_{75} = 40-5(75-1)$

$=-330$

(b) $S_n = \frac{n}{2}(2a+(n-1)d)$

$S_{100} = \frac{100}{2}(80+99(-5))$

$=-20750$

④ (a) A sequence is given as follows : $2x+1, 3x+4, 4x+7, 5x+10,\ldots$
State an expression for the common difference.
(b) Find an expression for the $12th$ term.
(c) Find an expression for the first $25$ terms of the series.

(a) Common difference $= x+3$

(b) $u_n = a+(n-1)d$

$u_{12} = (2x+1)+11(x+3)$

$= 13x+34$

(c) $S_{25} = \frac{25}{2}(2(2x+1)+24(x+3))$

$= \frac{25}{2}(28x+74)$

$= 350x+925$

⑤ If the $6th$ term of an arithmetic sequence is $-22$ and the $3rd$ is $-10$, define the sequence and find the first four terms.

$u_6 = a+5d \qquad u_3 = a+2d$

$-22 = a+5d \qquad -10 = a+2d$

Solve equations simultaneously to get $a=-2$, $d=-4$

$u_n = a+(n-1)d$

$u_n = -2-4(n-1)$

$u_1 = -2$, $u_2 = -6$, $u_3 = -10$, $u_4 = -14$

⑥ Find the value of $n$ when $a=-3$, $d=2$ and the term of the sequence is $15$.

$u_n = a+(n-1)d$

$15 = -3+2(n-1)$

$18 = 2(n-1)$

$n = 10$

⑦ Which term in the arithmetic sequence $4, 1, -2, \ldots$ is -14?

$u_n = a+(n-1)d$

$u_n = 4-3(n-1)$

$-14 = 4-3n+3$

$n = 7$

The $7th$ term is -14.

⑧ If an arithmetic sequence has $u_1 = 1$ and $u_2 = 1 \cdot 5$, which term has the value $31$?

$u_n = a + (n-1)d$

$31 = 1 + \frac{1}{2}(n-1)$

$60 = n - 1$

$n = 61$

The $61st$ term has the value $31$.

⑨ An arithmetic sequence starts with $3$. The $100th$ term is $-393$. What is the common difference between the terms?

$u_n = a + (n-1)d$

$u_n = 3 + (n-1)d$

$u_{100} = 3 + 99d$

$-393 = 3 + 99d$

$99d = -396$

$d = -4$

The common difference is $-4$.

⑩ The sum of the first $4$ terms of an arithmetic sequence is $26$. The sum of the first $12$ terms is $222$. What is the sum of the first $20$ terms?

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_4 : 26 = \frac{4}{2}(2a + 3d)$ $\qquad$ $S_{12} : 222 = \frac{12}{2}(2a + 11d)$

$26 = 4a + 6d$ $\qquad$ $222 = 12a + 66d$

Solve simultaneously to get $a = 2$ , $d = 3$.

$S_n = \frac{n}{2}(4 + 3(n-1))$

$S_{20} = 10(4 + 3 \times 19)$

$S_{20} = 610$

⑪ The first two terms of an arithmetic sequence are $9$ and $12$ in that order.

(a) Find the sum of the first (i) $18$ terms (ii) $19$ terms

(b) Hence calculate the $19th$ term.

(a) $a=9$, $d=3$ $\quad S_n=\frac{n}{2}\left(2a+(n-1)d\right)$

$$S_n=\frac{n}{2}\left(18+3(n-1)\right)$$

$S_{18}=9\left(18+3\times17\right)=621$ $\qquad S_{19}=\frac{19}{2}\left(18+3\times18\right)=684$

(b) $19th$ term $=684-621=63$

⑫ The first two terms of an arithmetic sequence are $2$ and $10$. After what term does this series exceed $300$?

$a=2, d=8$ $\quad S_n=\frac{n}{2}\left(2a+(n-1)d\right)$

$$S_n=\frac{n}{2}\left(4+8(n-1)\right)$$

$$S_n=2n+4(n-1)n$$

$$S_n=4n^2-2n$$

$S_n>300$

$4n^2-2n>300$

$2n^2-n-150>0$

When is $2n^2-n-150=0$

$$n=\frac{1\pm\sqrt{1+8\times150}}{4}$$

$n=8\cdot9$ or $n=-8\cdot4$ $\therefore$ Series exceeds $300$ at the $9th$ term.

⑬ The sum of the first $50$ terms of an arithmetic sequence is $2000$. The common difference is $2$. What is the first term?

$$S_{50}=\frac{50}{2}\left(2a+49\times2\right)$$

$$2800=25\left(2a+98\right)$$

$a=7$ $\qquad$ The first term is $7$.

**Questions**

① (a) Find the $25th$ term of the sequence $2, 6, 10, 14, \ldots$
(b) Find the sum of the first 12 terms.

② The 3rd term of an arithmetic sequence is $40$ and the 5th term is $30$.
(a) Find the first term and common difference.
(b) Find the 20th term of the sequence.

③ The 4th term of an arithmetic sequence is $22$ and the 7th term is $37$.
(a) Find a formula for the nth term of the sequence.
(b) What is the sum of the first $50$ terms?

④ If an arithmetic sequence has $u_1 = 8$ and $u_2 = 5$, which term in the sequence has the value $-49$?

⑤ The 1st term in an arithmetic sequence is 6 . The 40th term is 435.
What is the common difference between the terms?

⑥ In the arithmetic sequence beginning $2, 8, 14, 20, \ldots$, which term is the first to exceed $100$?

⑦ The 1st term of an arithmetic sequence is 3 and the 6th term is twice the 3rd term.
Find the common difference and the 12th term.

⑧ For the arithmetic series $5 + 7 + 9 + 11 + \ldots$
(a) Find a formula for $S_n$, the sum of the first $n$ terms.
(b) Hence, find how many terms must be taken to give a sum of $192$.

⑨ (a) Find a formula for $S_n$, the sum of the first $n$ terms of the arithmetic series

$$3 + 8 + 13 + 18 + \ldots$$

(b) Hence find the least number of terms that are required to make a sum exceeding 2000.

⑩ In an arithmetic sequence, the 8th term is twice the 4th term.
(a) If $a$ is the first term and $d$ is the common difference, show that $a = d$.
(b) Given that the 20th term is $40$, find the sum of the first $50$ terms of this sequence.

## Geometric

A geometric sequence is one where we establish the next number in the sequence by multiplying or dividing by a given number. This number is called the **common ratio**. It is the ratio between two consecutive terms.

Generally :

Let $a$ be the first term of the sequence and $r$ be the common ratio.

$$u_1 = a$$
$$u_2 = ar$$
$$u_3 = ar^2$$
$$\vdots$$
$$\boxed{u_n = ar^{n-1}}$$

Again, this result should be remembered. It allows us to find the $n^{th}$ term in a geometric sequence provided we know the first term and the common ratio.

If we sum the terms of this sequence we have a **geometric series**. The sum to $n$ terms of this geometric series is given by the formula

$$\boxed{S_n = \frac{a\left(1-r^n\right)}{(1-r)}, \quad r \neq 1}$$

**Proof**

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

If we multiply both sides by $r$ we get $\ldots$ $\quad rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$

Subtracting $\ldots$ $\quad S_n - rS_n = a - ar^n$

$$(1-r)S_n = a\left(1-r^n\right)$$

$$S_n = \frac{a\left(1-r^n\right)}{(1-r)}, \quad r \neq 1$$

(When $r = 1$, the series is $a + a + a + a + \ldots$ so $S_n = na$ )

**Examples**

① Find the $nth$ term and the $10th$ term of the geometric sequence $3, 12, 48, \ldots$

$$a = 3, \quad r = \frac{12}{3} = 4, \qquad u_n = ar^{n-1}$$

$$u_n = 3 \times 4^{n-1}$$
$$u_{10} = 3 \times 4^{10-1}$$
$$= 3 \times 4^9$$
$$= 786432$$

② Find the geometric sequence whose $3rd$ term is $18$ and whose $8th$ term is $4374$.

$u_3 = 18 \qquad u_8 = 4374$

$18 = ar^2 \qquad 4374 = ar^7$

$$\frac{ar^7}{ar^2} = \frac{4374}{18}$$
$$r^5 = 243$$
$$r = 3$$

$18 = a \times 3^2$

$a = 2$

So $u_n = 2 \times 3^{n-1}$

$u_1 = 2$, $u_2 = 6$, $u_3 = 18$, $u_4 = 54$, $\ldots$

So sequence is $2, 6, 18, 54, \ldots$

③ Given the geometric sequence $5, 10, 20, 40, \ldots$ find the value of $n$ for which $u_n = 20480$.

$a = 5$, $r = \frac{10}{5} = 2$, $\quad u_n = ar^{n-1}$

$$u_n = 5 \times 2^{n-1}$$
$$5 \times 2^{n-1} = 20480$$
$$2^{n-1} = 4096$$
$$\ln 2^{n-1} = \ln 4096$$
$$(n-1)\ln 2 = \ln 4096$$
$$n - 1 = \frac{\ln 4096}{\ln 2}$$
$$n = 13$$

④ For each of the following geometric sequences
(i) identify $a$ and $r$ (ii) find an expression for the $nth$ term.

(a) $1458, 486, 162, 54, \ldots$

(b) $4, -8, 16, -32, \ldots$

(c) $0\cdot16, 0\cdot28, 0\cdot1024, \ldots$

(d) $\frac{2}{3}, \frac{4}{15}, \frac{8}{75}, \frac{16}{375}, \ldots$

(a) (i) $a = 1458$, $r = \frac{486}{1458} = \frac{1}{3}$ (ii) $u_n = ar^{n-1}$

$$u_n = 1458\left(\frac{1}{3}\right)^{n-1}$$

(b) (i) $a = 4$, $r = \frac{-8}{4} = -2$

(ii) $u_n = ar^{n-1}$

$u_n = 4(-2)^{n-1}$

(c) (i) $a = 0\cdot16 = \frac{4}{25}$, $r = \frac{0\cdot128}{0\cdot16} = \frac{4}{5}$

(ii) $u_n = ar^{n-1}$

$u_n = \frac{4}{25}\left(\frac{4}{5}\right)^{n-1}$

(d) (i) $a = \frac{2}{3}$, $r = \frac{4/15}{2/3} = \frac{2}{5}$

(ii) $u_n = ar^{n-1}$

$u_n = \frac{2}{3}\left(\frac{2}{5}\right)^{n-1}$

⑤ (a) The first term of a geometric sequence is $25$. The tenth term is $139$. Calculate the common ratio to $2$ d.p.
(b) The common ratio of a geometric sequence is $0\cdot98$. The $8th$ term is $131$. What is the first term to the nearest whole number?

(a) $a = 25$, $u_{10} = 139$

$$u_{10} = 25 \times r^{n-1}$$
$$139 = 25 \times r^9$$
$$r^9 = \frac{139}{25}$$
$$r = \sqrt[9]{\frac{139}{25}}$$
$$r = 1\cdot21$$

(b) $r = 0\cdot98$, $u_8 = 131$

$$u_8 = a \times 0\cdot98^{8-1}$$
$$131 = a \times 0\cdot98^7$$
$$a = 151$$

⑥ Find the sum of the first $10$ terms of the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$

$a = 1$, $r = \frac{1/2}{1} = \frac{1}{2}$

$$S_{10} = \frac{1\left(1 - \left(1/2\right)^{10}\right)}{1 - 1/2}$$
$$S_{10} = 2\left(1 - \frac{1}{1024}\right)$$
$$S_{10} = \frac{1023}{512}$$

⑦ Find the sum to six terms of the geometric sequence whose first term is $4$ and whose common ratio is $\frac{3}{2}$.

$a = 4, \quad r = \frac{3}{2}$

$$S_6 = \frac{4\left(1-\left(\frac{3}{2}\right)^6\right)}{1-\frac{3}{2}}$$

$$S_6 = 83\cdot125$$

⑧ A geometric sequence starts $12, 15, 18\cdot75, \ldots$. What is the smallest value of $n$ for which $S_n$ is bigger than 100?

$a = 12, \quad r = \frac{15}{21} = \frac{5}{4}$

$$S_n > 100$$

$$\frac{12\left(1-\left(\frac{5}{4}\right)^n\right)}{1-\frac{5}{4}} > 100$$

$$-48\left(1-\left(\frac{5}{4}\right)^n\right) > 100$$

$$48\left(\left(\frac{5}{4}\right)^n - 1\right) > 100$$

$$\left(\frac{5}{4}\right)^n > \frac{100}{48} + 1$$

$$n\ln\left(\frac{5}{4}\right) > \ln\left(\frac{37}{12}\right)$$

$$n > \frac{\ln\left(\frac{37}{12}\right)}{\ln\left(\frac{5}{4}\right)}$$

$$n > 5\cdot046$$

$n = 6$ as $n \in \mathbb{N}$

⑨ A geometric series is such that $S_3 = 14$ and $S_6 = 126$. Identify the series.

$14 = \frac{a\left(1-r^3\right)}{1-r}$ and $126 = \frac{a\left(1-r^6\right)}{1-r}$ so $\frac{126}{14} = \frac{1-r^6}{1-r^3}$

$$9\left(1-r^3\right) = 1-r^6$$

$$r^6 - 9r^3 + 8 = 0$$

$$\left(r^3-1\right)\left(r^3-8\right) = 0$$

$r = 1$ or $r = 2$

$r \neq 1$ so $r = 2$

The series is $2+4+8+\ldots$. The $nth$ term is given by $u_n = 2\times2^{n-1}$.

⑩ Calculate each of these geometric series to the required number of terms.

(a) $\frac{1}{4}+\frac{1}{12}+\frac{1}{30}+\ldots$ to $6$ terms.

(b) $12-10+\frac{25}{3}-\ldots$ to $7$ terms.

(c) $50-20+8-\ldots$ to $8$ terms.

(a) $a=\frac{1}{4}$, $r=\frac{1/12}{1/4}=\frac{1}{3}$ $\qquad S_6=\frac{\frac{1}{4}\left(1-\left(1/3\right)^6\right)}{1-1/3}=\frac{91}{243}$

(b) $a=12$, $r=\frac{-10}{12}=\frac{-5}{6}$ $\qquad S_7=\frac{12\left(1-\left(-5/6\right)^7\right)}{1+5/6}=\frac{32551}{3888}$

(c) $a=50$, $r=\frac{-20}{50}=\frac{-2}{5}$ $\qquad S_8=\frac{50\left(1-\left(-2/5\right)^8\right)}{1+2/5}=\frac{111534}{3125}$

⑪ A pendulum is slowly coming to rest. With each swing the arc it sweeps is reduced by 2%. Each swing takes the same time, $1\cdot 2$ seconds. The first sweep is a $12^\circ$ arc.

(a) How many degrees has the pendulum swung through in five swings?

(b) How long will it take before a total exceeding $90^\circ$ has been swept through?

(a) The sequence is $12^\circ, 11\cdot 76^\circ, \ldots$

$a=12$, $r=0\cdot 98$ $\qquad S_5=\frac{12\left(1-0\cdot 98^5\right)}{1-0\cdot 98}=57\cdot 65^\circ$

(b) $\frac{12\left(1-0\cdot 98^n\right)}{1-0\cdot 98}>90$

$1-0\cdot 98^n>0\cdot 15$

$0\cdot 98^n<0\cdot 85$

$n\ln 0\cdot 98<\ln 0\cdot 85$

$n>\frac{\ln 0\cdot 85}{\ln 0\cdot 98}$ ($\ln 0\cdot 98$ is –ve so reverse inequality sign)

$n>8\cdot 04$

$\therefore 9$ whole swings are needed before a total exceeding $90^\circ$ has been swept through.
This will take $9\times 1\cdot 2=10\cdot 8$ seconds.

**Questions**

① (a) Find an expression for the $nth$ term of the geometric sequence $4, -8, 16, -32, \ldots$
(b) Calculate the $10th$ term.

② A geometric sequence of positive terms has 3rd term $18$ and 7th term 1458.
Find the 5th term of this sequence.

③ Given the geometric sequence $6, 12, 24, 48, \ldots$, find the value of $n$ for which $u_n = 98304$.

④ A geometric sequence of positive terms has 1st term $9$ and 3rd term $36$.
Find the 10th term of this sequence.

⑤ Find the sum to 9 terms of the series $64 + 32 + 16 + 8 + \ldots$

⑥ Evaluate the sum of the geometric series $4 + 20 + 100 + \ldots + 62500$

⑦ A geometric sequence has 3rd term $54$ and 6th term $1458$.
Find the sum of the first $8$ terms of this sequence.

⑧ A geometric sequence has 5th term $18$ and 8th term $\frac{2}{3}$.
Find the sum of the first 6 terms of this sequence.

⑨ (a) Find an expression for the sum of the first $n$ terms of the geometric sequence $10, 30, 90, 270, \ldots$
(b) Find the number of terms which must be added to give a sum of $32800$.
(c) Find the least number of terms which must be added to give a sum exceeding 1000000.

⑩ Find the least number of terms of the geometric series $2 + 4 + 8 + 16 + \ldots$
which must be added to give a sum exceeding 100000.

⑪ A geometric sequence has 1st term 5 and 4th term 40.
Find an expression for the sum to $n$ terms of this sequence and hence, find the least number of terms which must be added to give a sum exceeding 100000.

⑫ A line $315cm$ long is divided into $6$ parts such that the lengths of the parts form a geometric sequence.
Given that the length of the longest part is $32$ times the length of the shortest part, find the length of the shortest part.

⑬ For a certain sequence it has been found that the sum to $n$ terms is $S_n = \frac{3^n - 1}{2}, \; n \geq 1$.
Find the type of sequence an the value of $u_6$, the 6th term.

## Convergence and Divergence of Infinite Geometric Series

**Examples**

① Consider the geometric series $27+2\cdot7+0\cdot27+0\cdot027+\ldots$

$S_1=27$
$S_2=27+2\cdot7=29\cdot7$
$S_3=27+2\cdot7+0\cdot27=29\cdot97$
$S_4=27+2\cdot7+0\cdot27+0\cdot027=29\cdot997$

If we were to continue we would see that $S_n\to30$ as $n\to\infty$.
This is a **convergent** series. We write $S_\infty=30$.

② $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

$S_1=1$
$S_2=1+\frac{1}{2}=1\cdot5$
$S_3=1+\frac{1}{2}+\frac{1}{4}=1\cdot75$
$S_4=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=1\cdot875$
$S_5=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=1\cdot9375$

In this case $S_n\to2$ as $n\to\infty$. We write $S_\infty=2$

③ $1+2+4+8+16+\ldots$

$S_1=1$
$S_2=1+2=3$
$S_3=1+2+4=7$
$S_4=1+2+4+8=15$

This time as $n\to\infty$, $S_n$ continues to increase.
This is a **divergent** series. $S_\infty$ does not exist.

## Condition for Existence of Sum to Infinity

Consider the formula for $S_n$.

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$

When $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$. So as $n \to \infty$, $S_n = \frac{a(1-r^n)}{1-r} \to \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

The sum to infinity of a geometric series exists if $|r| < 1$ and can be calculated by

$$S_\infty = \frac{a}{1-r}$$

**Examples**

① Find the sum to infinity of the series $32 + 16 + 8 + 4 + 2 + \ldots$

$a = 32$, $\quad r = \frac{16}{32} = \frac{1}{2}$ $\qquad |r| < 1$ so $S_\infty$ exists.

$$S_\infty = \frac{a}{1-r} = \frac{32}{1-\frac{1}{2}} = 64$$

② Find the sum to infinity of the series $32 - 16 + 8 - 4 + 2 - \ldots$

$a = 32$, $\quad r = \frac{-16}{32} = \frac{-1}{2}$ $\qquad |r| < 1$ so $S_\infty$ exists.

$$S_\infty = \frac{a}{1-r} = \frac{32}{1+\frac{1}{2}} = \frac{64}{3}$$

**Questions**

① Explain why the geometric series $27 + 18 + 12 + \ldots$ has a sum to infinity and find the sum to infinity.

② A geometric sequence has 1st term 48 and a sum to infinity of 64.
Find the 4th term of the sequence.

③ A geometric series has a common ratio of $\frac{2}{5}$ and the sum to infinity of the series is $25$.
Find the 1st and the 4th terms of this series.

## Sigma Notation

The notation $\sum_{k=a}^{b} f(k)$ is shorthand for $f(a)+f(a+1)+f(a+2)+\ldots+f(b)$ where $a$ and $b$ are integers and $a \le b$.

### Examples

① $\sum_{k=1}^{5} k^2 = 1^2+2^2+3^2+4^2+5^2 = 55$

② $\sum_{r=2}^{4} \frac{1}{r} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{13}{12}$

③ $\sum_{k=0}^{4} (-2)^k = (-2)^0+(-2)^1+(-2)^2+(-2)^3+(-2)^4 = 1-2+4-8+16 = 11$

④ $\sum_{r=-1}^{1} (2r+5)(r-3) = (2(-1)+5)(-1-3)+(2(0)+5)(0-3)+(2(1)+5)(1-3)$

$$= (3)(-4)+(5)(-3)+(7)(-2)$$
$$= -12-15-14$$
$$= -41$$

⑤ Find $\sum_{k=1}^{50} (2k-3)$

It is best to write out the first few terms to see what kind of series it is and get the first term and common difference or ratio.

The series is $-1+1+3+5+7+\ldots$ summed to 50 terms.
It is arithmetic with $a=-1$, $d=2$ and $n=50$.

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$ so $S_{50} = \frac{50}{2}\left[2(-1)+(50-1)2\right] = 2400$

⑥ Find $\sum_{r=1}^{\infty} \left(\frac{-1}{2}\right)^r$ The series is $\left(\frac{-1}{2}\right)+\left(\frac{-1}{2}\right)^2+\left(\frac{-1}{2}\right)^3+\ldots$

It is geometric with $a = \frac{-1}{2}$, $r = \frac{-1}{2}$. $|r|<1$ so $S_\infty$ exists.

$S_\infty = \frac{a}{1-r}$ so $S_\infty = \frac{-1/2}{1-(-1/2)} = \frac{-1}{3}$

## Some Useful Properties

$$\sum_{k=1}^{n}\left(f(k)+g(k)\right)=\sum_{k=1}^{n}f(k)+\sum_{k=1}^{n}g(k)$$

$$\sum_{k=1}^{n}af(k)=a\sum_{k=1}^{n}f(k),$$ where $a$ is a constant.

## Some Useful Standard Formulae

$$\sum_{k=1}^{n}1=\underbrace{1+1+1+\ldots\ldots+1}_{\longleftarrow\; n \text{ terms} \;\longrightarrow}=n$$

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n}(ak+b)=a\sum_{k=1}^{n}k+b\sum_{k=1}^{n}1=\frac{an}{2}(n+1)+bn$$

No need to memorise these as they are in the formula list! Notice that $\sum_{k=1}^{n}k^3=\left(\sum_{k=1}^{n}k\right)^2$

### Examples

① Evaluate $\sum_{k=1}^{10}\left(2k^3-3k\right)$

$$\begin{aligned}\sum_{k=1}^{10}\left(2k^3-3k\right)&=2\sum_{k=1}^{10}k^3-3\sum_{k=1}^{10}k\\&=2\left(\frac{10^2\times11^2}{4}\right)-3\left(\frac{10\times11}{2}\right)\quad\text{(using formulae from above)}\\&=2(3025)-3(55)\\&=5885\end{aligned}$$

② Evaluate the sum of the series $(1\times4^2)+(2\times5^2)+(3\times6^2)+\ldots+(12\times15^2)$

$$=\sum_{k=1}^{12}k(k+3)^2$$
$$=\sum_{k=1}^{12}k(k^2+6k+9)$$
$$=\sum_{k=1}^{12}k^3+6k^2+9k$$
$$=\sum_{k=1}^{12}k^3+6\sum_{k=1}^{12}k^2+9\sum_{k=1}^{12}k$$
$$=\left(\frac{12^2\times13^2}{4}\right)+6\left(\frac{12\times13\times25}{6}\right)+9\left(\frac{12\times13}{2}\right)$$
$$=6084+3900+702$$
$$=10686$$

③ Obtain an expression for $\sum_{k=1}^{n}(2k^2-5)$.

Give your answer as a single algebraic fraction in its simplest form.

$$\sum_{k=1}^{n}(2k^2-5)=2\sum_{k=1}^{n}k^2-5\sum_{k=1}^{n}1$$
$$=2\left(\frac{n(n+1)(2n+1)}{6}\right)-5n$$
$$=\frac{n(n+1)(2n+1)}{3}-\frac{15n}{3}$$
$$=\frac{n((n+1)(2n+1)-15)}{3}$$
$$=\frac{n(2n^2+3n-14)}{3}$$
$$=\frac{n(2n+7)(n-2)}{3}$$

④ Evaluate $\sum_{k=25}^{40}k^2$

$$\sum_{k=25}^{40}k^2=\sum_{k=1}^{40}k^2-\sum_{k=1}^{24}k^2$$
$$=\left(\frac{40\times41\times81}{6}\right)-\left(\frac{24\times25\times49}{6}\right)$$
$$=22140-4900$$
$$=17240$$

**Questions**

① Evaluate

(a) $\sum_{k=1}^{12}\left(2k^2+3k-1\right)$ (b) $\sum_{k=1}^{8}k^2\left(4k+1\right)$ (c) $\sum_{k=1}^{10}\left(k^3+3k^2-5k+2\right)$ (d) $\sum_{k=1}^{8}k^2\left(4k+1\right)$

② Evaluate the sum of these series by writing them in the form $\sum_{k=1}^{n} f\left(k\right)$

(a) $\left(1\times5\right)+\left(2\times6\right)+\left(3\times7\right)+\left(4\times8\right)+\ldots+\left(15\times19\right)$

(b) $\left(1\times1\right)+\left(2\times4\right)+\left(3\times7\right)+\left(4\times10\right)+\ldots+\left(25\times73\right)$

(c) $\left(1^2\times3\right)+\left(2^2\times4\right)+\left(3^2\times5\right)+\left(4^2\times6\right)+\ldots+\left(20^2\times22\right)$

③ Find an expression for these summations in simplest form.

(a) $\sum_{k=1}^{n}\left(2k^2-5\right)$ (b) $\sum_{k=1}^{n}\left(k-1\right)\left(k+1\right)$ (c) $\sum_{k=1}^{n}k\left(k+1\right)^2$ (d) $\sum_{k=1}^{n}k\left(k+1\right)\left(2k-1\right)$

④ Evaluate (a) $\sum_{k=36}^{60}k$ (b) $\sum_{k=15}^{30}k^2$ (c) $\sum_{k=16}^{24}k^3$

## Differentiation and Integration of Series

**Examples**

① Find the sum to infinity of $1-x+x^2-x^3+x^4-\ldots,\ |x|<1$

Hence, use integration to find a series for $\ln\left(1+x\right)$.

$1-x+x^2-x^3+x^4-\ldots$ is geometric with $a=1$ and $r=-x$,

$|-x|<1$ so a sum to infinity exists. $S_\infty=\frac{1}{1-(-x)}=\frac{1}{(1+x)}$

So $\frac{1}{(1+x)}=1-x+x^2-x^3+x^4-\ldots$

Integrating both sides.... $\int\frac{1}{(1+x)}\,dx=\int\left(1-x+x^2-x^3+x^4-\ldots\right)dx$

$$\ln\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots+C$$

Put $x=0$ to get $C=0$ so $\ln\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$ , when $|x|<1$

② Find the sum of $1+e^{x}+e^{2x}+e^{3x}+\ldots\ldots+e^{99x},\quad x\neq 0$
Use differentiation to find a formula for the series $e^{x}+2e^{2x}+3e^{3x}+\ldots\ldots+99e^{99x},\quad x\neq 0$

The series is geometric with $a=1$, $r=e^{x}$ and $n=100$.

$S_n=\dfrac{a\left(1-r^{n}\right)}{1-r}$ so $S_{100}=\dfrac{1\left(1-e^{100x}\right)}{1-e^{x}}=\dfrac{1-e^{100x}}{1-e^{x}},\quad x\neq 0$

So $1+e^{x}+e^{2x}+e^{3x}+\ldots\ldots+e^{99x}=\dfrac{1-e^{100x}}{1-e^{x}},\quad x\neq 0$

Differentiating each side ……

$$e^{x}+2e^{2x}+3e^{3x}+\ldots\ldots+99e^{99x}=\frac{-100e^{100x}\left(1-e^{x}\right)-\left(1-e^{100x}\right)\cdot -e^{x}}{\left(1-e^{x}\right)^{2}}$$

$$e^{x}+2e^{2x}+3e^{3x}+\ldots\ldots+99e^{99x}=\frac{99e^{101x}-100e^{100x}+e^{x}}{\left(1-e^{x}\right)^{2}}$$

## Past Paper Questions

### 2002 – A10

Define $S_n(x)$ by

$$S_n(x) = 1 + 2x + 3x^2 + \ldots + nx^{n-1}$$

where $n$ is a positive integer.

Express $S_n(1)$ in terms of $n$.

By considering $(1-x)S_n(x)$, show that

$$S_n(x) = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{(1-x)}, \quad x \neq 1.$$

Obtain the value of $\lim_{n\to\infty}\left\{\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots+\frac{n}{3^{n-1}}+\frac{3}{2}\cdot\frac{n}{3^n}\right\}$. **(2, 4, 3 marks)**

### 2003 – A2

Given that $u_k = 11 - 2k, \quad (k \geq 1)$, obtain a formula for $S_n = \sum_{k=1}^{n} u_k$.

Find the values of $n$ for which $S_n = 21$. **(3, 2 marks)**

### 2004 – Q16

(a) Obtain the sum of the series $8 + 11 + 14 + \ldots + 56$.

(b) A geometric sequence of positive terms has first term $2$, and the sum of the first three terms is $266$, Calculate the common ratio.

(c) An arithmetic sequence, $A$, has first term $a$ and common difference $2$, and a geometric sequence, $B$, has first term $a$ and common ratio $2$. The first four terms of each sequence have the same sum. Obtain the value of $a$.

Obtain the smallest value of $n$ such that the sum to $n$ terms for sequence $B$ is more than **twice** the sum to $n$ terms for the sequence $A$. **(2, 3, 3, 2 marks)**

### 2005 – Q4

The sum, $S(n)$, of the first $n$ terms of a sequence, $u_1, u_2, u_3, \ldots$ is given by

$$S(n) = 8n - n^2, \quad n \geq 1.$$

Calculate the values of $u_1, u_2, u_3$ and state what type of sequence it is.

Obtain a formula for $u_n$ in terms of $n$, simplifying your answer. **(3, 2 marks)**

**2006 – Q16**

The first three terms of a geometric sequence are

$\frac{x(x+1)}{(x-2)}, \frac{x(x+1)^2}{(x-2)^2}$ and $\frac{x(x+1)^3}{(x-2)^3}$ , where $x<2$.

(a) Obtain expressions for the common ratio and the $nth$ term of the sequence.

(b) Find an expression for the sum of the first $n$ terms of the sequence.

(c) Obtain the range of values of $x$ for which the sequence has a sum to infinity and find an expression for the sum to infinity.

**(3, 3, 4 marks)**

**2007 – Q9**

Show that $\sum_{r=1}^{n}(4-6r)=n-3n^2$.

Hence write down a formula for $\sum_{r=1}^{2q}(4-6r)$.

Show that $\sum_{r=q+1}^{2q}(4-6r)=q-9q^2$. **(2, 1, 2 marks)**

**2008 – Q1**

The first term of an arithmetic sequence is $2$ and the $20th$ term is $97$. Obtain the sum of the first $50$ terms.

**(4 marks)**

**2009 – Q12**

The first two terms of a geometric sequence are $a_1 = p$ and $a_2 = p^2$.

Obtain expressions for $S_n$ and $S_{2n}$ in terms of $p$, where $S_k = \sum_{j=1}^{k} a_j$ .

Given that $S_{2n} = 65S_n$ show that $p^n = 64$.

Given also that $a_3 = 2p$ and that $p>0$, obtain the exact value of $p$ and hence the value of $n$.

**(1, 1, 2, 1, 1 marks)**

**2010 – Q2**

The second and third terms of a geometric series are $-6$ and $3$ respectively.
Explain why the series has a sum to infinity and obtain this sum. **(5 marks)**

**2011 – Q8**

① Write down an expression for $\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2$

and an expression for $\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2$. **(1, 3 marks)**

**2011 – Q13**

The first three terms of an arithmetic sequence are $a, \frac{1}{a}, 1$ where $a < 0$.

Obtain the value of $a$ and the common difference.

Obtain the smallest value of $n$ for which the sum of the first $n$ terms is greater than $1000$.

**(5, 4 marks)**

**2012 – Q2**

The first and fourth terms of a geometric sequence are $2048$ and $256$ respectively.
Calculate the value of the common ratio.

Given that the sum of the first $n$ terms is $4088$, find the value of $n$.

**(2, 3 marks)**

**2013 – Q17**

Write down the sums to infinity of the geometric series

$$1 + x + x^2 + x^3 + \ldots\ldots$$

and

$$1 - x + x^2 - x^3 + \ldots\ldots$$

valid for $|x| < 1$.

Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x| < 1$,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right)$$

Show how this series can be used to evaluate $\ln 2$.
Hence determine the value of $\ln 2$ correct to 3 decimal places. **(7, 3 marks)**

**2015 – Q3**

The sum of the first twenty terms of an arithmetic sequence is 320.

The twenty-first term is 37.

What is the sum of the first ten terms? **(5 marks)**

**2016 – Q2**

A geometric sequence has second and fifth terms 108 and 4 respectively.

(a) Calculate the value of the common ratio.

(b) State why the associated geometric series has a sum to infinity.

(c) Find the value of this sum to infinity. **(3, 1, 2 marks)**

**2017 – Q4**

The fifth term of an arithmetic sequence is -6 and the twelfth term is -34.

(a) Determine the values of the first term and the common difference.

(b) Obtain algebraically the value of $n$ for which $S_n = -144$.

**(2, 3 marks)**