Systems of Equations

Matrices are useful in mathematics to represent a system of equations.
Say we have the following:

$$3x+4y=10$$
$$5x-3y=7$$

We can set up a matrix equation to solve this for x and y. It may not seem worth it for a 2 × 2 system as we know how to solve these equations simultaneously but for 3 × 3 systems (and larger) it makes the process easy to do.

First we take the coefficients of the LHS and put them into a matrix.

$A=\begin{pmatrix} 3 & 4 \\ 5 & -3 \end{pmatrix}$ A is called our coefficient matrix.

Then we put the unknowns into a column matrix $X=\begin{pmatrix} x \\ y \end{pmatrix}$ and the RHS into another column matrix $B=\begin{pmatrix} 10 \\ 7 \end{pmatrix}$.

The system of equations can now be written as a matrix equation.

$$AX=B$$
$$\begin{pmatrix} 3 & 4 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

To solve this for x and y it is more useful to write this as an **augmented matrix.**

$$(A \vdots B)=\left(\begin{array}{cc|c} 3 & 4 & 10 \\ 5 & -3 & 7 \end{array}\right)$$

Before we look at how to solve this let's construct another matrix equation with an augmented matrix for a 3 × 3 system.

$$x+y+z=2$$
$$4x+2y+z=4$$
$$x-y+z=4$$

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix} \qquad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad B=\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

The augmented matrix is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right)$

Gaussian Elimination

When we solve a system of equations by Gaussian elimination we use **Elementary Row Operations (ERO's)**.
These alter the system to produce a new system with the same solutions.

- The order of the equations can be changed.
- An equation can be multiplied by a constant.
- One equation can be added to or subtracted from another.

To solve a system of equations we must convert the coefficient part of the augmented matrix to **upper triangular form**. This is where all the elements below the main diagonal (which runs from the top left to the bottom right) are zero.

Examples

1 Solve $3x + 4y = 10$
$5x - 3y = 7$

$$\left(\begin{array}{cc|c} 3 & 4 & 10 \\ 5 & -3 & 7 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\begin{array}{r} \\ 5R_1 - 3R_2 \end{array}\left(\begin{array}{cc|c} 3 & 4 & 10 \\ 0 & 29 & 29 \end{array}\right)$$

From here $29y = 29$, $y = 1$

$3x + 4y = 10$
$3x + 4 = 10$
$x = 2$

2 Solve $x + y + 2z = 2$
$4x + 2y + z = 4$
$x - y + z = 4$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ 4R_1 - R_2 \\ R_1 - R_3 \end{array}\left(\begin{array}{ccc:c} 1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 4 \\ 0 & 2 & 1 & -2 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ \\ R_2 - R_3 \end{array}\left(\begin{array}{ccc:c} 1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 4 \\ 0 & 0 & 6 & 6 \end{array}\right)$$

$$6z = 6 \qquad 2y + 7z = 4 \qquad x + y + 2z = 2$$

$$z = 1 \qquad 2y + 7 = 4 \qquad x - \frac{3}{2} + 2 = 2$$

$$y = \frac{-3}{2} \qquad x = \frac{3}{2}$$

This process of using ERO's to reduce an augmented matrix to upper triangular form is **Gaussian Elimination**.

More examples:-

③ Use Gaussian Elimination to solve the following equations for x, y and z.

$$2x - y + z = 5$$
$$x - 3y + 2z = 2$$
$$2x + y + 4z = -3$$

$$\left(\begin{array}{ccc:c} 2 & -1 & 1 & 5 \\ 1 & -3 & 2 & 2 \\ 2 & 1 & 4 & -3 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ R_1 - 2R_2 \\ R_3 - R_1 \end{array}\left(\begin{array}{ccc:c} 2 & -1 & 1 & 5 \\ 0 & 5 & -3 & 1 \\ 0 & 2 & 3 & -8 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ \\ 5R_3 - 2R_2 \end{array}\left(\begin{array}{ccc:c} 2 & -1 & 1 & 5 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 21 & -42 \end{array}\right)$$

$$21z = -42 \qquad 5y - 3z = 1 \qquad 2x - y + z = 5$$

$$z = -2 \qquad 5y + 6 = 1 \qquad 2x + 1 - 2 = 5$$

$$y = -1 \qquad x = 3$$

④ Solve
$$2x+3y-z=-1$$
$$x-3y-2z=4$$
$$5x+y+3z=4$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ 1 & -3 & -2 & 4 \\ 5 & 1 & 3 & 4 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ R_1-2R_2 \\ R_3-5R_2 \end{array}\left(\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ 0 & 9 & 3 & -9 \\ 0 & 16 & 13 & -16 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ \\ 9R_3-16R_2 \end{array}\left(\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ 0 & 9 & 3 & -9 \\ 0 & 0 & 69 & 0 \end{array}\right)$$

$$\begin{aligned} 69z&=0 \\ z&=0 \end{aligned} \qquad \begin{aligned} 9y+3z&=-9 \\ y&=-1 \end{aligned} \qquad \begin{aligned} 2x+3y-z&=-1 \\ 2x-3&=-1 \\ x&=1 \end{aligned}$$

Questions

Use Gaussian elimination to solve the following systems of equations.

(a) $x+y+z=6$
$-x+y+2z=5$
$3x+2z=12$

(b) $a+b+c=2$
$4a+2b+c=4$
$a-b+c=4$

(c) $2y+z=-8$
$x-2y-3z=0$
$-x+y+2z=3$

(d) $3x-4y+z=24$
$x-2y-2z=7$
$x+y+z=4$

(e) $3x+y=5$
$x+2y-3z=-12$
$x+2z=10$

(f) $2a+3c=5$
$4a+2b+c=3$
$a+3b+5c=3$

Infinite and No Solutions

So far these systems of equations have been straightforward and our Gaussian elimination has given us a unique solution. Geometrically, the 3 equations are equations of planes and the unique solution is the point of intersection of the 3 planes. You will study the equation of planes in the vectors chapter later in the course.

There are cases when a unique solution is not possible.

Consider the following system of equations for a 2 × 2 case.

$$2x+3y=6$$
$$4x+6y=12$$

You will see immediately that the second equation is just double the first and so contributes no new information to the system. If we try to solve this by Gaussian Elimination:

$$\left(\begin{array}{cc:c} 2 & 3 & 6 \\ 4 & 6 & 12 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\begin{array}{r} \\ R_2-2R_1 \end{array}\left(\begin{array}{cc:c} 2 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right)$$

We end up with $0y=0$ which gets us nowhere.

If you consider the geometric situation here, the first equation is a straight line and the second equation is exactly the same straight line so the system has an **infinite** number of solutions, with each solution being a point lying on the straight line.

The solutions can be represented **parametrically** as follows:

Let $y=\lambda$.
Then from the first equation we get $2x+3\lambda=6$.

$$\Rightarrow x=3-\frac{3}{2}\lambda$$

So the **parametric solution** is $x=3-\frac{3}{2}\lambda,\ y=\lambda$.
Every real number λ gives a solution and every solution corresponds to a value of λ.
(More of this in the vectors chapter).

Now consider the following 2 × 2 system.

$$2x+3y=6$$
$$4x+6y=15$$

You should see straight away that the equations are **inconsistent**.
If $2x+3y=6$ then $4x+6y$ should $=12$, not 15.
If we try to solve this by Gaussian Elimination:

$$\left(\begin{array}{cc:c} 2 & 3 & 6 \\ 4 & 6 & 15 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\begin{array}{c} \\ R_2 - 2R_1 \end{array}\left(\begin{array}{cc:c} 2 & 3 & 6 \\ 0 & 0 & 3 \end{array}\right)$$

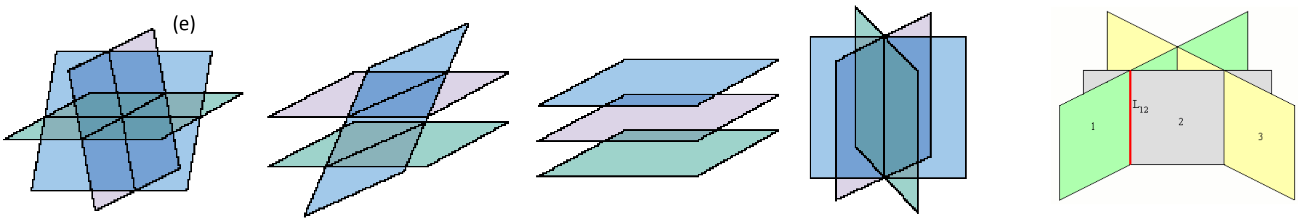
We end up with $0y = 3$ which makes no sense.
If you consider the geometric situation here, the first equation is a straight line and the second equation is a straight line parallel to the first so the system has no solutions as the lines never intersect.

So when solving a pair of simultaneous linear equations we can end up with:
(1) A unique solution (lines intersect).
(2) An infinite number of solutions (lines are coincident).
(3) No solutions (lines are parallel).

We will now consider a 3 × 3 system.

An equation such as $x + 2y + 3z = 1$ represents a plane in 3-D space and if we have a system of 3 such equations then we have 3 planes, one for each equation. There are several geometric possibilities:

(1) All 3 planes intersect at one point - diagram (a) – one unique solution.
(2) 2 planes are parallel and the other is a transversal - diagram (b) – no solutions.
(3) The 3 planes are parallel - diagram (c) – no solutions
(4) All 3 planes meet in one straight line - diagram (d) – infinite number of solutions (points on the straight line).
(5) The 3 planes meet in pairs, in 3 straight lines - diagram (e) – no solutions.
(6) Two of the planes are coincident, cut by a third distinct plane – infinite number of solutions (points on a straight line).
(7) All 3 planes are coincident – infinite number of solutions (points in the plane).



(a) (b) (c) (d)

Examples

① Solve by Gaussian Elimination

$$x+y+z=1$$
$$2x+2y+z=3$$
$$2x+2y+2z=2$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \\ 2 & 2 & 2 & 2 \end{array}\right)\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{c} \\ R_2-2R_1 \\ R_3-2R_1 \end{array}\left(\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The last equation is **redundant.**

The system has an **infinite number of solutions**

$$-z=1 \qquad x+y+z=1$$
$$z=-1 \qquad x+y-1=1$$
$$x+y=2$$

You should see that equation 3 is exactly double equation 1 so these represent identical planes.
Here we have 2 coincident planes which intersect with the third along a straight line.
The general solution can be given in parametric form.
Let $y=\lambda$. $\quad x+\lambda=2$

$$\Rightarrow x=2-\lambda$$

So the parametric solution is $x=2-\lambda$, $y=\lambda$, $z=-1$.

② Solve by Gaussian Elimination

$$x+y+z=-1$$
$$2x-2y+z=5$$
$$3x-y+2z=4$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & -1 \\ 2 & -2 & 1 & 5 \\ 3 & -1 & 2 & 4 \end{array}\right)\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{c} \\ 2R_1-R_2 \\ 3R_1-R_3 \end{array}\left(\begin{array}{ccc:c} 1 & 1 & 1 & -1 \\ 0 & 4 & 1 & -7 \\ 0 & 4 & 1 & -7 \end{array}\right)\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{c} \\ \\ R_3 - R_2 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 4 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This last equation is **redundant**.
This system has an **infinite number of solutions**.
The 3 planes intersect in one line.
The general solution can be given in parametric form.

Let $z = \lambda$

$$4y + \lambda = -7$$
$$y = \frac{-7-\lambda}{4}$$

$$x + y + z = -1$$
$$x + \frac{-7-\lambda}{4} + \lambda = -1$$
$$4x - 7 - \lambda = -4 - 4\lambda$$
$$4x = 3 - 3\lambda$$
$$x = \frac{3(1-\lambda)}{4}$$

The parametric solution is $x = \frac{3(1-\lambda)}{4}$, $y = \frac{-7-\lambda}{4}$, $z = \lambda$.

③ Solve by Gaussian Elimination

$$x + 2y + 3z = -2$$
$$2x + 4y + 6z = -4$$
$$3x + 6y + 9z = -6$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 2 & 4 & 6 & -4 \\ 3 & 6 & 9 & -6 \end{array}\right)\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}\left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This has an **infinite number of solutions**.
The second and third equations are **redundant**.
You should see that equation 2 is exactly double equation 1 and equation 3 is exactly 3 times equation 1 so these represent 3 coincident planes.
The solutions are all the points on the plane $x + 2y + 3z = -2$.

④ Solve by Gaussian Elimination

$$3x - y + z = 1$$
$$-x + 2y + 3z = 0$$
$$3x - y + z = 3$$

$$\begin{pmatrix} 3 & -1 & 1 & \vdots 1 \\ -1 & 2 & 3 & \vdots 0 \\ 3 & -1 & 1 & \vdots 3 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$\begin{matrix} \\ R_1 + 3R_2 \\ R_3 - R_1 \end{matrix} \begin{pmatrix} 3 & -1 & 1 & \vdots 1 \\ 0 & 5 & 10 & \vdots 1 \\ 0 & 0 & 0 & \vdots 2 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

There are **<u>no solutions</u>** to this system of equations.
The equations are **<u>inconsistent</u>.**
The first and third planes are parallel so will never intersect.

⑤ Solve by Gaussian Elimination

$$x + y + z = -1$$
$$2x - 2y + z = 5$$
$$3x - y + 2z = -1$$

$$\begin{pmatrix} 1 & 1 & 1 & \vdots -1 \\ 2 & -2 & 1 & \vdots\ 5 \\ 3 & -1 & 2 & \vdots -1 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$\begin{matrix} \\ 2R_1 - R_2 \\ 3R_1 - R_3 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & \vdots -1 \\ 0 & 4 & 1 & \vdots -7 \\ 0 & 4 & 1 & \vdots -2 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$\begin{matrix} \\ \\ R_3 - R_2 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & \vdots -1 \\ 0 & 4 & 1 & \vdots -7 \\ 0 & 0 & 0 & \vdots\ 5 \end{pmatrix}$$

This system of equations is **<u>inconsistent</u>**.
There are **<u>no solutions</u>**.
None of the planes are parallel so the planes intersect in 3 straight parallel lines - as in diagram (e).

⑥ Solve by Gaussian Elimination

$$-3x+3y-6z=7$$
$$x-y+2z=1$$
$$2x-2y+4z=2$$

$$\left(\begin{array}{ccc:c} -3 & 3 & -6 & 7 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 2 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ 3R_2+R_1 \\ 3R_3+2R_1 \end{array}\left(\begin{array}{ccc:c} -3 & 3 & -6 & 7 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

There are **<u>no solutions</u>** to this system of equations.
The equations are **<u>inconsistent</u>.**
The second and third planes are coincident and the first plane is parallel to the other two.

⑦ Solve by Gaussian Elimination

$$x+2y+3z=-2$$
$$x+2y+3z=0$$
$$x+2y+3z=1$$

$$\left(\begin{array}{ccc:c} 1 & 2 & 3 & -2 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 1 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ R_2-R_1 \\ R_3-R_1 \end{array}\left(\begin{array}{ccc:c} 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

There are **<u>no solutions</u>** to this system of equations. The equations are **<u>inconsistent</u>.**
The three planes are all parallel.

Another type of question:

⑧ The following system of equations has no solutions. What is the value of k?

$$x+2y-z=8$$
$$3x+y+2z=-1$$
$$x+y+kz=-6$$

$$\left(\begin{array}{ccc:c} 1 & 2 & -1 & 8 \\ 3 & 1 & 2 & -1 \\ 1 & 1 & k & -6 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ 3R_1-R_2 \\ R_3-R_1 \end{array}\left(\begin{array}{cccccc} 1 & 2 & -1 & \vdots & 8 \\ 0 & 5 & -5 & \vdots & 25 \\ 0 & -1 & k+1 & \vdots & -14 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ \\ 5R_3+R_2 \end{array}\left(\begin{array}{ccccc} 1 & 2 & -1 & \vdots & 8 \\ 0 & 5 & -5 & \vdots & 25 \\ 0 & 0 & 5k & \vdots & -45 \end{array}\right)$$

This has no solutions when $5k=0$.

$$\therefore\ k=0$$

⑨ Find the value of k which makes the system of equations have infinitely many solutions

$$\begin{aligned} x+y+z&=1 \\ 2x+3y+2z&=-1 \\ x-y+kz&=7 \end{aligned}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 1 \\ 2 & 3 & 2 & \vdots & -1 \\ 1 & -1 & k & \vdots & 7 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ 2R_1-R_2 \\ R_1-R_3 \end{array}\left(\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 0 & \vdots & 3 \\ 0 & 2 & 1-k & \vdots & -6 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ \\ 2R_2+R_3 \end{array}\left(\begin{array}{ccccc} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 0 & \vdots & 3 \\ 0 & 0 & 1-k & \vdots & 0 \end{array}\right)$$

This has infinitely many solutions when $1-k=0 \therefore\ k=1$.

Questions

① For each of these systems of equations, use Gaussian Elimination to reduce the system to upper triangular form.

- Where possible, find the unique solution.
- Where there is inconsistency, state that there are no solutions.
- Where there is a redundant equation, find a general solution in parametric form.

(a) $\begin{aligned} 2x-y+z&=1 \\ 3x+2y-4z&=4 \\ -6x+3y-3z&=2 \end{aligned}$

(b) $\begin{aligned} x-y+2z&=-3 \\ 4x+4y-2z&=1 \\ -2x+2y-4z&=6 \end{aligned}$

(c) $\begin{aligned} x-2y-6z&=12 \\ 2x+4y+12z&=-17 \\ x-4y-12z&=22 \end{aligned}$

(d) $x-3y+z=4$
$2x-8y+8z=-2$
$-6x+3y-15z=9$

(e) $x-3y=-7$
$-3x+10y+z=23$
$4x-10y+2z=-24$

(f) $x+y-z=4$
$2x-y+2z=-2$
$x-3y-4z=-1$

② A parabola passes through the points $(1,2)$, $(2,5)$ and $(-1,8)$.
Form a system of equations and solve it to find the equation of the parabola.

③ For the system of equations:

$$x+2y+z=60$$
$$2x+3y+z=85$$
$$3x+y+pz=105$$

find the value of p such that there is inconsistency and hence no solutions.

④ For what values of a and b will the system of equations

$$2x+y-3z=5$$
$$x+2y+3z=1$$
$$2x-y+az=b$$

(a) have no solutions?
(b) have infinitely many solutions?

Ill-Conditioning

Consider this 2 × 2 system.

$$x+y=2$$
$$x+0\cdot 99y=1\cdot 99$$

This system has the exact solution $x=1,\ y=1$.
Now make a small change to the y-coefficient in the second equation:

$$x+y=2$$
$$x+1\cdot 01y=1\cdot 99$$

The exact solution to this system is $x=3,\ y=-1$.
This may not look like a huge rise but a change of around 1% in a single coefficient has produced changes of several hundred % in the exact solution.
This is known as **ill-conditioning,** where a small change in the input data produces a much larger change in the computed solution.

Geometrically, this occurs when the two lines are nearly coincident and they intersect at a very small angle. In fact, if the coefficient of y in the second equation becomes 1 exactly, then we have two parallel lines and no solution at all.
Ill-conditioning means that our % error is very high and so we can have little confidence in the solutions obtained.

Examples

① Is this system ill-conditioned?

$$x+3y=5$$
$$x+3\cdot01y=5\cdot01$$

$$\left(\begin{array}{cc:c} 1 & 3 & 5 \\ 1 & 3\cdot01 & 5\cdot01 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\begin{array}{l} \\ R_2-R_1 \end{array}\left(\begin{array}{cc:c} 1 & 3 & 5 \\ 0 & 0\cdot01 & 0\cdot01 \end{array}\right)$$

$0\cdot01y=0\cdot01 \qquad x+3y=5$
$y=1 \qquad x=2$

Now make a small change.

$$x+3y=5$$
$$x+2\cdot99y=5\cdot01$$

$$\left(\begin{array}{cc:c} 1 & 3 & 5 \\ 1 & 2\cdot99 & 5\cdot01 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\begin{array}{l} \\ R_2-R_1 \end{array}\left(\begin{array}{cc:c} 1 & 3 & 5 \\ 0 & -0\cdot01 & 0\cdot01 \end{array}\right)$$

$-0\cdot01y=0\cdot01 \qquad x+3y=5$ This system **is ill-conditioned** as a very small change in
$y=-1 \qquad x=8$ the input data has produced a large change in the solution.

② Is this system ill-conditioned?

$$2x+9y=17$$
$$3x-5y=7$$

$$\left(\begin{array}{cc:c} 2 & 9 & 17 \\ 3 & -5 & 7 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$3R_1 - 2R_2 \left(\begin{array}{cc:c} 2 & 9 & 17 \\ 0 & 37 & 37 \end{array}\right)$$

$$37y = 37 \qquad 2x + 9y = 17$$
$$y = 1 \qquad x = 4$$

Now make a small change.

$$2x + 9y = 17$$
$$3x - 5\cdot 01y = 7$$

$$\begin{array}{c} \left(\begin{array}{cc:c} 2 & 9 & 17 \\ 3 & -5\cdot 01 & 7 \end{array}\right) \end{array} \begin{array}{l} R_1 \\ R_2 \end{array}$$

$$3R_1 - 2R_2 \left(\begin{array}{cc:c} 2 & 9 & 17 \\ 0 & 37\cdot 02 & 37 \end{array}\right)$$

$$37\cdot 02y = 37 \qquad 2x + 9y = 17$$
$$y = 0\cdot 999 \qquad x = 4\cdot 005$$

This system is **not ill-conditioned** as a very small change in the input data has produced a very small change in the solution.

Questions

Are these systems ill-conditioned?

(a) $3x + 4y = 15$
$3\cdot 01x + 4y = 15\cdot 01$

(b) $7x + 5y = 2$
$4x - 3y = 13$

(c) $400x - 201y = 200$
$-800x + 401y = -200$

(d) $2x + 7y = 5$
$3x + 10y = 6$

(e) $2x + 9y = 17$
$3x - 5y = 7$

(f) $9x + 8y = 1$
$8x + 7y = 1$

Past Paper Questions

2001 – A1

Use Gaussian Elimination to solve the following system of equations.

$$x+y+z=10$$
$$2x-y+3z=4$$
$$x+2z=20$$

(5 marks)

2002 – Q1

Use Gaussian Elimination to solve the following system of equations.

$$x+y+3z=2$$
$$2x+y+z=2$$
$$3x+2y+5z=5$$

(5 marks)

2003 – Q6

Use elementary row operations to reduce the following system of equations to upper triangular form.

$$x+y+3z=1$$
$$3x+ay+z=1$$
$$x+y+z=-1$$

Hence express x, y and z in terms of the parameter a.

Explain what happens when $a=3$. **(2, 2, 2 marks)**

2005 – Q6

Use Gaussian Elimination to solve the system of equations below when $\lambda \neq 2$.

$$x+y+2z=1$$
$$2x+\lambda y+z=0$$
$$3x+3y+9z=5$$

Explain what happens when $\lambda=2$. **(4, 2 marks)**

2006 – QQ9

Use Gaussian elimination to obtain solutions of the equations

$$2x-y+2z=1$$
$$x+y-2z=2$$
$$x-2y+4z=-1$$

(5 marks)

2009 – Q16a

Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x+y-z&=6 \\ 2x-3y+2z&=2 \\ -5x+2y-4z&=1 \end{aligned}$$

(5 marks)

2010 – Q14a

Use Gaussian elimination to show that the set of equations

$$\begin{aligned} x-y+z&=1 \\ x+y+2z&=0 \\ 2x-y+az&=2 \end{aligned}$$

has a unique solution when $a \neq 2\cdot5$.

Explain what happens when $a = 2\cdot5$.

Obtain the solution when $a = 3$.

(5, 1, 1 marks)

2012 – Q14

Use Gaussian elimination to obtain the solution of the following system of equations in terms of the parameter λ.

$$\begin{aligned} 4x+6z&=1 \\ 2x-2y+4z&=-1 \\ -x+y+\lambda z&=2 \end{aligned}$$

Describe what happens when $\lambda = -2$.

When $\lambda = -1\cdot9$ the solution is $x = -22\cdot5$, $y = 8\cdot25$, $z = 15$.

Find the solution when $\lambda = -2\cdot1$.

Comment on these solutions. **(5, 1, 2, 1 marks)**

2014 – Q3

Use Gaussian elimination on the system of equations below to give an expression for z in terms of λ.

$$\begin{aligned} x+y+z&=2 \\ 4x+3y-\lambda z&=4 \\ 5x+6y+8z&=11 \end{aligned}$$

For what values of λ does this system have a solution?

Determine the solution to this system of equations when $\lambda = 2$. **(6 marks)**

<u>2016 – Q4</u>

Below is a system of equations.

$$x+2y+3z=3$$
$$2x-y+4z=5$$
$$x-3y+2\lambda z=2$$

Use Gaussian elimination to find the value of λ which leads to redundancy. **(4 marks)**

<u>2017 – Q5</u>

(a)(i) Use Gaussian elimination on the system of equations below to give an expression for *z* in terms of λ.

$$x+2y-z=-3$$
$$4x-2y+3z=11$$
$$3x+y+2\lambda z=8$$

(ii) For what value of λ is this system of equations inconsistent?

(b) Determine the solution of this system when $\lambda=-2.5$

(4,1,1 marks)