Vectors

Reminders from Higher

Position Vector $\underline{p} = (x, y, z)$ starts at the origin and is denoted $\overrightarrow{OP}$.

A unit vector has a magnitude (length) of 1. $|p| = \sqrt{x^2 + y^2 + z^2}$.

Scalar Product $\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$
$= |\underline{a}||\underline{b}|\cos\theta$

Unit Vectors $\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

The Vector Product

The scalar product is where two vectors multiply to produce a number (scalar).
The vector product is where two vectors multiply to produce a vector.

The vector product (or cross product) is denoted $\underline{a} \times \underline{b}$

$$\boxed{\underline{a} \times \underline{b} = \underline{n}|\underline{a}||\underline{b}|\sin\theta}$$

Where θ is the angle between the positive directions of $\underline{a}$ and $\underline{b}$.
If we use our right hand $\underline{a} =$ 1st finger
$\underline{b} =$ 2nd finger
$\underline{n} =$ thumb $\rightarrow$ perpendicular (normal) to the plane.

If $\underline{a} = 0$ or $\underline{b} = 0$, n is not defined $\rightarrow \underline{a} \times \underline{b} = 0$

Properties
$\underline{a} \times \underline{b}$ is a vector and is perpendicular to the plane contained by $\underline{a}$ & $\underline{b}$.
$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$ (non commutative).
$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\theta$

In component form if $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ & $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$\boxed{= (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}}$$

Examples

① If $\underset{\sim}{a} = \begin{pmatrix} 4 \\ -5 \\ -6 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$ find $\underset{\sim}{a} \times \underset{\sim}{b}$.

$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{b} &= (a_2b_3 - a_3b_2)\underset{\sim}{i} - (a_1b_3 - a_3b_1)\underset{\sim}{j} + (a_1b_2 - a_2b_1)\underset{\sim}{k} \\
&= (-5\times3 - (-6)\times(-2))\underset{\sim}{i} - (4\times3 - (-6)\times(-1))\underset{\sim}{j} + (4\times(-2) - (-5)\times(-1))\underset{\sim}{k} \\
&= -27\underset{\sim}{i} - 6\underset{\sim}{j} - 13\underset{\sim}{k} \\
&= \begin{pmatrix} -27 \\ -6 \\ -13 \end{pmatrix}
\end{aligned}$$

② Find $\underset{\sim}{a} \times \underset{\sim}{b}$ for the following:

(a) $\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j}$

(b) $\underset{\sim}{a} = -5\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

(c) $\underset{\sim}{a} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - 4\underset{\sim}{j} - 8\underset{\sim}{k}$

(a)
$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{b} &= ((-1)(0) - (3)(2))\underset{\sim}{i} - ((-2)(0) - (1)(3))\underset{\sim}{j} + ((-2)(2) - (-1)(1))\underset{\sim}{k} \\
&= -6\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k} \\
&= \begin{pmatrix} -6 \\ 3 \\ -3 \end{pmatrix}
\end{aligned}$$

(b)
$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{b} &= ((-3)(-1) - (1)(4))\underset{\sim}{i} - ((-5)(-3) - (2)(4))\underset{\sim}{j} + ((-5)(1) - (2)(-1))\underset{\sim}{k} \\
&= -\underset{\sim}{i} - 7\underset{\sim}{j} - 3\underset{\sim}{k} \\
&= \begin{pmatrix} -1 \\ -7 \\ -3 \end{pmatrix}
\end{aligned}$$

(c)
$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{b} &= ((2)(-8) - (-4)(4))\underset{\sim}{i} - ((-1)(-8) - (4)(2))\underset{\sim}{j} + ((-1)(-4) - (2)(2))\underset{\sim}{k} \\
&= 0
\end{aligned}$$

$\therefore \underset{\sim}{a}\ \&\ \underset{\sim}{b}$ are parallel

Questions

(a) Find a vector perpendicular to $\underline{u} = 2\underline{i} - \underline{j} + 3\underline{k}$ & $\underline{v} = \underline{i} + 2\underline{j}$.

(b) Find the <u>length</u> of $\underline{a} \times \underline{b}$, given that $\underline{a}$ has length 4, $\underline{b}$ has length 5,& the angle between $\underline{a}$ & $\underline{b}$ is 45°.

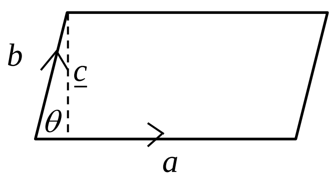
(c) Given that $\underline{a} = 3\underline{i} - \underline{j} + 2\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} - \underline{k}$ & $\underline{c} = \underline{i} - 2\underline{j} + 2\underline{k}$ find i) $(\underline{a} \times \underline{b}) \times \underline{c}$; ii) $\underline{a} \times (\underline{b} \times \underline{c})$. Comment on your results.

(d) Determine two unit vectors perpendicular to the plane of $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ & $\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$.

(e) $\underline{a} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ & $\underline{b} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$. Find i) $\underline{a} \times \underline{b}$ ii) $\underline{b} \times \underline{a}$ iii) $(\underline{a} + \underline{b}) \times (\underline{a} - \underline{b})$.

Areas of Triangles and Parallelograms

The length of the vector product $\underline{a} \times \underline{b}$ has the same formula as the area of the parallelogram in which the vectors $\underline{a}$ & $\underline{b}$ are the sides of the parallelogram as demonstrated below.



Area of Parallelogram = $|\underline{a}||\underline{c}|$

$$|\underline{c}| = |\underline{b}|\sin\theta$$

$$\therefore \quad = |\underline{a}||\underline{b}|\sin\theta \quad \text{Area}$$

$$= |\underline{a} \times \underline{b}|$$

Triangles are similar, but obviously their area is given by $\frac{1}{2}|\underline{a} \times \underline{b}|$

Examples

① Find the area of the parallelogram which has adjacent edges $\underline{u} = 3\underline{i} + \underline{j} + 2\underline{k}$ & $\underline{v} = 2\underline{i} - \underline{j}$.

$$|\underline{u} \times \underline{v}| = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & 2 \\ 2 & -1 & 0 \end{vmatrix}$$

$$= (0 - (-2))\underline{i} - (0 - 4)\underline{j} + (-3 - 2)\underline{k}$$

$$= 2\underline{i} + 4\underline{j} - 5\underline{k}$$

$$|\underline{u} \times \underline{v}| = \sqrt{2^2 + 4^2 + (-5)^2}$$

$= 6.7$ sq. units

② Calculate the area of the triangle whose vertices are $A(-2,1,3)$, $B(5,-1,2)$ & $C(2,3,4)$.

$$\overrightarrow{AB} = \begin{pmatrix} 7 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 7 & -2 & -1 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= (-2-(-2))\underline{i} - (7-(-4))\underline{j} + (14-(-8))\underline{k}$$

$$= -11\underline{j} + 22\underline{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}\left|-11\underline{j} + 22\underline{k}\right|$$

$$= \frac{1}{2}\sqrt{121+484}$$

$= 12.3$ sq. units

<u>Questions</u>

(a) Find the area of the parallelogram whose adjacent sides have the vectors
$\underline{u} = -\underline{i} + 3\underline{j} + 4\underline{k}$ & $\underline{v} = 3\underline{i} + \underline{j} - \underline{k}$.

(b) Find the area of the triangle whose vertices are:

i) $A(2,-1,4)$, $B(-1,0,2)$ & $C(4,4,0)$.

ii) $X(-3,1,1)$, $Y(1,-1,0)$ & $Z(2,0,3)$.

<u>Scalar Triple Product</u>

The scalar triple product, denoted $\underline{a}.(\underline{b} \times \underline{c})$ gives the volume of the parallelepiped in 3D space, bounded by 3 pairs of parallel planes. It is a number (scalar) not a vector and is given by:

$$\boxed{\underline{a}.(\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

In component form

$$\boxed{\underline{a}.(\underline{b} \times \underline{c}) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_3)}$$

<u>Examples</u>

① Given $\underline{a} = 2\underline{i} + \underline{j} + 3\underline{k}$, $\underline{b} = 5\underline{i} + 3\underline{j} - 2\underline{k}$, $\underline{c} = -\underline{i} + 2\underline{j} + 4\underline{k}$; Find

i) $\underline{a}.(\underline{b} \times \underline{c})$

$$\underline{a}.(\underline{b} \times \underline{c}) = \begin{vmatrix} 2 & 1 & 3 \\ 5 & 3 & -2 \\ -1 & 2 & 4 \end{vmatrix}$$

$$= 2(12-(-4)) - 1(20-2) + 3(10-(-3))$$

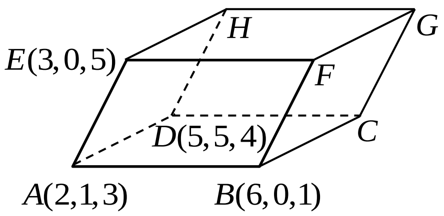
$$= 32 - 18 + 39$$

$$= 53$$

ii) $(\underline{a}\times\underline{b}).\underline{c}$

$$\underline{c}.(\underline{a}\times\underline{b}) = \begin{vmatrix} -1 & 2 & 4 \\ 2 & 1 & 3 \\ 5 & 3 & -2 \end{vmatrix}$$
$$= -1(-2-9)-2(-4-15)+4(6-5)$$
$$= 11+38+4$$
$$= 53$$

② Calculate the volume of the parallelepiped shown.



Let $\underline{a} = \overrightarrow{AE}$, $\underline{b} = \overrightarrow{AD}$, $\underline{c} = \overrightarrow{AB}$ $\Rightarrow$ $\underline{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ & $\underline{c} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$

$$\underline{a}.(\underline{b}\times\underline{c}) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & 1 \\ 4 & -1 & -2 \end{vmatrix}$$
$$= 1(-8-(-1))+1(-6-4)+2(-3-16)$$
$$= (-)55$$

Volume = 55 units^3

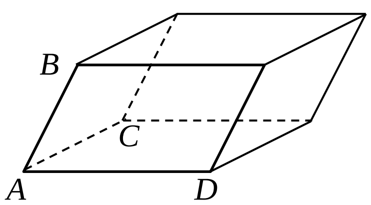
Questions

(a) Given $\underline{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\underline{s} = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$, $\underline{t} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$ & $\underline{u} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ find:

i) $\underline{r}.(\underline{s}\times\underline{t})$

ii) $\underline{s}.(\underline{t}\times\underline{u})$ iii) $\underline{u}.(\underline{s}\times\underline{r})$

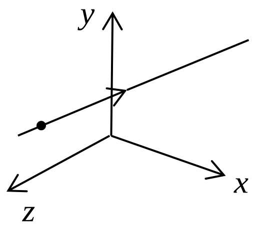
(b) Calculate the volume of the parallelepiped.



$A(1,1,2)$
$B(2,3,6)$
$C(4,4,3)$
$D(5,3,4)$

Lines in 3D Space

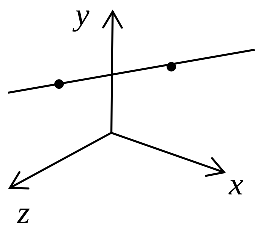
A line in 3D space can be defined in 2 ways…



1 point & a direction

OR

2 points



There are three ways of expressing this line:

Vector Form $\quad \underline{p} = \underline{a} + \lambda \underline{u}$ where λ is a scalar

$\underline{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ (point) & $\underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ (direction vector)

Parametric Form

$$x = x_1 + a\lambda$$
$$y = y_1 + b\lambda$$
$$z = z_1 + c\lambda$$

Symmetric Form $\quad \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = \lambda$

Examples

① Write down the symmetric and parametric form of the equation of a line which passes through $(1,-2,8)$ and is parallel to $3\underline{i} + 5\underline{j} + 11\underline{k}$. Determine whether or not the point $(-2,-7,-3)$ lies on the line.

Symmetric - $\dfrac{x-1}{3} = \dfrac{y-(-2)}{5} = \dfrac{z-8}{11} = \lambda \quad \Rightarrow \quad \dfrac{x-1}{3} = \dfrac{y+2}{5} = \dfrac{z-8}{11} = \lambda$

Parametric -

$$x = 1 + 3\lambda$$
$$y = -2 + 5\lambda$$
$$z = 8 + 11\lambda$$

If the point lies on the line, subbing $(-2,-7,-3)$ into the equation will give same value for λ.

$\dfrac{-2-1}{3} = -1 \qquad \dfrac{-7+2}{5} = -1 \qquad \dfrac{-3-8}{11} = -1 \qquad$ Point lies on line.

Consistent results.

②Find the symmetric equation of the line passing through the points $A(4,1,3)$ & $B(2,5,-2)$.

For direction vector find $\overrightarrow{AB}$. $\overrightarrow{AB}=\begin{pmatrix}-2\\4\\-5\end{pmatrix}$

Then, in symmetric form

$$\frac{x-4}{-2}=\frac{y-1}{4}=\frac{z-3}{-5}=\lambda$$

Questions

Find the symmetric and parametric equations of the lines defined as follows.

(a) Passing through $(4,2,1)$ with direction vector $\underline{i}+\underline{j}+3\underline{k}$.

(b) Passing through the points $(2,1,4)$ & $(5,5,5)$.

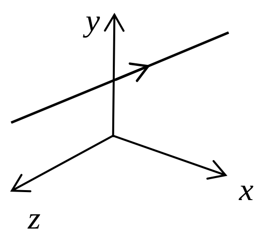
(c) Passing through $(3,2,7)$ with direction vector $3\underline{i}+\underline{4j}-\underline{k}$.

(d) Passing through the points $(0,a,0)$ & $(a,0,2a)$.

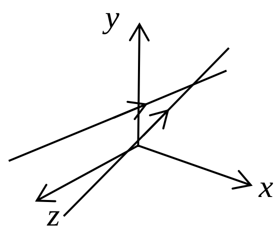
(e) Find the vector equation of the line with direction vector $\underline{d}=(1,2,-8)^T$ and point $P(2,-1,6)$. Change this vector equation into symmetric form.

Intersection of, & Angle Between, Two Lines

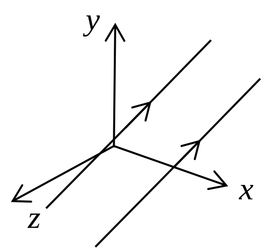
Lines in 3D space can



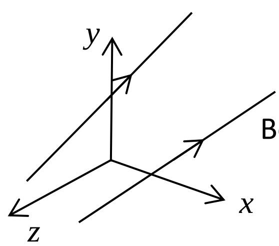
Be identical



Intersect



Be Parallel



Be Skew

Lines are parallel if their directions are proportional (one is a multiple of the other).

If the lines intersect we should be able to calculate the point of intersection and the acute angle between them. (Subtract from 180° if an obtuse angle is first obtained.)

Examples

①Show that the lines $\frac{x+10}{1}=\frac{y+20}{3}=\frac{z-15}{-2}$ & $\frac{x-18}{5}=\frac{y+12}{-4}=\frac{z-11}{3}$ intersect and find the angle between them.

Let $\frac{x+10}{1}=\frac{y+20}{3}=\frac{z-15}{-2}=\lambda$ $\qquad\frac{x-18}{5}=\frac{y+12}{-4}=\frac{z-11}{3}=\mu$

This gives

$$\begin{aligned}x&=-10+\lambda\\y&=-20+3\lambda\\z&=15-2\lambda\end{aligned}\qquad\qquad\begin{aligned}x&=18+5\mu\\y&=-12-4\mu\\z&=11+3\mu\end{aligned}$$

Equate these

$\underline{x}$

$$\begin{aligned}-10+\lambda&=18+5\mu\\\lambda&=28+5\mu\end{aligned}$$

$\underline{y}$

$$\begin{aligned}-20+3\lambda&=-12-4\mu\\3\lambda&=8-4\mu\end{aligned}$$

$\underline{z}$

$$\begin{aligned}15-2\lambda&=11+3\mu\\2\lambda&=4-3\mu\end{aligned}$$

Taking $\lambda=28+5\mu$ & $2\lambda=4-3\mu$.

$2\lambda=56+10\mu$ Equating gives

$$\begin{aligned}56+10\mu&=4-3\mu\\13\mu&=-52\\\mu&=-4\end{aligned}$$

Sub into x

$$\begin{aligned}\lambda&=28+5\times(-4)\\\lambda&=28-20\\\lambda&=8\end{aligned}$$

Check in y and z for consistency.

$$\begin{array}{ll}3\lambda & 8-4\mu\\=24 & =24\end{array}\qquad\qquad\begin{array}{ll}2\lambda & 4-3\mu\\=16 & =16\end{array}$$

Consistent in y $\qquad$ Consistent in z

Point of Intersection

Using λ (we could equally use μ.)

$$\begin{aligned}x&=-10+8\\&=-2\end{aligned}\qquad\begin{aligned}y&=-20+24\\&=4\end{aligned}\qquad\begin{aligned}z&=15-16\\&=-1\end{aligned}$$

$\therefore$ The Point of Intersection is $(-2,4,-1)$.

<u>The Angle…</u>
Find the direction vectors of both lines and calculate the acute angle between them.

$\cos\theta = \dfrac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$ $\qquad \underline{a} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \underline{b} = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$ from denominators of symmetric form.

$$\cos\theta = \frac{5-12-6}{\sqrt{14}\sqrt{50}}$$
$$= \frac{-13}{10\sqrt{7}}$$
$$\theta = 60.57^\circ$$

We may also have to prove that two lines do not intersect. In this case there will be no consistent solutions for λ and μ.

② Determine whether the lines $L_1 \ \dfrac{x-4}{3} = \dfrac{y}{-1} = \dfrac{z-2}{-1}$ and $L_2 \ x = \mu,\ y = \mu,\ z = 3 + \mu$ intersect.
Write both in the same form.

$L_1 \quad x = 4 + 3\lambda$, $y = -\lambda$, $z = 2 - \lambda$

If they intersect then $\mu = 4 + 3\lambda$, $\mu = -\lambda$, $3 + \mu = 2 - \lambda$

Equating the first two

$$-\lambda = 3\lambda + 4$$
$$-4\lambda = 4$$
$$\lambda = -1$$

Sub into 2nd

$$\mu = 1$$

Sub into 3rd

$$3 + \mu = 2 - (-1)$$
$$\mu = 0$$

Inconsistent
$\therefore$ no solns
Lines do not
Intersect.

③ Show that the lines L_1 and L_2 intersect & calculate the angle between them.

$L_1 \Rightarrow \dfrac{x+4}{2} = \dfrac{y-5}{-4} = \dfrac{z-3}{1} = \lambda$ $\qquad L_2 \Rightarrow \dfrac{x}{-1} = \dfrac{y-3}{-1} = \dfrac{z-2}{1} = \mu$

Parametric form…

$x = -4 + 2\lambda$, $y = 5 - 4\lambda$, $z = 3 + \lambda$ $\qquad x = -\mu$, $y = 3 - \mu$, $z = 2 + \mu$

Equating…

$-4+2\lambda=-\mu$ $\quad\mu=4-2\lambda$

$5-4\lambda=3-\mu$ $\quad\mu=4\lambda-2$

$3+\lambda=2+\mu$ $\quad\mu=1+\lambda$

Equating 1st two… $4-2\lambda=4\lambda-2$, $6\lambda=6$, $\lambda=1$

Sub into 2nd $\mu=4-2$ $=2$

Sub into 3rd $\mu=1+1$ $=2$

$\therefore \lambda=1, \mu=2$

So $x=-2, y=1, z=4$ $\therefore$ Point of Intersection $(-2,1,4)$

Direction Vectors

$$\underset{\sim}{a}=\begin{pmatrix}2\\-4\\1\end{pmatrix}, \underset{\sim}{b}=\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$$\cos\theta=\frac{\underset{\sim}{a}.\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$=\frac{-2+4+1}{\sqrt{21}\sqrt{3}}$$

$$=\frac{3}{\sqrt{63}}$$

$$\theta=67.8^\circ$$

Questions

(a) Find the intersection of & the angle between the lines L_1 & L_2.

$$L_1 \Rightarrow \frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{1}=\lambda \qquad L_2 \Rightarrow \frac{x+1}{2}=\frac{y-2}{3}=\frac{z-7}{-1}=\mu$$

(b) Determine whether the two lines L_1 & L_2 intersect when

$$L_1 \Rightarrow \frac{x+3}{1}=\frac{y-2}{1}=\frac{z-1}{3}=\lambda \qquad L_2 \Rightarrow x=-4-2\mu,\ y=1+\mu,\ z=\mu.$$

(c) Find the intersection of and the angle between the line L_1 with parametric equations $x=-2+2\lambda,\ y=1-3\lambda,\ z=-1+\lambda,$ and the line L_2 which passes through the point $(-3,4,0)$ and is parallel to $-\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$.

(d) Find the point of intersection of the line through the point $(1,3,-2)$ and parallel to $4\underset{\sim}{i}+\underset{\sim}{k}$ with the line through the point $(5,3,8)$ and parallel to $-\underset{\sim}{i}+2\underset{\sim}{k}$. Then find the angle between them.

Equation of a Plane

We can identify a plane if we know:

- 3 points on the plane
- 2 lines on the plane & a point (intersection or otherwise.)
- 1 point on the plane & a normal to the plane.

When 3 points are known…

① Find the equation of the plane which passes through the points $A(-2,1,2)$, $B(0,2,5)$ and $C(2,0,3)$.

$$\overrightarrow{AB}=\begin{pmatrix}2\\1\\3\end{pmatrix},\overrightarrow{AC}=\begin{pmatrix}4\\-1\\1\end{pmatrix}\Rightarrow \qquad \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\2 & 1 & 3\\4 & -1 & 1\end{vmatrix}$$

$$=(1-(-3))\underline{i}-(2-12)\underline{j}+(-2-4)\underline{k}$$
$$=4\underline{i}+10\underline{j}-6\underline{k}$$

$$\therefore \underline{n}=\begin{pmatrix}4\\10\\-6\end{pmatrix}$$

Equation of Plane $\quad \underline{p}.\underline{n}=\underline{p}_o.\underline{n}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}4\\10\\-6\end{pmatrix}=\begin{pmatrix}-2\\1\\2\end{pmatrix}.\begin{pmatrix}4\\10\\-6\end{pmatrix}$$
$$4x+10y-6z=-8+10-12$$
$$4x+10y-6z=-10$$
$$2x+5y-3z=-5$$

② Find the equation of the plane which passes through the points $M(1,0,1)$, $N(1,1,1)$ and $Q(2,1,-1)$. Determine whether or not the point $(1,5,1)$ lies on the plane.

$$\overrightarrow{MN}=\begin{pmatrix}0\\1\\0\end{pmatrix},\overrightarrow{MQ}=\begin{pmatrix}1\\1\\-2\end{pmatrix}\Rightarrow$$

$$\overrightarrow{MN} \times \overrightarrow{MQ} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{vmatrix} \qquad \therefore \underline{n} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$

$$= -2\underline{i} - \underline{k}$$

Equation of Plane $\underline{p}.\underline{n} = \underline{p}_o.\underline{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$

$$-2x - z = -3$$
$$2x + z = 3$$

Considering the point $(1,5,1) \Rightarrow 2x + z$

$$= 2\times 1 + 1$$
$$= 3$$

Point satisfies Equation of the Plane
$\therefore$ Point lies on the Plane.

Questions

(a) Find the equation of the plane through the points:

i) $O(0,0,0)$, $B(1,2,1)$, $C(-2,1,2)$. ii) $G(-1,1,0)$, $H(3,3,3)$, $I(2,-1,2)$.

(b) Find the equation of the plane through $A(-1,3,1)$, $B(1,-3,-3)$, $C(3,-1,5)$. Verify that the point $(0,2,2)$ also lies on the plane.

(c) Prove that the 4 points below are coplanar.

$$S(5,7,-1),\ T(2,-3,6),\ U(1,-4,7),\ V(6,1,2).$$

When 2 lines and a point are known…

1 A plane is parallel to the vectors $3\underline{i} + 2\underline{j} - \underline{k}$ and $4\underline{i} - 2\underline{k}$. The plane contains the point $(1,1,0)$. Find the equation of the plane.

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -1 \\ 4 & 0 & -2 \end{vmatrix} \qquad \therefore \underline{n} = \begin{pmatrix} -4 \\ 2 \\ -8 \end{pmatrix}$$

$$= -4\underline{i} - (-6 - (-4))\underline{j} + (-8)\underline{k}$$
$$= -4\underline{i} + 2\underline{j} - 8\underline{k}$$

Equation of Plane $\underline{p}.\underline{n} = \underline{p}_o.\underline{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} -4 \\ 2 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} . \begin{pmatrix} -4 \\ 2 \\ -8 \end{pmatrix}$$

$$-4x + 2y - 8z = -4 + 2$$
$$2x - y + 4z = 1$$

Questions

(a) Find the equation of the plane parallel to $\underline{i}-\underline{k}$ and $6\underline{j}+5\underline{k}$ passing through the point $(-2,3,7)$.

(b)There exists two lines

$$L_1 \Rightarrow \frac{x-1}{3}=\frac{y-4}{-1}=\frac{z+7}{2}=\lambda \text{ and } L_2 \Rightarrow \frac{x+4}{4}=\frac{y-3}{-1}=\frac{z-3}{1}=\mu$$

Find the point of intersection, and hence the equation for the plane containing them.

<u>When a normal and a point are known…</u>

① Find the equation for the plane through $(4,-1,3)$ with normal vector $\underline{n}=(1,5,2)^T$.

$$\underline{p}.\underline{n}=\underline{p}_o.\underline{n}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\5\\2\end{pmatrix}=\begin{pmatrix}4\\-1\\3\end{pmatrix}.\begin{pmatrix}1\\5\\2\end{pmatrix}$$

$$x+5y+2z=4-5+6$$

$$x+5y+2z=5$$

② Find the equation for the plane through $(-1,2,1)$ with normal vector $\underline{i}-3\underline{j}+2\underline{k}$.

$$\underline{p}.\underline{n}=\underline{p}_o.\underline{n}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\-3\\2\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}.\begin{pmatrix}1\\-3\\2\end{pmatrix}$$

$$x-3y+2z=-1-6+2$$

$$x-3y+2z=-5$$

Questions

Find the equation of the plane perpendicular to the vector and containing the point given.

(a) $2\underline{i}+3\underline{j}+\underline{k}$ & $(0,2,6)$

(b) $5\underline{i}+4\underline{j}-3\underline{k}$ & $(2,1,-1)$

(c) $2\underline{i}-3\underline{j}+\underline{k}$ & $(5,3,-2)$

(d) $-4\underline{i}+6\underline{j}+7\underline{k}$ & $(-4,6,7)$

Equations of a Plane in Parametric/Symmetric Form/Vector Equation of a Plane

Unlike for the equation of a line, the equation of a plane does not have distinct parametric and symmetric forms; the two words are interchangeable for the same format of an equation.

This form is $\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$

Where $\underline{a}$ is a position vector of a point on the plane.
$\underline{b}$ & $\underline{c}$ are two non-parallel vectors, each parallel to the plane.
λ & μ are real numbers.

Example

① Find the parametric equation of the plane through $(2,3,1)$ and parallel to the vectors $-2\underline{i}+3\underline{j}+\underline{k}$ and $6\underline{i}-2\underline{j}+\underline{k}$.

$$\underline{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$$

Calculating the Angle Between Two Planes

When 2 planes intersect on a line, we can calculate both the line of intersection and the acute angle between the two planes.
The angle is calculated in the same way as we calculated the angle between two lines. We take the acute angle between the two direction vectors(normal) of the planes.

Example

① Find the angle between the two planes with equations $2x+y-2z=5$ and $3x-6y-2z=7$.

Let $\underline{a} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \underline{b} = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$

$$\cos\theta = \frac{6-6+4}{\sqrt{9}\sqrt{49}}$$

$$= \frac{4}{21}$$

$\theta = 79^\circ$ (remember to check angle is acute.)

Questions

(a) Find the acute angle between the planes $2x - y = 0$ and $x + y + z = 0$.

(b) Find the angle between the planes $x + y - 4z = -1$ and $2x - 3y + 4z = -5$.

(c) The plane π_1 contains the vectors $2\underline{i} + \underline{j}$ and $3\underline{i} + 2\underline{k}$.
The plane π_2 contains the vectors $\underline{i} + 3\underline{j} - \underline{k}$ and $\underline{i} + \underline{j} - \underline{k}$.
Find the angle between the planes π_1 and π_2.

(d) Find the acute angle between the planes;
π_1, passing through $O(0,0,0)$, $P(1,0,1)$ and $Q(0,1,1)$
and π_2, passing through $A(1,-1,1)$, $B(3,1,-2)$ and $C(0,2,-1)$.

Intersection of 2 Planes - the Line of Intersection

If two planes intersect, we can use Gaussian elimination to find and equation of the line of intersection.

Example

① Find the equation of the line of intersection of the planes $\pi_1; 4x + y - 2z = 3$ and $\pi_2; x + y - z = 1$.

$$\left(\begin{array}{ccc|c} 4 & 1 & -2 & 3 \\ 1 & 1 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\begin{array}{r} \\ 4R_2 - R_1 \end{array} \left(\begin{array}{ccc|c} 4 & 1 & -2 & 3 \\ 0 & 3 & -2 & 1 \end{array}\right)$$

Let $z = \lambda$ $\qquad 3y - 2\lambda = 1 \qquad\qquad 4x + y - 2\lambda = 3$

$$y = \frac{1}{3} + \frac{2}{3}\lambda \qquad\qquad 4x = \frac{8}{3} + \frac{4}{3}\lambda$$

$$x = \frac{2}{3} + \frac{1}{3}\lambda$$

$\therefore$ Equation of line of Intersection

in Parametric Form is:

$$x = \frac{2}{3} + \frac{1}{3}\lambda$$

$$y = \frac{1}{3} + \frac{2}{3}\lambda$$

$$z = \lambda$$

Questions

Find the equations of the line of intersection of each of these pairs of planes.

(a) $x-y-3z=-7$ & $2x+3y-z=-4$ (b) $2x-y-2z-1=0$ & $x-2y-2z=-8$.

Intersection of 3 Planes

We do exactly the same thing for the intersection of 3 planes…we've been doing this for a long time!

Examples

Determine how these planes intersect.

①

$$\begin{aligned} x+2y+3z&=3 \\ 2x-y+4z&=5 \\ x-3y+2z&=2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & -1 & 4 & 5 \\ 1 & -3 & 2 & 2 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ 2R_1-R_2 \\ R_1-R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 5 & 2 & 1 \\ 0 & 5 & 1 & 1 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ \\ R_2-R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$z=0 \quad 5y=1 \quad x+\dfrac{2}{5}+0=3$

$y=\dfrac{1}{5} \qquad x=\dfrac{13}{5}$

Planes intersect at a Point.

Point of Intersection is $\left(\dfrac{13}{5},\dfrac{1}{5},0\right)$.

②

$$\begin{aligned} x+2y+3z&=3 \\ 2x-y+4z&=5 \\ x-3y+z&=2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & -1 & 4 & 5 \\ 1 & -3 & 1 & 2 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ 2R_1-R_2 \\ R_1-R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 5 & 2 & 1 \\ 0 & 5 & 2 & 1 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{array}{r} \\ \\ R_2-R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z=\lambda \qquad 5y+2\lambda=1$

$y=\dfrac{1}{5}-\dfrac{2}{5}\lambda$

$x+2\left(\dfrac{1}{5}-\dfrac{2}{5}\lambda\right)+3\lambda=3$

$x=\dfrac{13}{5}-\dfrac{11}{5}\lambda$

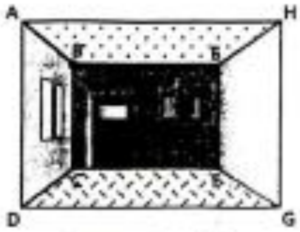
Planes intersect along a line given by

$x=\dfrac{13}{5}-\dfrac{11}{5}\lambda,\ y=\dfrac{1}{5}-\dfrac{2}{5}\lambda,\ z=\lambda$

The intersection of three planes

When examining the intersection of three planes we must consider six cases. The intersection could be:

- a single point
- a line
- two lines
- three lines
- a plane
- undefined.



Various examples can be found by considering the structure of a room, including the diagonal planes.

Since each plane has an equation of the form $ax + by + cz + d = 0$, the intersection is the solution of a 3×3 system of equations. We can use the planar equations to form an augmented matrix and solve for x, y and z using Gaussian elimination.
The six cases are best described through example.

A point of intersection

$$\begin{aligned} x - 2y + z &= 8 \\ 3x + y - z &= 1 \\ 2x - 2y + 3z &= 18 \end{aligned} \quad \begin{pmatrix} 1 & -2 & 1 & 8 \\ 3 & 1 & -1 & 1 \\ 2 & -2 & 3 & 18 \end{pmatrix} \text{ which reduces by suitable EROs to } \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

This unique solution indicates that the three planes all intersect at $(2, -1, 4)$.

A line of intersection

$$\begin{aligned} x + 2y - 2z &= -7 \\ x - 2y + z &= 6 \\ 3x + 2y - 3z &= -8 \end{aligned} \quad \begin{pmatrix} 1 & 2 & -2 & -7 \\ 1 & -2 & 1 & 6 \\ 3 & 2 & -3 & -8 \end{pmatrix} \text{ which reduces to } \begin{pmatrix} 1 & 2 & -2 & -7 \\ 0 & -4 & 3 & 13 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The bottom line yields no information and arises because the third equation is redundant. So we have two equations in three unknowns, which will yield an infinite number of solutions.

From the second equation,

$$-4y + 3z = 13 \quad \Rightarrow \quad y = \frac{3z - 13}{4}$$

It is common at this stage to let z arbitrarily equal t, but it is more convenient to let $z = 4t - 1$

$$\Rightarrow y = \frac{3(4t - 1) - 13}{4} = 3t - 4$$

Now

$$x = -7 - 2y + 2z \Rightarrow x = 2t - 1$$

The intersection is the line with equations $x = 2t - 1$, $y = 3t - 4$, $z = 4t - 1$.

or, in symmetric form: $\frac{x+1}{2} = \frac{y+4}{3} = \frac{z+1}{4}$.

Two lines of intersection

$$\begin{aligned} x - y + z &= 10 \\ 2x - y + 3z &= 5 \\ 4x - 2y + 6z &= 7 \end{aligned} \quad \begin{pmatrix} 1 & -1 & 1 & 10 \\ 2 & -1 & 3 & 5 \\ 4 & -2 & 6 & 7 \end{pmatrix} \text{ which reduces to } \begin{pmatrix} 1 & -1 & 1 & 10 \\ 0 & 1 & 1 & -15 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$0 = -3 \Rightarrow$ no solutions, the system being inconsistent.
The first two planes intersect where $y + z = -15$.
Let $z = t$, so $y = -15 - t$, and $x = 10 + y - z = 10 - 15 - t - t = -5 - 2t$. Therefore the planes meet in the line with equation

$$\frac{x+5}{-2} = \frac{y+15}{-1} = z$$

The first and third planes can be shown similarly to intersect on the line

$$\frac{x+6.5}{-2} = \frac{y+16.5}{-1} = z$$

Note

- These two lines of intersection are parallel.
- The second and third planes are parallel, having the same normal vector.

Three lines of intersection

$$\begin{aligned} x+2y-2z&=-7\\ 3x+2y-3z&=-15\\ 5x+2y-4z&=-9 \end{aligned} \quad \begin{pmatrix} 1 & 2 & -2 & -7\\ 3 & 2 & -3 & -15\\ 5 & 2 & -4 & -9 \end{pmatrix} \text{ which reduces to } \begin{pmatrix} 1 & 2 & -2 & -7\\ 0 & -4 & 3 & 6\\ 0 & 0 & 0 & 14 \end{pmatrix}$$

$0 = 14 \Rightarrow$ the system is inconsistent and so has no solutions. However, examining each pair of planes in turn gives

Plane 1 with plane 2: with $-4y + 3z = 6$ and, arbitrarily, letting $z = 4t + 2$, we get $x = 2t - 3$, $y = 3t$, $z = 4t + 2$, so

$$\frac{x+3}{2} = \frac{y}{3} = \frac{z-2}{4}$$

Plane 1 with plane 3: with $-4y + 3z = 13$ and, arbitrarily, letting $z = 4t - 1$, we get $x = 2t - 1$, $y = 3t - 4$, $z = 4t - 1$, so

$$\frac{x+1}{2} = \frac{y+4}{3} = \frac{z+1}{4}$$

Plane 2 with plane 3: subtract to eliminate y; this gives $2x - z = 6$. Arbitrarily, let $z = 4t$.

We get $x = 3 + 2t$, $y = 3t - 12$, $z = 4t$, so

$$\frac{x-3}{2} = \frac{y+12}{3} = \frac{z}{4}$$

Notice that the three lines of intersection are parallel, with direction vector $\begin{pmatrix} 2\\ 3\\ 4 \end{pmatrix}$.

A plane of intersection

$$\begin{aligned} 2x-y+3z&=4\\ 6x-3y+9z&=12\\ 8x-4y+12z&=16 \end{aligned} \quad \begin{pmatrix} 2 & -1 & 3 & 4\\ 6 & -3 & 9 & 12\\ 8 & -4 & 12 & 16 \end{pmatrix} \text{ which reduces to } \begin{pmatrix} 2 & -1 & 3 & 4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Two redundant equations. The three planes coincide. Thus the intersection is the plane $2x - y + 3z = 4$.

No intersection

$$\begin{aligned} 4x-8y+12z&=12\\ 2x-4y+6z&=2\\ 3x-6y+9z&=6 \end{aligned} \quad \begin{pmatrix} 4 & -8 & 12 & 12\\ 2 & -4 & 6 & 2\\ 3 & -6 & 9 & 6 \end{pmatrix} \text{ which reduces to } \begin{pmatrix} 1 & -2 & 3 & 3\\ 0 & 0 & 0 & -4\\ 0 & 0 & 0 & -3 \end{pmatrix}$$

A completely inconsistent set of equations. Thus there are no solutions. Even a cursory glance at these equations reveals a set of three parallel planes.

Questions

Determine how each of these sets of planes intersect

(a) $\begin{aligned} x+y+z&=3\\ 2x+3y+z&=5\\ x-y-2z&=-5 \end{aligned}$

(b) $\begin{aligned} x-y+2z&=3\\ x+2y-z&=-3\\ 2y-2z&=1 \end{aligned}$

(c) $\begin{aligned} 2x+y-z&=1\\ 3x-2y+3z&=-2\\ 4x+2y-2z&=-5 \end{aligned}$

Intersection Between a Line and a Plane

To find the coordinates of the point where the line intersects the plane

- Substitute in the parametric equations of the line into the equation of the plane.
- Solve for λ
- Substitute in this value of λ to find the coordinates of the point of intersection.

Example

① Find the coordinate of the intersection of L_1 and π_1 when $\pi_1 \Rightarrow 2x + y + 3z = 41$ and

$$L_1 \Rightarrow x = 1 + \lambda$$
$$y = 2 + 3\lambda$$
$$z = 5 + 2\lambda$$

$$2(1+\lambda) + (2+3\lambda) + 3(5+2\lambda) = 41$$
$$19 + 11\lambda = 41$$
$$\lambda = 2$$

When $\lambda = 2$

$x = 1 + 2$ $\quad y = 2 + 6$ $\quad z = 5 + 4$

$= 3$ $\quad = 8$ $\quad = 9$

$\therefore$ Point of Intersection is $(3,8,9)$.

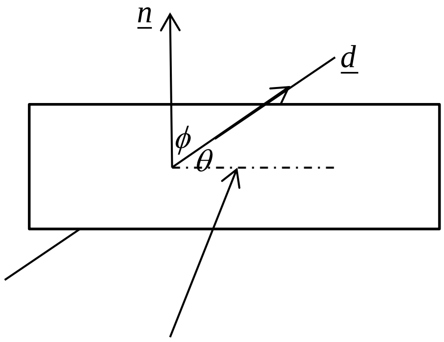
Questions

(a) Find the point of intersection between L_1 $\dfrac{x-12}{5} = \dfrac{y+7}{4} = \dfrac{z-5}{3}$ and π_1 $5x + 3y - z = 0$.

(b) The plane $8x + 5y - 2z + 2 = 0$ and line $\dfrac{x+7}{3} = \dfrac{y-6}{-1} = \dfrac{z-17}{-5}$ intersect.

Find their point of intersection.

The Angle Between a Line and a Plane



Projection of line on plane.

When we calculate the angle between a line and a plane we need to find θ. We do not find θ directly, we find ϕ and $\theta = 90^\circ - \phi$ (as usual ϕ needs to be acute.)

$\underline{d} \Rightarrow$ direction vector of the line

$\underline{n} \Rightarrow$ normal to the plane

$$\cos\phi = \frac{\underline{d}.\underline{n}}{|\underline{d}||\underline{n}|} \qquad \theta = 90^\circ - \phi$$

Example

① Find the size of the acute angle between L_1 and π_1 when $\pi_1 \Rightarrow 2x + y + 3z = 41$ and

$$L_1 \Rightarrow \begin{aligned} x &= 1 + \lambda \\ y &= 2 + 3\lambda \\ z &= 5 + 2\lambda \end{aligned}$$

$\underline{d} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \underline{n} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix},$

$$\cos\phi = \frac{\underline{d}.\underline{n}}{|\underline{d}||\underline{n}|} = \frac{2+3+6}{\sqrt{14}\sqrt{14}} = \frac{11}{14}$$

$\phi = 38.2^\circ$

$$\therefore \theta = 90 - 38.2 = 51.8^\circ$$

Questions

Find the coordinates of intersection and the angle between the following lines and planes.

(a) $\pi_1 \Rightarrow 2x + y + 3z = 41$ and $L_1 \Rightarrow \dfrac{2x+7}{1} = \dfrac{y+5}{2} = \dfrac{3z+9}{2}$

(b) $\pi_2 \Rightarrow 8x + 5y - 2z = -14$ and $L_2 \Rightarrow \begin{aligned} x &= -9 + 2\lambda \\ y &= -13 + 5\lambda \\ z &= 3 - \lambda \end{aligned}$

Past Paper Questions

2001 – QB6

Let L_1 and L_2 be the lines

$$L_1: \ x = 8 - 2t, \ \ y = -4 + 2t, \ \ z = 3 + t$$

$$L_2: \ \frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2}.$$

(a)(i) Show that L_1 and L_2 intersect and find their point of intersection.

(ii) Verify the acute angle between them is $\cos^{-1}\left(\frac{4}{9}\right)$.

(b) (i) Obtain an equation of the plane Π that is perpendicular to L_2 and passes through the point $(1, -4, 2)$.

(ii) Find the coordinates of the point of intersection of the plane Π and the line L_1.

(4, 2, 3, 2 marks)

2002 – Q11

(a) Find an equation for the plane π_1 which contains the points $A(1,1,0)$, $B(3,1,-1)$ and $C(2,0,-3)$.

(b) Given that π_2 is the plane whose equation is $x + 2y + z = 3$, calculate the size of the acute angle between the plane π_1 and π_2.

(4, 3 marks)

2003 – Q12

Find the point of intersection of the line $\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2}$
and the plane with equation $2x + y - z = 4$. **(4 marks)**

2004 – Q14

(a) Find an equation of the plane π_1 containing the points $A(1,0,3)$, $B(0,2,-1)$ and $C(1,1,0)$. Calculate the size of the acute angle between π_1 and the plane π_2 with equation $x + y - z = 0$.

(b) Find the point of intersection of the plane π_2 and the line $\frac{x-11}{4} = \frac{y-15}{5} = \frac{z-12}{2}$.

(4, 3, 3 marks)

2005 – Q8

The equations of two planes are $x - 4y + 2z = 1$ and $x - y - z = -5$. By letting $z = t$ or otherwise, obtain parametric equations for the line of intersection of the planes.
Show that this line lies in the plane with equation $x + 2y - 4z = -11$.

(4, 1 marks)

2006 – Q15

Obtain an equation for the plane passing through the point $P(1,1,0)$ which is perpendicular to the line L given by $\frac{x+1}{2}=\frac{y-2}{1}=\frac{z}{-1}$.

Find the coordinates of the point Q where the plane and L intersect.

Hence, or otherwise, obtain the shortest distance from P to L and explain why this is the shortest distance.

(3, 4, 2, 1 marks)

2007 – Q15

Lines L_1 and L_2 are given by the parametric equations

$$L_1 : x=2+s,\ \ y=-s,\ \ z=2-s \qquad\qquad L_2 : x=-1-2t,\ \ y=t,\ \ z=2+3t.$$

(a) Show that L_1 and L_2 do not intersect.

(b) The line L_3 passes through the point $P(1,1,3)$ and its direction is perpendicular to the directions of both L_1 and L_2. Obtain parametric equations for L_3.

(c)Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1.

(d) PQ is the shortest distance between the lines L_1 and L_2. Calculate PQ.

(3, 3, 3, 1 marks)

2008 – Q14

(a) Find an equation of the plane π_1 through the point $A(1,1,1)$, $B(2,-1,1)$ and $C(0,3,3)$.

(b) The plane π_2 has equation $x+3y-z=2$.

Given that the point $(0,a,b)$ lies on both the planes π_1 and π_2, find the values of a and b. Hence find an equation of the line of intersection of the planes π_1 and π_2.

(c) Find the size of the acute angle between the planes π_1 and π_2.

(3, 4, 3 marks)

2009 – Q16

(a) Use Gaussian elimination to solve the following system of equations

$$x+y-z=6$$
$$2x-3y+2z=2$$
$$-5x+2y-4z=1$$

(b) Show that the line of intersection, L, of the planes $x+y-z=6$ and $2x-3y+2z=2$ has parametric equations

$$x=\lambda$$
$$y=4\lambda-14$$
$$z=5\lambda-20.$$

(c) Find the acute angle between line L and the plane $-5x+2y-4z=1$.

(5, 2, 4 marks)

2010 – Q6

Given $\underset{\sim}{u}=-2\underset{\sim}{i}+5\underset{\sim}{k}$, $\underset{\sim}{v}=3\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}$ and $\underset{\sim}{w}=-\underset{\sim}{i}+\underset{\sim}{j}+4\underset{\sim}{k}$.

Calculate $\underset{\sim}{u}.(\underset{\sim}{v}\times\underset{\sim}{w})$. **(4 marks)**

2011 – Q15

The lines L_1 and L_2 are given by the equations $\frac{x-1}{k}=\frac{y}{-1}=\frac{z+3}{1}$ and $\frac{x-4}{1}=\frac{y+3}{1}=\frac{z+3}{2}$ respectively.

Find

(a) The value of k for which L_1 and L_2 intersect and the point of intersection.

(b) The acute angle between L_1 and L_2.

(6, 4 marks)

2012 – Q5

Obtain an equation for the plane passing through the points $P(-2,1,-1)$, $Q(1,2,3)$ and $R(3,0,1)$.

(5 marks)

2013 – Q15

(a) Find an equation of the plane π_1 through the points $A(0,-1,3)$, $B(1,0,3)$ $C(0,0,5)$.

(b) π_2 is the plane through A with normal in the direction $-\underset{\sim}{j}+\underset{\sim}{k}$.

Find an equation of the plane π_2.

(c) Determine the acute angle between the planes π_1 and π_2.

(4, 2, 3 marks)

2014 – Q5

Three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ are given by $\underline{u}, \underline{v}$ and $\underline{w}$ where

$\underline{u} = 5\underline{i} + 13\underline{j},\ \underline{v} = 2\underline{i} + \underline{j} + 3\underline{k},\ \underline{w} = \underline{i} + 4\underline{j} - \underline{k}.$

Calculate $\underline{u}.(\underline{v} \times \underline{w})$.

Interpret your result geometrically.

(3, 1 marks)

2015 – Q15

A line L_1, passes through the point P$(2,4,1)$ and is parallel to

$$\underline{u}_1 = \underline{i} + 2\underline{j} - \underline{k}$$

and a second line, L_2, passes through Q$(-5,2,5)$ and is parallel to

$$\underline{u}_2 = -4\underline{i} + 4\underline{j} + \underline{k}.$$

(a) Write down the vector equations for L_1 and L_2.

(b) Show that the lines L_1 and L_2 intersect and find their point of intersection.

(c) Determine the equation of the plane containing L_1 and L_2.

(2, 4, 4 marks)

2016 – Q14

Two lines L_1 and L_2 are given by the equations:

$$L_1 : x = 4 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = -7\lambda$$

$$L_2 : \frac{x-3}{-2}, = \frac{y-8}{1}, = \frac{z+1}{3}$$

(a) Show that the lines L_1 and L_2 intersect and find the point of intersection.

(b) Calculate the obtuse angle between the lines L_1 and L_2.

(5, 4 marks)

2017 – Q15

(a) A beam of light passes through the points B(7, 8, 1) and T(-3, -22, 6).
Obtain parametric equations of the line representing the beam of light.

(b) A sheet of metal is represented by a plane containing the points P(2, 1, 9), Q(1, 2, 7) and R(-3, 7, 1). Find the Cartesian equation of the plane.

(c) The beam of light passes through a hole in the metal at point H.
Find the coordinates of H.

(2, 4, 3 marks)