Differentiation
First Principles

QUESTION

Differentiate the following functions from First Principles :-

(a) $f(x) = x^3$ (b) $f(x) = x^2 + 2x$ (c) $f(x) = 3x^2 + 4x - 5$

ANSWER

(a). $3x^2$ (b) $2x + 2$ (c) $6x + 4$

Differentiation
Revision

QUESTION 1

Differentiate the following functions with respect to x :-

1. $f(x) = x^3 - x^2 + 5x - 6$
2. $f(x) = 3x^2 + 7 - \frac{4}{x}$
3. $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$
4. $f(x) = x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}$
5. $f(x) = \frac{1}{x^2} - \frac{1}{x^3}$
6. $f(x) = \frac{\sqrt{x}}{x^2} + \frac{x^2}{\sqrt{x}}$
7. $f(x) = (4x + 5)^5$
8. $f(x) = \left(2x^4 - 3\right)^{\frac{1}{2}}$
9. $f(x) = \frac{3}{\sqrt{(4 - x^2)}}$
10. $f(x) = \frac{4}{\left(x^3 + 3x\right)^{\frac{1}{3}}}$
11. $f(x) = \cos^3 x$
12. $f(x) = \sqrt{\sin x}$

ANSWER 1

1. $3x^2 - 2x + 5$
2. $6x + \frac{4}{x^2}$
3. $\frac{1}{2x^{\frac{1}{2}}} - \frac{1}{2x^{\frac{3}{2}}}$
4. $\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{2x^{\frac{3}{2}}}$
5. $-\frac{2}{x^3} + \frac{3}{x^4}$
6. $-\frac{3}{2x^{\frac{5}{2}}} + \frac{3}{2}x^{\frac{1}{2}}$
7. $20(4x + 5)^4$
8. $\frac{4x^3}{\left(2x^4 - 3\right)^{\frac{1}{2}}}$
9. $\frac{3x}{\left(4 - x^2\right)^{\frac{3}{2}}}$
10. $-\frac{4(x^2 + 1)}{\left(x^3 + 3x\right)^{\frac{4}{3}}}$
11. $-3\sin x\cos^2 x$
12. $\frac{\cos x}{2\sqrt{\sin x}}$

QUESTION 1

Differentiate the following functions using the Chain Rule.

1. $y = \left(x^2 + 4x - 5\right)^3$
2. $y = \sqrt{x^3 + 5}$
3. $y = \left(1 + 2\sqrt{x}\right)^4$
4. $y = \frac{3}{\sqrt{(4 - x^2)}}$

ANSWER 1

. $(6x + 12)\left(x^2 + 4x - 5\right)^2$
2. $\frac{3x^2}{2\sqrt{(x^3 + 5)}}$
3. $\frac{4\left(1 + 2\sqrt{x}\right)^3}{\sqrt{x}}$
4. $\frac{3x}{\left(4 - x^2\right)^{\frac{3}{2}}}$

Differentiation
Revision

QUESTION

Differentiate: (1) $\sqrt{2x+3}$ (2) $(4-5x)^{\frac{3}{2}}$ (3) $\dfrac{1}{\sqrt{2-x}}$ (4) $(x-\frac{1}{x})^3$

(5) $\dfrac{1}{(3x-5)^2}$ (6) $(2x^2-x)^{-\frac{1}{2}}$ (7) $(ax^2-b)^4$ (8) $\dfrac{a}{\sqrt{ax+b}}$

(9) $2\sin x + 3\cos x$ (10) $\sin 3x$ (11) $\cos 5x$ (12) $\cos(2x+1)$

(13) $\sin(4x-7)$ (14) $\sin^2 x$ (15) $\cos^3 x$ (16) $\sin^2 3x$

Answers

(1) $(2x+3)^{-½}$ (2) $-\dfrac{15}{2}(4-5x)^{½}$ (3) $\dfrac{1}{2}(2-x)^{-3/2}$ (4) $3(1+\dfrac{1}{x^2})(x-\dfrac{1}{x})^2$

(5) $-6(3x-5)^{-3}$ (6) $-\dfrac{1}{2}(4x-1)(2x^2-x)^{-3/2}$ (7) $8ax(ax^2-b)^3$

(8) $-\dfrac{1}{2}a^2(ax+b)^{-3/2}$ (9) $2\cos x - 3\sin x$ (10) $3\cos 3x$ (11) $-5\sin x$

(12) $-2\sin(2x+1)$ (13) $4\cos(4x-7)$ (14) $2\sin x \cos x$

(15) $-3\cos^2 x \sin x$ (16) $6\sin 3x \cos 3x$

QUESTION

1. Find the derivative of each of the following

(a) $x^3+\cos x$ (b) $x^{-2}+\sqrt{x}$ (c) $\dfrac{3}{x}+\cos 3x$ (d) $(5x+1)^4$ (e) $\sin(x^2-2x)$

(f) $(x^2+3x+1)^3$ (g) $\sin^2 x$ (h) $\cos^4 x$

2. Find $\dfrac{dy}{dx}$ when $y=\sin^2 5x$

ANSWER

1. (a) $3x^2-\sin x$ (b) $-2x^{-3}+\dfrac{1}{2}x^{-1/2}$ (c) $-3x^{-2}-3\sin 3x$

(d) $20(5x+1)^3$ (e) $(2x-2)\cos(x^2-2x)$ (f) $3(2x+3)(x^2+3x+1)^2$

(g) $2\sin x\cos x$ (h) $-4\cos^3 x\sin x$

2. $10\sin 5x\cos 5x$

Differentiation
Higher Order Derivatives

QUESTIONS

(1) Write down the first and second derivatives of :

(a) $x^2(x-1)$ (b) $5x^4 - 3x^3 + 2x^2 - x + 1$ (c) $10x^4 - 4x^3 + 5x + 2$

(d) $\frac{1}{x}$ (e) $\sqrt{x}$ (f) $\sqrt{2x+1}$ (g) $\frac{1}{x^3}$

ANSWERS

Answers

(1) (a) $\frac{dy}{dx} = 3x^2 - 2x$; $\frac{d^2y}{dx^2} = 6x - 2$

(b) $\frac{dy}{dx} = 20x^3 - 9x^2 + 4x - 1$; $\frac{d^2y}{dx^2} = 60x^2 - 18x + 4$

(c) $\frac{dy}{dx} = 40x^3 - 12x^2 + 5$; $\frac{d^2y}{dx^2} = 120x^2 - 24x$

(d) $\frac{dy}{dx} = -x^{-2}$; $\frac{d^2y}{dx^2} = 2x^{-3}$

(e) $\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$; $\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2}$

(f) $\frac{dy}{dx} = \frac{1}{2}(2x+1)^{-1/2}$; $\frac{d^2y}{dx^2} = -(2x+1)^{-3/2}$

(g) $\frac{dy}{dx} = -3x^{-4}$; $\frac{d^2y}{dx^2} = 12x^{-5}$

QUESTIONS

(1) Find $\frac{d^3y}{dt^3}$ when $y = t^4 + 3t^2 + 5t - 6$

ANSWERS

(1)

24t

* DO HIGHER S. POINT QUESTIONS USING THE 2ND DERIVATIVE TEST (EG FROM T.BOOK AND PAST PAPERS). AFTER AUGUST, WHEN WE KNOW MORE RULES OF DIFFERENTIATION WE WILL REVISE IT

Differentiation
Product Rule

QUESTION

Differentiate the following functions using the Product rule as above :-

1. $y = x^2(x-3)^2$
2. $y = x(2x+3)^3$
3. $y = x\sqrt{(x-6)}$
4. $y = \sqrt{x}(x-3)^3$
5. $y = (x+1)^2(x-1)^4$
6. $y = x^3\sqrt{(x-1)}$
7. $y = x\sin x$
8. $y = x^2\sin x$
9. $y = \sin x\cos x$
10. $y = \sin 2x\cos 5x$

ANSWERS

1. $2x(x-3)(2x-3)$
2. $(8x+3)(2x+3)^2$
3. $\dfrac{3(x-4)}{2\sqrt{x-6}}$
4. $\dfrac{(x-3)^2(7x-3)}{2\sqrt{x}}$
5. $2(x+1)(3x+1)(x-1)^3$
6. $\dfrac{x^2(7x-6)}{2\sqrt{x-1}}$
7. $\sin x + x\cos x$
8. $x(2\sin x + x\cos x)$
9. $\cos 2x$
10. $2\cos 2x\cos 5x - 5\sin 2x\sin 5x$

QUESTION

Find the derivatives of:

(1) $(3x+1)(2x+1)$ (2) $(x^2+1)(\frac{1}{2}x+1)$ (3) $(3x-5)(x^2+2x)$

(4) $(x^2+3)(2x^2-1)$ (5) $(x^2-x+1)(x+1)$ (6) $(x^2+4x+5)(x^2-2)$

(7) $(x^2-x+1)(x^2+x-1)$ (8) $(x-2)(x^2+2x+4)$ (9) $(2x^2-3)(3x^2+x-1)$

(10) $\sqrt{x}\,(x^2+x+1)$ (11) $x\sin x$ (12) $\sin 2x \,.\, \cos 2x$

(13) $x^2\cos 2x$ (14) $x\sqrt{\sin x}$ (15) $(x^2+x+1)(x-1)$

(16) $(x^2+4x)(3x^2-x)$ (17) $(1-\cos x)(1+\cos x)$

(18) $2x^{\frac{3}{2}}(\sqrt{x}+2)(\sqrt{x}-1)$

Answers

(1) $12x+5$ (2) $\dfrac{3}{2}x^2+2x+\dfrac{1}{2}$ (3) $9x^2+2x-10$ (4) $8x^3+10x$ (5) $3x^2$

(6) $4x^3+12x^2+6x-8$ (7) $4x^3+2x$ (8) $3x^2$ (9) $24x^3+6x^2-22x-3$

(10) $\dfrac{5}{2}x^{3/2}+\dfrac{3}{2}x^{1/2}+\dfrac{1}{2}x^{-1/2}$ (11) $\sin x + x\cos x$ (12) $-2\sin^2 2x + 2\cos^2 2x$

Differentiation
Product Rule

(13) $2x\cos 2x - 2x^2\sin 2x$ (14) $\sqrt{\sin x} + \dfrac{x\cos x}{2\sqrt{\sin x}}$ (15) $3x^2$

(16) $12x^3 + 33x^2 - 8x$ (17) $-2\sin x\cos x$ (18) $5x^{1/2} + 4x - 6x^{1/2}$

QUESTION

1. Use the product rule to differentiate

(a) $y = x^2 \sin x$ (b) $y = 5x^4 \cos x$ (c) $f(x) = (x+1)^4 (x-1)^3$

(d) $f(t) = (t-1)^4 \cos t$ (e) $y = (1-5x)^3 (1+5x)^3$ (f) $y = \sin x^4 \cos x^3$

2. (a) If $f(x) = x^3 (x-1)^2$ find $f'(1)$

(b) If $f(x) = (x-1)^2 \sin x$ find $f'\left(\dfrac{\pi}{2}\right)$

ANSWER

1. (a) $2x\sin x + x^2\cos x$ (b) $20x^3\cos x - 5x^4\sin x$

(c) $3(x+1)^4(x-1)^2 + 4(x+1)^3(x-1)^3$ (d) $4\cos t(t-1)^3 - (t-1)^4\sin t$

(e) $15(1+5x)^2(1-5x)^3 - 15(1-5x)^2(1+5x)^3$

(f) $4x^3\cos x^4\cos x^3 - 3x^2\sin x^3\sin x^4$

2. (a) 0 (b) $\pi - 2$

Differentiation
Quotient Rule

QUESTION

Differentiate the following functions using the Quotient rule as above :-

1. $y = \frac{x^2}{x + 3}$
2. $y = \frac{4 - x}{x^2}$
3. $y = \frac{4x}{(1 - x)^3}$
4. $y = \frac{2x^2}{x - 2}$
5. $y = \frac{(1 - 2x)^3}{x^3}$
6. $y = \frac{\sqrt{(x + 1)}}{x^2}$

ANSWERS

1. $\frac{x^2 + 6x}{(x + 3)^2}$
2. $\frac{x - 8}{x^3}$
3. $\frac{4(2x + 1)}{(1 - x)^4}$
4. $\frac{2x(x - 4)}{(x - 2)^2}$
5. $-\frac{3(1 - 2x)^2}{x^4}$
6. $-\frac{(3x + 4)}{2x^3\sqrt{x + 1}}$

QUESTION

Differentiate

(1) $\frac{3}{2x-1}$ (2) $\frac{1}{1-3x^2}$ (3) $\frac{x}{x-2}$ (4) $\frac{x^2}{x-4}$ (5) $\frac{x^2+x+1}{x^2-x+1}$

(6) $\frac{2x^2-x+1}{3x^2+x-1}$ (7) $\frac{1+x+x^2}{x}$ (8) $\frac{1-\cos x}{1+\cos x}$ (9) $\frac{\sin^2 x}{1+\sin x}$

Answers

(1) $\frac{-6}{(2x-1)^2}$ (2) $\frac{6x}{(1-3x^2)^2}$ (3) $\frac{-2}{(x-2)^2}$ (4) $\frac{x^2-8x}{(x-4)^2}$ (5) $\frac{-2x^2+2}{(x^2-x+1)^2}$

(6) $\frac{5x^2-7x}{(3x^2+x-1)^2}$ (7) $1 - \frac{1}{x^2}$ (8) $\frac{2\sin x}{(1+\cos x)^2}$ (9) $\frac{\sin x\cos x(2+\sin x)}{(1+\sin x)^2}$

Differentiation
Quotient Rule

QUESTION

Differentiate

(1) $\dfrac{\sqrt{x}}{x+1}$ (2) $\dfrac{\sqrt{x+1}}{\sqrt{x-1}}$ (3) $\dfrac{x^{\frac{1}{2}}+2}{x^{\frac{3}{2}}}$

(4) $\dfrac{1}{\sqrt{x^2-1}}$ (5) $\dfrac{1}{\sqrt{2x^2-3x+4}}$ (6) $\dfrac{\sqrt{x}}{\sin x}$

Answers

(1) $\dfrac{x^{-1/2}-x^{1/2}}{2(x+1)^2}$ (2) $-\dfrac{1}{(x-1)^{1/2}(x+1)^{3/2}}$ (3) $-\dfrac{1}{x^2}-\dfrac{3}{x^{5/2}} \equiv \dfrac{-x^{1/2}-3}{x^{5/2}}$

(4) $\dfrac{-x}{(x^2-1)^{3/2}}$ (5) $\dfrac{-4x+3}{2(2x^2-3x+4)^{3/2}}$ (6) $\dfrac{x^{1/2}(\frac{1}{2}x\sin x-\cos x)}{\sin^2 x}$

QUESTION

1 Use the quotient rule to differentiate

(a) $y=\dfrac{2x-7}{3x+5}$ (b) $f(x)=\dfrac{\cos x}{x}$ (c) $y=\dfrac{\sqrt{t}-3}{3t}$ (d) $f(x)=\dfrac{x^2}{(x+4)^{3/2}}$

(e) $y=\dfrac{x^2}{\sqrt{x-1}}$ (f) $f(t)=\dfrac{2t^2+3t-6}{t-2}$ (g) $y=\dfrac{(3x+2)^2}{(2x-1)}$ (h) $f(x)=\dfrac{\cos x}{\sin 2x}$

2 If $y=\dfrac{2x+3}{x^2+4}$ evaluate $\dfrac{dy}{dx}$ at $x=0$.

3 Differentiate

(a) $\dfrac{x^2(x-3)^5}{x+2}$ (b) $\dfrac{(x-1)^3(x+2)}{x-2}$

4 A curve has equation given by $y=\dfrac{\cos x}{\cos x+\sin x}$. Find the gradient of the tangent of this curve at $x=\dfrac{\pi}{4}$.

Differentiation
Quotient Rule

ANSWERS

1. (a) $\dfrac{31}{(3x+5)^2}$ (b) $-\dfrac{x\sin x+\cos x}{x^2}$ (c) $\dfrac{6-t}{6t^2\sqrt{t-3}}$ (d) $\dfrac{x(x+16)}{2(x+4)^{5/2}}$

   (e) $\dfrac{x(3x-4)}{2(x-1)^{3/2}}$ (f) $\dfrac{2t(t-4)}{(t-2)^2}$ (g) $\dfrac{2(3x+2)(3x-5)}{(2x-1)^2}$

   (h) $\dfrac{-\sin x\sin 2x-2\cos x\cos 2x}{\sin^2 2x}$

2. 1/2

3. (a) $\dfrac{x(x-3)^4(6x^2+11x-12)}{(x+2)^2}$ (b) $\dfrac{(x-1)^2(3x^2-4x-8)}{(x-2)^2}$

4. $-1/2$

Differentiation
tan, sec, cosec and cot

QUESTION

Differentiate using the Chain, Product and Quotient Rules as above :-

1. $y = \tan^3 2x$
2. $y = -2\text{cosec}^4 x$
3. $y = \sec x \tan x$
4. $y = x^2 \cot x$

ANSWERS

1. $6\tan^2 2x \sec^2 2x$
2. $8\text{cosec}^4 x \cot x$
3. $\sec x\left(\tan^2 x + \sec^2 x\right)$
4. $x\left(2\cot x - x\text{cosec}^2 x\right)$

QUESTIONS

1. Evaluate

(1) $\text{cosec}^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{6}$ (2) $\cos\frac{\pi}{3}.\text{cosec}^2 \frac{\pi}{4}.\text{cosec}^2 \frac{\pi}{4}$

(3) $\sin\frac{\pi}{4}.\sin\frac{\pi}{6} + \cos\frac{\pi}{4}.\sin\frac{\pi}{3}.\tan\frac{\pi}{6}$ (4) $(1+\sin\frac{\pi}{4})(1-\cos\frac{\pi}{4})$

2. Evaluate

(1) $\cos\frac{\pi}{3}.\text{cosec}^2 \frac{\pi}{4}$ (2) $\sec^2 \frac{\pi}{3} - \dfrac{\text{cosec}\frac{\pi}{6}}{\sin^2 \frac{\pi}{4}}$

(3) $\text{cosec}^2 \frac{\pi}{4}.\sin\frac{\pi}{6} - \sin^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{6}$ (4) $(\sec\frac{\pi}{6} + \cot\frac{\pi}{3})(\text{cosec}\frac{\pi}{3} - \tan\frac{\pi}{6})$

3. Simplify

(1) $\cot\theta \sin\theta$ (2) $\cot\theta \sec\theta$ (3) $\dfrac{\sec\theta}{\text{cosec}\theta}$

(4) $\sin\theta \sec\theta$ (5) $\cos\theta\text{cosec}\theta$ (6) $\dfrac{\cos\theta}{\sec\theta} + \dfrac{\sin\theta}{\text{cosec}\theta}$

Answers

1. (1) 1 (2) 2 (3) $\dfrac{1}{\sqrt{2}}$ (4) $\dfrac{1}{2}$

2. (1) 1 (2) 0 (3) $-\dfrac{3}{4}$ (4) 1

3. (1) $\cos\theta$ (2) $\dfrac{1}{\sin\theta}$ (3) $\tan\theta$ (4) $\tan\theta$ (5) $\cot\theta$ (6) 1

Differentiation
tan, sec, cosec and cot

QUESTION

Differentiate

(1) $\tan x$ (2) $\sec x$ (3) $\operatorname{cosec} x$

ANSWERS

(1) $\sec^2 x$ (2) $\dfrac{\sin x}{\cos^2 x}\ (\equiv \tan x \sec x)$ (3) $-\dfrac{\cos x}{\sin^2 x}\ (\equiv -\cot x \operatorname{cosec} x)$

QUESTION

1. Prove that the derivative of $\sec x$ is $\sec x \tan x$.

2. If $y = x^2 \tan x$ find the derivative of y with respect to x.

ANSWERS

1. Proof

2. $2x \tan x + x^2 \sec^2 x$

Differentiation
Exponential Functions

QUESTION

(1) $y = (x + 2)e^{-x}$ (2) $y = xe^{-2x^2}$ (3) $y = \dfrac{e^x}{x + 2}$ (4) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

ANSWER

(1) $-(x + 1)e^{-x}$ (2) $e^{-2x^2}\left(1 - 4x^2\right)$ (3) $\dfrac{(x + 1)e^x}{(x + 2)^2}$ (4) $\dfrac{4}{\left(e^x + e^{-x}\right)^2}$

QUESTION

Differentiate with respect to x:

1) (a) e^{5x} (b) $e^{\frac{x}{2}}$ (c) $e^{\sqrt{x}}$

2) (a) e^{-2x} (b) $e^{\frac{-5x}{2}}$ (c) $e^{(5-2x)}$

3) (a) xe^x (b) xe^{-x} (c) x^2e^{-x}

4) (a) $(x+4)e^x$ (b) $e^x \sin x$ (c) $10e^x$

5) (a) $e^{\sin x}$ (b) $e^{\cos x}$ (c) $e^{\tan x}$

6) $\dfrac{e^{ax}}{x^{\frac{1}{2}}}$ 7) x^2e^{4x} 8) $e^{ax}\sin^2 x$

9) xe^{x-x^2} 10) $x^2e^{\sin^2 x}$

ANSWER

1) (a) $5e^{5x}$ (b) $\dfrac{1}{2}e^{x/2}$ (c) $\dfrac{1}{2}x^{-1/2}e^{\sqrt{x}}$

2) (a) $-2e^{-2x}$ (b) $-\dfrac{5}{2}e^{-5/2x}$ (c) $-2e^{(5-2x)}$

3) (a) $(1+x)e^x$ (b) $(1-x)e^{-x}$ (c) $2xe^{-x} - x^2e^{-x}$

4) (a) $(5+x)e^x$ (b) $(\cos x + \sin x)e^x$ (c) $10e^x$

5) (a) $\cos x\, e^{\sin x}$ (b) $-\sin x\, e^{\cos x}$ (c) $\sec^2 x\, e^{\tan x}$

6) $\left(-\dfrac{1}{2}x^{-3/2} + ax^{-1/2}\right)e^{ax}$ 7) $(2x + 4x^2)e^{4x}$ 8) $(a\sin^2 x + 2\sin x\cos x)e^{ax}$

9) $(1 + x - 2x^2)e^{x-x^2}$ 10) (1 [illegible]

Differentiation
Exponential Functions

QUESTION

(1) Find the derivative of

(a) e^{5x} (b) e^{7x^2} (c) $5e^{\sin x}$

(2) Use the product or quotient rule to help differentiate

(a) $(2x+4)e^{x+2}$ (b) $3xe^{x}$ (c) $\dfrac{e^{x^4}}{x}$ (d) $\dfrac{x^2+3x+1}{e^x}$ (e) $e^{2x}\cos(2x+1)$

(3) A curve has equation $y=\dfrac{\sin x}{e^x}$. Find the gradient of its tangent at $x=0$.

ANSWER

(1) (a) $5e^{5x}$ (b) $14xe^{7x^2}$ (c) $5\cos x\, e^{\sin x}$

(2) (a) $2(x+3)e^{x+2}$ (b) $3(1+x)e^{3x}$ (c) $\dfrac{e^{x^4}(4x^4-1)}{x^2}$ (d) $\dfrac{2-x-x^2}{e^x}$

(e) $2e^{2x}(\cos(2x+1)-\sin(2x+1))$

(3) 1

Differentiation
Logarithmic Functions

QUESTIONS

(1) $y = \ln(3x + 2)$ (2) $y = \dfrac{x^2}{\ln x}$ (3) $y = \ln\left(\dfrac{1 + x}{1 - x}\right)$ (4) $y = \ln\left(\sqrt{(x^2 + 1)}\right)$

ANSWERS

(1) $\dfrac{3}{3x + 2}$ (2) $\dfrac{x(2\ln x - 1)}{(\ln x)^2}$ (3) $\dfrac{2}{1 - x^2}$ (4) $\dfrac{x}{x^2 + 1}$

QUESTIONS

Differentiate

1) (a) $\ln\dfrac{x}{a}$ (b) $\ln(ax^2 + bx + c)$

2) (a) $\ln x^2$ (b) $\ln(x^3 + 3)$

3) $x\ln x$ 4) (a) $\ln\sin x$ (b) $\ln\cos x$ 5) $\ln\left(\dfrac{a+x}{a-x}\right)$

6) $\ln(e^x + e^{-x}$ 7) $\ln(x + \sqrt{x^2+1}$ 8) $\ln\sqrt{x^2+1}$

9) $\ln\tan\left(\dfrac{x}{2}\right)$ 10) $\ln\sqrt{\sin x}$ 11) $\ln\left(\dfrac{e^x}{1+e^x}\right)$

12) $\ln\{\sqrt{x-1} + \sqrt{x+1}$ 13) $\ln(\sec x + \tan)$

14) $\ln(\operatorname{cosec} x + \cot x)$ 15) $\ln(1 - 2\cos 2x)$

ANSWERS

1) (a) $\dfrac{1}{x}$ (b) $\dfrac{2ax+b}{ax^2+bx+c}$ 2) (a) $\dfrac{2}{x}$ (b) $\dfrac{3x^2}{x^3+3}$

3) $1 + \ln x$ 4) (a) $\dfrac{\cos x}{\sin x} \equiv \cot x$ (b) $-\dfrac{\sin x}{\cos x} \equiv -\tan x$

5) $-\dfrac{2x}{(a+x)(a-x)}$ 6) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ 7) $\dfrac{1}{(x^2+1)^{½}}$

8) $\dfrac{x}{x^2+1}$ 9) $\dfrac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} \equiv \dfrac{1}{\sin x}$ 10) $\dfrac{\cos x}{2\sin x} \equiv \dfrac{1}{2}\cot x$

11) $\dfrac{1}{1+e^x}$ 12) $\dfrac{1}{2\sqrt{(x-1)(x+1)}}$ 13) $\sec x$ 14) $\dfrac{\cos x - 1}{\sin x(\cos x + 1)}$ 15) $\dfrac{4\sin 2x}{1 - 2\cos 2x}$

Differentiation
Logarithmic Functions

QUESTIONS

(1) Find the derivative of

(a) $\ln(x+5)$ (b) $6\ln x^6$ (c) $\ln(4x^3-1)$ (d) $\ln(\sin 2x)$

(2) Use the product or quotient rule to help differentiate

(a) $\dfrac{x}{\ln x}$ (b) $x\ln x$ (c) $\dfrac{\ln(\cos x)}{x^2}$ (d) $\ln(xe^x)$

ANSWERS

(1) (a) $\dfrac{1}{x+5}$ (b) $\dfrac{36}{x}$ (c) $\dfrac{12x^2}{4x^3-1}$ (d) $\dfrac{2\cos 2x}{\sin 2x}$

(2) (a) $\dfrac{\ln x-1}{(\ln x)^2}$ (b) $1+\ln x$ (c) $-\dfrac{1}{x^3}(x\tan x+2\ln(\cos x))$ (d) $\dfrac{1+x}{x}$

Differentiation

Identities & More Practice at S.P. Nature Using 2nd Derivatives

QUESTION

(1) If $y = x^2 \ln x$ show that $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = kx$, stating the value of the constant k.

(2) If $y = \operatorname{cosec}^3 x$ show that $\dfrac{dy}{dx} + 3y\cot x = 0$.

ANSWERS

(1) Proof. Find $k = 2$.

(2) Proof

QUESTIONS

Use the second derivative to find the stationary values and their nature for the following functions.

1. $y = x - \ln x$
2. $y = x\ln x$
3. $y = xe^{-x}$
4. $y = \frac{1}{2}\sin\theta + \sin 2\theta$

ANSWERS

1. Min at $(1, 1)$
2. Min at $(^1/_e, {}^1/_e)$
3. Max at $(1, {}^1/_e)$
4. Max at $(^{\pi}/_3, {}^{3\sqrt{3}}/_4)$, P of I at $(\pi, 0)$, Min at $(^{5\pi}/_3, {}^{-3\sqrt{3}}/_4)$

Differentiation
Parametric Differentiation

QUESTION

1. Find $\dfrac{dy}{dx}$ in terms of the parameter.

(a) $x = t^3 + t^2, \quad y = t^2 + t$

(b) $x = 4\cos\theta, \quad y = 3\sin\theta$

(c) $x = \dfrac{1}{1+t}, \quad y = \dfrac{t}{1-t}$

(d) $x = \dfrac{t-1}{t+1}, \quad y = \dfrac{2t-1}{t-2}$

(e) $x = (t+1)^2, \quad y = t^2 - 1$

(f) $x = \dfrac{t}{1-t}, \quad y = \dfrac{t^2}{t+3}$

(g) $x = \cos 2\theta, \quad y = 4\sin\theta$

(h) $x = a\cos^2\theta, \quad y = a\sin^3\theta$

(i) $x = e^t\cos t, \quad y = e^t\sin t$

(j) $x = a(t - \cos t), \quad y = a(1 + \sin t)$

2. Find the equations of the tangents to these curves at the point P(*x*,*y*) :-

(a) $x = ct, \quad y = \dfrac{c}{t}$

(b) $x = at^2, \quad y = at(t^2 - 1)$

(c) $x = \dfrac{a}{2}(t + \dfrac{1}{t}), \quad y = \dfrac{a}{2}(t - \dfrac{1}{t})$

(d) $x = \sec\theta, \quad y = \tan\theta$

ANSWERS

1.(a) $\dfrac{2t+1}{t(3t+2)}$ (b) $\dfrac{-3}{4}\cot\theta$ (c) $-\dfrac{(1+t)^2}{(1-t)^2}$ (d) $-\dfrac{3(t+1)^2}{2(t-2)^2}$

(e) $\dfrac{t}{t+1}$ (f) $\dfrac{(t+6)(1-t)^2}{(t+3)^2}$ (g) $-\mathrm{cosec}\,\theta$ (h) $\dfrac{-3}{2}\sin(\theta)$

(i) $\dfrac{\cos t + \sin t}{\cos t - \sin t}$ (j) $\dfrac{\cos t}{1 + \sin t}$

2.(a) $x + t^2 y = 2ct$ (b) $(3t^2 - 1)x - 2ty = at^2(t^2 + 1)$

(c) $(t^2 + 1)x - (t^2 - 1)y = 2at$ (d) $x\sec\theta - y\tan\theta = 1$

Differentiation
Parametric Differentiation

QUESTION

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t :-

(a) $x = \frac{1}{t^2}$, $y = 1 + t$ (b) $x = (t+1)^2$, $y = t^2 - 1$

(c) $x = 4\cos t$, $y = 3\sin t$ (d) $x = \cos^2 t$, $y = \sin t$

(e) $x = 2\cos t - \cos 2t$, $y = 2\sin t + \sin 2t$

2. A curve has parametric equations $x = t - \cos t$, $y = \sin t$.
Find the coordinates of the points at which the gradient of the curve is zero.

3. Find the coordinates of the stationary points on the curves and determine their nature:-

(a) $x = 4 - t^2$, $y = 4t - t^3$ (b) $x = (5 - 3t)^2$, $y = 6t - t^2$

(c) $x = t^2 + 1$, $y = t(t-3)^2$

4. Given that $x = t - \sin t$, $y = 1 - \cos t$, show that $y^2\frac{d^2y}{dx^2} + 1 = 0$

ANSWER

1.(a) $-\frac{1}{2}t^3, \frac{3}{4}t^5$ (b) $\frac{t}{1+t}, \frac{1}{2(1+t)^3}$ (c) $-\frac{3}{4}\cot t, -\frac{3}{16}\operatorname{cosec}^3 t$

(d) $-\frac{1}{2}\operatorname{cosec} t, -\frac{1}{4}\operatorname{cosec}^3 t$ (e) $\frac{\cos t + 1}{\sin t}, \frac{-1}{\sin^3 t(2\cos t - 1)}$

2. $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -1\right)$

3.(a) Maximum S.Pt. at $\left(\frac{8}{3}, \frac{16}{9}\sqrt{3}\right)$, Minimum S.Pt. at $\left(\frac{8}{3}, \frac{-16}{9}\sqrt{3}\right)$

(b) Maximum S.Pt. at (16, 9)

(c) Maximum S.Pt. at (2, 4), Minimum S. Pt. at (10, 0)

4. Proof

Differentiation
Parametric Differentiation

QUESTION

(1) Eliminate t from $x = ct,\ y = \frac{c}{t}$

(2) Eliminate θ from $x = a\cos\theta,\ y = a\sin\theta$

Answers

(1) $y = \frac{c^2}{x}$ (2) $x^2 + y^2 = a^2$

QUESTION

(1) Find the equation of the tangent to each of the following curves at the given points:

(a) $x = t^2 - 2t,\ y = 1 - t^4$ at the point $t = -1$.

(b) $x = \sin t,\ y = \cos 2t\ \left(-\frac{1}{2}\pi \le t \le \frac{1}{2}\pi\right)$ at the point $t = \frac{1}{4}\pi$.

(2) Show that there are four points on the curve $x = \frac{t^3}{t^2+1},\quad y = \frac{t}{t^2+1}$

at which the gradient is equal to $-\frac{1}{10}$.

(3) Find the equation of the tangent at the point with parameter $t = 2$ on the curve

$x = \frac{4t}{t^3+1},\quad y = \frac{t^2}{t^3+1}\quad (t \ne -1)$.

Find the parameter of the other point at which the tangent meets the curve.

Answers

(1) (a) $y = -x + 3$ (b) $y = -2\sqrt{2}x + 2$

(2) Proof

(3) $y - \frac{4}{9} = \frac{1}{5}(x - \frac{8}{9})$

$t = -\frac{1}{4}$

Differentiation
Parametric Differentiation

QUESTION

Find $\frac{d^2y}{dx^2}$ in terms of t:

(1) $x = t^2, y = t^3$ (2) $x = at,\ y = \frac{a}{t}$ (3) $x = \frac{1}{1+t},\ \ y = \frac{t}{1-t}$

(4) $x = \frac{t-1}{t+1},\ \ y = \frac{2t-1}{t-2}$ (5) $x = 3t - t^3,\ y = 1 - t^2$

(6) $x = \tan\theta$, $y = \sin 2\theta$ (7) $x = t^2 + t,\ y = 3 - t^3$

Answers

(1) $\frac{3}{4t}$ (2) $\frac{2}{at^3}$ (3) $\frac{4(1+t)^3}{(1-t)^3}$ (4) $\frac{9(t+1)^3}{2(t-2)^3}$

(5) $\frac{-2(1-3t)(1+t)}{(1-t)^2}$ (6) $2\sin 2\theta\ (\cos^2\theta - 4\cos^4\theta)$ (7) $\frac{-6t(t+1)}{(2t+1)^3}$

QUESTION

1. By eliminating t, find y in terms of x for the following parametric equations

(a) $\begin{array}{l} x = 1 - t \\ y = 1 + 2t \end{array}$ (b) $\begin{array}{l} x = 5t^2 \\ y = 10t \end{array}$ (c) $\begin{array}{l} x = 4\cos t \\ y = 3\sin t \end{array}$ (d) $\begin{array}{l} x = \frac{2+3t}{4} \\ y = \frac{3-4t}{5} \end{array}$

2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following pairs of parametric equations

(a) $\begin{array}{l} x = t \\ y = \frac{1}{t} \end{array}$ (b) $\begin{array}{l} x = 3t^3 - t \\ y = 4t^2 \end{array}$ (c) $\begin{array}{l} x = 2t + 1 \\ y = 2t(t-1) \end{array}$

3. A curve is defined by $x = t^2 - \frac{1}{t^2}$ and $y = t^2 + \frac{1}{t^2}$. Find its stationary point and determine its nature.

Differentiation
Parametric Differentiation

ANSWERS

1. (a) $y = 3 - 2x$ (b) $y = \pm\sqrt{20x}$ (c) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ (d) $y = \dfrac{17 - 16x}{15}$

2. (a) $\dfrac{dy}{dx} = -\dfrac{1}{t^2}$ $\dfrac{d^2y}{dx^2} = \dfrac{2}{t^3}$ (b) $\dfrac{dy}{dx} = \dfrac{8t}{9t^2 - 1}$ $\dfrac{d^2y}{dx^2} = -\dfrac{8(9t^2 + 1)}{(9t^2 - 1)^3}$

   (c) $\dfrac{dy}{dx} = 2t - 1$ $\dfrac{d^2y}{dx^2} = 1$

3. SP when $t^4 = 1$ i.e. when $t = \pm 1$. In both cases $x = 0$ and $y = 2$. Use nature table (or maybe better second derivative) to show it is a minimum SP.

Differentiation
Implicit Differentiation

QUESTIONS

1. Find $\frac{dy}{dx}$ in each of these relations :

(a) $x^2 - y^2 = 0$ (b) $y^2 = 2x + 2y$

(c) $xy^2 = 9$ (d) $4x^2 - y^3 + 2x + 3y = 0$

(e) $x^3y + xy^3 = x - y$ (f) $\sin x \cos y = 1$

(g) $e^{xy} = 2$ (h) $e^x \ln y = x$

2. Find the equations of the tangents at the given points on the following curves:-

(a) $x^3 - 2y^3 = 3xy$ at $(2, 1)$

(b) $x^2y^2 = x^2 + 5y^2$ at $(3, \frac{3}{2})$.

(c) $y(x + y)^2 = 3(x^3 - 5)$ at $(2,1)$.

3. For the curve $xy(x + y) = 84$, find $\frac{dy}{dx}$ at $(3, 4)$.

4. Find the gradient of the curve $x^2 + 3xy + y^2 = x + y + 8$ at the point $(1, 2)$.

5. For the curve, $4x^2 + y^3 = 2x + 7y$, find the values of :-

$\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-1, 2)$.

6. Show that $(-1, 3)$ and $(0, 0)$ are stationary points on the curve:-
$3x^2 + 2xy - 5y^2 + 16y = 0$ and find the nature of each.

ANSWERS

1. (a) $\frac{x}{y}$ (b) $\frac{1}{y-1}$ (c) $\frac{-y}{2x}$ (d) $\frac{2(4x+1)}{3(y^2-1)}$

(e) $\frac{1-3x^2y-y^3}{1+3xy^2+x^3}$ (f) $\cot x \cot y$ (g) $\frac{-y}{x}$ (h) $ye^{-x} - y\ln y$

2. (a) $3x - 4y = 2$ (b) $5x + 8y = 27$ (c) $2x - y = 3$

3. $\frac{-40}{33}$ 4. $\frac{-7}{6}$ 5. $2, \frac{-56}{5}$ 6. $(-1, 3)$ Minimum, $(0,0)$ Maximum.

Differentiation
Implicit Differentiation

QUESTIONS

Find $\frac{dy}{dx}$ for each of the following:

(1) $x^2 - y^2 = 4$ (2) $y^2 = 2ax + 2by$ (3) $x^2 - 4x + 6y = 0$ (4) $y^2 = 4ax$

(5) $x^2 = 4ay$ (6) $(x^2 + y^2)^2 - (x^2 - y^2) = 0$ (7) $x^3 + y^3 = 3xy$

(8) $\frac{x^2}{4} + \frac{y^2}{5} = 1$ (9) If $(x + y)^3 - 5x + y = 1$ defines y, find $\frac{dy}{dx}$

(10) If $(x + y)^4 = 4xy$, show that $\frac{dy}{dx} = \frac{y(3x - y)}{x(x - 3y)}$

ANSWERS

Answers

(1) $\frac{x}{y}$ (2) $\frac{a}{y-b}$ (3) $\frac{2}{3} - \frac{1}{3}x$ (4) $\frac{2a}{y}$

(5) $\frac{x}{2a}$ (6) $\frac{x - 2x^3 - 2xy}{2x^2y + 2y^3 + y}$ (7) $\frac{y - 3x^2}{y^2 - 3x}$ (8) $\frac{5x}{4y}$

(9) $\frac{5 - 3(x + y)^2}{1 + 3(x + y)^2}$ (10) Proof

Differentiation
Inverse Trig. Functions

QUESTION

Find the derivatives of the following, using the Chain , Product and Quotient Rules.

1. $y = \sin^{-1}(\sqrt{x})$
2. $y = \tan^{-1}(\sqrt{x})$
3. $y = x\tan^{-1}x$
4. $y = x\tan^{-1}(\frac{x}{2})$
5. $y = x\sin^{-1}x + \sqrt{(1-x^2)}$
6. $y = \cos^{-1}(2x-1)$
7. $y = \sin^{-1}(\frac{x-1}{x+1})$
8. $y = \tan^{-1}(\frac{x-1}{x+1})$
9. $y = \tan^{-1}(\frac{2x}{\sqrt{(1-x^2)}})$
10. $y = \sin^{-1}(\frac{2x}{\sqrt{(1-x^2)}})$
11. $y = \tan^{-1}(\sec x)$
12. $y = \cos(\sin^{-1}x)$
13. $y = \tan^{-1}(e^x)$
14. $y = \sin^{-1}(x^2) - xe^x$

ANSWER

1. $\dfrac{1}{2\sqrt{x(1-x)}}$
2. $\dfrac{1}{2(1-x)\sqrt{x}}$
3. $\tan^{-1}x + \dfrac{x}{1+x^2}$
4. $\tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{2x}{4+x^2}$
5. $\sin^{-1}x$
6. $\dfrac{-1}{\sqrt{(x-x^2)}}$
7. $\dfrac{1}{(x+1)\sqrt{x}}$
8. $\dfrac{1}{1+x^2}$
9. $\dfrac{2}{(1+3x^2)\sqrt{(1-x^2)}}$
10. $\dfrac{2}{(1-x^2)\sqrt{(1-5x^2)}}$
11. $\dfrac{\sec x\tan x}{1+\sec^2 x}$
12. $\dfrac{-x}{\sqrt{1-x^2}}$
13. $\dfrac{e^x}{1+e^{2x}}$
14. $\dfrac{2x}{\sqrt{1-x^4}} - e^{x^2}[2x^2+1]$

QUESTION

Differentiate with respect to x

(1) $\sin^{-1}(1-x)$ (2) $\sin^{-1}\left(\dfrac{x-1}{x+1}\right)$ (3) $\sin^{-1}\sqrt{1-x^2}$ (4) $x\sin^{-1}x^2$ (5) $\dfrac{1}{\sin^{-1}x}$

(6) $\dfrac{\sin^{-1}2x}{\sin x}$ (7) $\sin^{-1}4x$ (8) $\sin^{-1}(\sqrt{x}$ (9) $\sin^{-1}\left(\dfrac{x}{2}\right)$ (10) $\sin^{-1}\left(\dfrac{1}{x}\right)$

(11) $\sin^{-1}(\sin x)$ (12) $x\sin^{-1}x$ (13) $\sin^{-1}(\sqrt{\sin x}$ (14) $\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$

Differentiation
Inverse Trig. Functions

Answers

(1) $-\dfrac{1}{\sqrt{2x-x^2}}$ (2) $\dfrac{1}{\sqrt{x}}$ (3) $-\dfrac{1}{2x\sqrt{1-x}}$ (4) $\sin^{-1} x^2 + \dfrac{2x^2}{\sqrt{1-x^4}}$

(5) $-\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$ (6) $\dfrac{2}{\sqrt{1-4x^2}\,\sin x} - \dfrac{\cos x \sin^{-1} 2x}{\sin x}$ (7) $\dfrac{4}{\sqrt{1-16x^2}}$

(8) $\dfrac{1}{2\sqrt{x-x^2}}$ (9) $\dfrac{1}{\sqrt{4-x^2}}$ (10) $\dfrac{-1}{x\sqrt{x^2-1}}$ (11) 1

(12) $\sin^{-1} x + \dfrac{x}{\sqrt{1-x^2}}$ (13) $\dfrac{\cos x}{2\sqrt{\sin x(1-\sin x)}}$ (14) $\dfrac{1}{1+x^2}$

Differentiate with respect to x:

(1) $\cos^{-1}(3x+5)$ (2) $\cos^{-1}\dfrac{1}{\sqrt{x}}$ (3) $\cos^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)$ (4) $b\cos^{-1}\left(\dfrac{x}{a}\right)$

(5) $\cos^{-1}\left(\dfrac{x}{3}\right)$ (6) $\cos^{-1}(2x^2)$

Answers

(1) $-\dfrac{3}{\sqrt{24-30x-9x^2}}$ (2) $\dfrac{1}{x\sqrt{x-1}}$ (3) $\dfrac{x^{1/2}}{(1+x)(2-2x)^{1/2}}$

(4) $-\dfrac{b}{\sqrt{a^2-x^2}}$ (5) $-\dfrac{1}{\sqrt{9-x^2}}$ (6) $-\dfrac{4x}{\sqrt{1-4x^4}}$

Differentiation
Inverse Trig. Functions

QUESTION

Differentiate with respect to x

(1) $\tan^{-1}\frac{1}{x}$ (2) $\tan^{-1}(1 - 2x)$ (3) $\tan^{-1}\sqrt{1-x}$ (4) $\tan^{-1}(3\sin 2x)$

(5) $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ (6) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ (7) $\tan^{-1}(a-x)$

(8) $(x^2+1)\tan^{-1}x$ (9) $\tan^{-1}\left(\frac{2x}{\sqrt{1-x^2}}\right)$ (10) $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

(11) $x\tan^{-1}x$

Answers

(1) $-\frac{1}{1+x^2}$ (2) $-\frac{1}{1-2x+2x^2}$ (3) $-\frac{1}{2x(1-x)^{½}}$

(4) $\frac{6\cos 2x}{1+9\sin^2 2x}$ (5) $\frac{1}{(1-x^2)^{½}}$ (6) $\frac{3(1+x^2)^2}{(1-3x^2)^2+(3x-x^3)^2}$

(7) $-\frac{1}{1+(a-x)^2}$ (8) $2x\tan^{-1}x + 1$ (9) $\frac{2}{(1+3x^2)(1-x^2)^{½}}$

(10) $\frac{x-2}{2(1-x)^{½}(x^2-x+1)}$ (11) $\tan^{-1}x + \frac{x}{1+x^2}$

Differentiation
Logarithmic Differentiation

QUESTION

Differentiate using logarithmic differentiation.

1. $y = 10^x$
2. $y = 2^{x^2}$
3. $y = x^{-x}$
4. $y = x^{\sin x}$
5. $y = x^{\frac{1}{x}}$
6. $y = x^{\ln x}$
7. $y = (\ln x)^x$
8. $y = (\ln x)^{\ln x}$
9. $y = \dfrac{x^5}{\sqrt{(3x+5)}}$
10. $y = \dfrac{x^3(2x-1)^5}{(x+1)^2}$

ANSWERS

1. $10^x \ln 10$
2. $2^{x^2} \times 2x \ln x$
3. $-x^{-x}(\ln x + 1)$
4. $x^{\sin x}(\cos x \ln x + \dfrac{\sin x}{x})$
5. $x^{\left(\frac{1}{x}-2\right)}(1 - \ln x)$
6. $2x^{\ln x - 1}\ln x$
7. $(\ln x)^x[\ln(\ln x) + \dfrac{1}{\ln x}]$
8. $\dfrac{(\ln x)^{\ln x}}{x}[\ln(\ln x) + 1]$
9. $\dfrac{x^4(27x+50)}{2(3x+5)^{3/2}}$
10. $\dfrac{3x^2(2x-1)^4(4x^2+5x-1)}{(x+1)^3}$

QUESTION

Use logarithmic differentiation to differentiate the following:

(1) 10^x (2) 2^x (3) $x\,a^x$ (4) $x^{\log x}$ (5) $(\sin x)^x$ (6) $2^{\sin x}$

(7) $(\log x)^x$ (8) x^{x^2} (9) $x^2 . 2^x$ (10) $(\log x)^{\log x}$

Answers

(1) $10^x.\ln x$ (2) $2^x.\ln 2$ (3) $a^x + x.a^x.\ln a$ (4) $2\ln x\, x^{\ln x - 1}$

(5) $(\sin x)^x(\ln(\sin x) + \dfrac{x\cos x}{\sin x})$ (6) $2^{\sin x}.\ln 2.\cos x$

(7) $(\ln x)^x[\ln(\ln x) + \dfrac{1}{\ln x}]$ (8) $x^{x^2}(2x\ln x + x)$

(9) $2^x(2x + x^2\ln x)$ (10) $(\ln x)^{\ln x}\dfrac{1}{x}(1 + \ln(\ln x))$

Differentiation
Logarithmic Differentiation

QUESTION

Differentiate

(1) $\dfrac{x^{\frac{1}{2}}(3-x)^{\frac{1}{2}}}{(2x+1)^{\frac{3}{4}}}$ (2) $\dfrac{(2x+1)^{\frac{1}{2}}(4x+5)^{\frac{1}{4}}}{(4x-3)^{\frac{3}{4}}}$ (3) $\dfrac{x^{\frac{3}{4}}(2-x)^{2}(1+x)}{(5+2x)^{3}}$

Answers

(1) $\dfrac{x^{1/2}(3-x)^{1/2}}{(2x+1)^{3/4}}\left(\dfrac{1}{2x}-\dfrac{1}{2(3-x)}-\dfrac{3}{2(2x+1)}\right)$

(2) $\dfrac{(4x+5)^{1/4}}{(2x+1)^{1/2}(4x-3)^{3/4}}+\dfrac{(2x+1)^{1/2}}{((4x-3)(4x+5))^{3/4}}-\dfrac{3(2x+1)^{1/2}(4x+5)^{1/4}}{(4x-3)^{7/4}}$

(3) $\dfrac{x^{3/4}(2-x)^{2}(1+x)}{(5+2x)^{3}}\left(\dfrac{3}{4x}-\dfrac{2}{2-x}+\dfrac{1}{1+x}-\dfrac{6}{5+2x}\right)$

Differentiation
Rectilinear Motion

QUESTIONS

1. A body moves in a straight line and the motion is such that x, the number of metres from a fixed point after t secs, is given by

$$x = 3 - 4t + t^2.$$

   (a) How far is the body from the fixed point at the start ?
   (b) What is its position after 4 seconds ?
   (c) What is its velocity after 3 seconds ?
   (d) What is the initial acceleration ?

2. If $x = 4t^3 - 3t^2 - 2t - 1$, where x is in metres and t in seconds, find
   (a) The velocity at the end of the 3rd and 4th seconds.
   (b) The acceleration at the end of the 3rd and 4th seconds.
   (c) The average velocity during the 4th second.
   (d) The average acceleration during the 4th second.

3. A motor bike starts from rest and its displacement x m after t secs is given by:-

$$x = \frac{1}{6}t^3 + \frac{1}{4}t^2$$

Calculate the initial acceleration and the acceleration at the end of the 2nd second.

4. A body is moving in a straight line, so that after t seconds its displacement x metres from a fixed point O, is given by

$$x = 9t + 3t^2 - t^3.$$

   (a) Find the initial displacement, velocity and acceleration of the body.
   (b) Find the time at which the body is instantaneously at rest.

5. A body moves along a straight line so that after t seconds its displacement from a fixed point O on the line is x metres.
If $x = 3t^2(3 - t)$, find:–
   (a) the initial velocity and acceleration.
   (b) the velocity and acceleration after 3 seconds.

Differentiation
Rectilinear Motion

Answers

1.	(a)	3 m	(b)	3 m	(c)	2 m/s	(d)	2 m/s^2
2.	(a)	88 m/s, 166 m/s	(b)	66 m/s2, 90 m/s^2				
	(c)	127 m/s	(d)	78 m/s^2				
3.	(a)	$\frac{1}{2}$ m/s^2	(b)	$2\frac{1}{2}$ m/s^2				
4.	(a)	0, 9 m/s, 6 m/s^2	(b)	3 secs				
5.	(a)	0 m/s, 18 m/s^2	(b)	–27m/s, –36 m/s^2				

Questions

1. A body travels along a straight line such that $s = t^3 - 6t^2 + 9t + 1$ where s represents its displacement in metres from the origin t seconds after observations began.

   (a) Find when the velocity is zero.

   (b) When is the displacement, s, decreasing (i.e. when is the velocity negative)?

   (c) When is the velocity of the body decreasing (i.e. when is the acceleration negative)?

   (d) Describe the motion of the particle during the first 4 seconds of its journey.

2. Find the position and acceleration of a particle, moving in a straight line, when it comes to rest given that its displacement s is given by $s(t) = 120t - 16t^2$.

3. The height, s metres, reached in t seconds by a body thrown vertically upwards with initial velocity u m/s is given by the formula $s = ut - 5t^2$. Find an expression for the maximum height reached.

4. A particle moves in a straight line, and its position s(t) at time $t \geq 0$ with respect to a fixed point on the line is given by $s(t) = -2t^3 + 7t^2 - 4t + 1$. Find the velocity of the particle, and determine when it is moving to the left and when it is moving to the right. Compute the total distance travelled by the particle between $t = 0$ and $t = 5$.

Differentiation
Rectilinear Motion

ANSWERS

1. (a) $t = 1$ and $t = 3$ (b) $1 < t < 3$ (c) $t < 2$

   (d) At $t = 0$, $s = 1$. Body moves in positive direction till $t = 1\ (s = 5)$. Body stops at $t = 1$. Body moves in negative direction till $t = 3\ (s = 1)$. Body stops at $t = 3$. Body moves in positive direction for $t > 3$. At $t = 4$, $s = 5$.

   N.B. Displacement is always positive.

2. Rests when $t = 3.75$ i.e. $s = 225$ and $a = -32$.

3. Max. height at $t = u/10$ i.e. max height $= u^2/20$.

4. $v = -6t^2 + 14t - 4$. Moving to the left when $t < 1/3$ and $t > 2$. Moving to the right when $1/3 < t < 2$. Total distance travelled in first 5 seconds is 104.

Differentiation
Closed Interval Extrema

QUESTION

1 Find the global extrema for the following functions.

a $f(x) = x^2 - 3x + 2$ in the interval $[1, 3]$

b $f(x) = x^2 - 3x + 2$ for $-1 \le x \le 2$

c $f(x) = 4x^3 - x^4$ for $1 \le x \le 4$

d $f(x) = \dfrac{x}{x^4 + 3}$ for $-2 \le x \le 2$

e $f(x) = \ln(x + 1) - x + 1$ for $0.5 \le x \le 1.5$

f $f(x) = \dfrac{1}{\sqrt{x^2 + 1}}$ for $-1 \le x \le 3$

2 Explore the global extrema of $f(x) = \dfrac{(2x - 3)(3x - 2)}{x}$ when the domain is:

a $0.5 \le x \le 2$

b $0 < x \le 2$

c $-1 \le x < 0$

d $-2 \le x \le -0.5$

3 Explore the global maxima and minima of these functions

a $f(x) = e^{2x} + 4$ for $-2 \le x \le 0$

b $f(x) = \ln(3x - 5)$ in the interval $[2, 3]$

4 Find the global extrema of $f(x) = x^2 + x^{-1}$ in the interval:

a $[-2, 0)$

b $(0, 2]$

5 Examine the following piecewise functions for global extrema.

a $f(x) = \begin{cases} x & \text{when } -1 \le x \le 2 \\ 4 - x & \text{when } 2 \le x \le 4 \\ x - 1 & \text{when } 4 \le x \le 5 \end{cases}$

b $f(x) = \begin{cases} x(2 - x) & \text{when } -1 \le x \le 2 \\ 2 - x & \text{when } 2 \le x \le 3 \\ x^2 - 6x - 1 & \text{when } 3 \le x \le 5 \end{cases}$

c $f(x) = \begin{cases} 3^x & \text{when } 0 \le x \le 1 \\ 3^{-x+2} & \text{when } 1 \le x \le 2 \\ 1 & \text{when } 2 \le x < 4 \end{cases}$

careful!

d $f(x) = \begin{cases} -x & \text{when } -1 < x \le 1 \\ x - 2 & \text{when } 1 \le x \le 2 \\ \frac{1}{2}x + 1 & \text{when } 2 \le x < 4 \end{cases}$

e $f(x) = \begin{cases} -x^{-1} & \text{when } -1 \le x < 0 \\ x & \text{when } 0 \le x \le 2 \\ x - 4 & \text{when } 2 \le x \le 3 \end{cases}$

f $f(x) = \begin{cases} \dfrac{x}{\sqrt{(x)^2}} & \text{when } -5 \le x < 0 \\ 0 & \text{when } x = 0 \\ \dfrac{x}{\sqrt{(x)^2}} & \text{when } 0 < x \le 5 \end{cases}$

6 The functions $f(x) = \dfrac{e^x + e^{-x}}{2}$ and $g(x) = \dfrac{e^x - e^{-x}}{2}$ are both important functions in mathematics.
Examine these functions for critical points and possible global extrema.
The functions are defined over all real x

7 Study the function $f(x) = x^2 \ln x$ in the domain $0 < x \le 2$ for global extrema, given that there is a critical point in the region $0.4 < x < 0.8$.

Differentiation
Closed Interval Extrema

ANSWERS

1 a global max = 2; global min = −0.25
b global max = 6; global min = −0.25
c global max = 27; global min = 0
d global max = 0.24; global min = −0.25
e global max = 0.416 29;
global min = 0.905 47
f global max = 1; global min = $1/\sqrt{10}$

2 a global max = 2; global min = −1
b no global max; global min = −1
c global max = −25; no global min
d global max = −25; global min = −28

3 a global max = 5; global min = 4.0183
b global max = 1.3863; global min = 0

4 a global max = 3.5; no global min
b no global max; global min = 1.8899

5 a global max = 4; global min = −1
b global max = 1; global min = −10
c global max = 3; global min = 1
d global max = 3; global min = −1
e no global max; global min = −2
f global max = 1; global min = −1

6 a local min at (1, 0); no global max;
global min = 0
b no local max or min; no global max or min

7 endpt max at (2, ln 16); local min at
(0.603, −0.184); global max = 2.77

Differentiation
Optimisation

QUESTION

1. Four squares each of side s cm are cut from the corners of a metal square of side 16 cm. The metal is then bent to make an open topped tray of volume, V cm^3.

   (a) Prove that $V = 4s^3 - 64s^2 + 256s$.

   (b) Find the value of s which makes the volume a maximum.

2. A sector of a circle with radius r cm has an area of 16 cm^2.

   (a) Show that the perimeter P cm of the sector is given by

$$P(r) = 2(r + \frac{16}{r})$$

   (b) Find the minimum value of P.

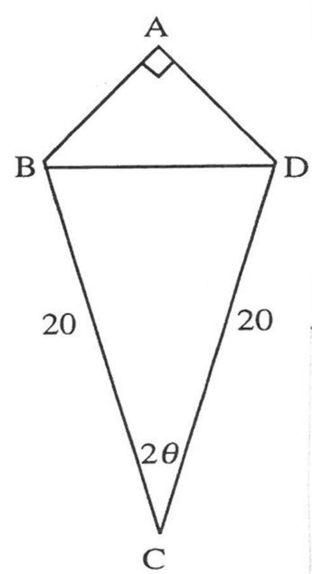
3. A cylindrical tank has a radius of r metres and a height of h metres. The sum of the radius and the height is 2 metres.

   (a) Prove that the volume, in m^3, is given by

$$V = \pi r^2(2 - r)$$

   (b) Find the maximum volume.

4. ABCD is a kite which has AC as its axis of symmetry. Angle BAD is right angled and BC and DC are 20 cm.



   (a) Show that the area of triangle BCD is given by the expression $200 \sin 2\theta$ and find an expression for BD^2.

   (b) Use this expression for BD^2 to show that the area of triangle BAD is given by the expression $200 - 200 \cos 2\theta$ and hence show that the area of the kite is given by the expression
   $A(\theta) = 200(1 - \cos 2\theta + \sin 2\theta)$

   (c) Find the value of θ which makes the area a maximum and find this maximum area.

ANSWER

1.	(a)	Proof	(b)	$s = {}^{8}/_{3}$ cm	4.	(a)	Proof	(b)	Proof
2.	(a)	Proof	(b)	P = 16 cm		(c)	$\theta = {}^{8\pi}/_{3}$, $A = 200(1 + \sqrt{2})$		
3.	(a)	Proof	(b)	$V = {}^{32\pi}/_{27}$ cm3					

Differentiation
Optimisation

QUESTION

(1) The turning effect, T, of a power boat, is given by the formula $T = 8\cos x \sin^2 x$, $0 < x < \pi/2$ where x is the angle (in radians) between the rudder and the central line of the boat. Find the size of x which maximises the turning effect.

(2) Find the point on the curve $y = x^2 + x - 1$ which is closest to $(-1, 0)$.

(3) A rectangular box with a square base and no top is to be made to contain 750cm^3. The material for the base costs $30\text{p} / \text{cm}^2$, while the material for the sides cost $20\text{p} / \text{cm}^2$. Find the dimensions for the minimal cost box.

(4) A man is on an oilrig which has a perpendicular distance of 4km to the straight shore (A). He must reach a point B on the shore which is 10km away from A. The man rows at 5km/h and runs at 13km/h. Which point (C) on the shore should he aim to arrive at so as to reach B in the shortest possible time?

ANSWER

(1) SPs occur when $\tan x = \pm\sqrt{2}$ and $\sin x = 0$. Therefore, for given domain, $x = 0.96$ radians is the only SP and this can be shown via a nature table to be a maximum.

(2) $(-3/2, -1/4)$

(3) Cost, in £s, of box is $C = \dfrac{3x^2}{10} + \dfrac{600}{x}$ where x is length of square base, measured in cm. Minimum cost occurs when $x = 10\,\text{cm}$ so $h = 7.5\,\text{cm}$.

(4) Time, in hours, of journey is $T = \dfrac{\sqrt{16 + x^2}}{5} + \dfrac{10 - x}{13}$, where x is the distance from A along the shore at which the man starts to run, measured in km. Shortest time corresponds to $x = 5/3\,\text{km}$ i.e. He should aim to reach shore 5/3km from A.

Differentiation
Motion in a Plane

QUESTIONS

1. At time t, the position of a moving point is given by

$$x = t(2 - t), \quad y = t(3 - t).$$

Find the speed when $t = 0$ and when $t = 2$.

2. At time t, the position of a moving point is given by

$$x = t + 1, \quad y = t^2 - 1.$$

Find the speed when $t = 2$.

3. At time t, the position of a moving point is given by

$$x = \cos 2t, \quad y = 2\sin t.$$

Find the speed when $t = 0$.

4. At time t, the position of a moving point is given by

$$x = e^t, \quad y = e^{-2t}.$$

Find the speed when $t = \ln 3$.

5. At time t, the position of a moving point is given by

$$x = \sec t, \quad y = \tan t.$$

Find the speed when $t = \dfrac{\pi}{6}$.

6. At time t, the position of a moving point is given by

$$x = \ln(t + 1), \quad y = t^2.$$

Find the speed when $t = 1$.

7. A particle moves so that its position at time t is given by:- $x = 4\cos t$, $y = 3\sin t$.

Show that its speed is $\sqrt{9 + 7\sin^2 t}$.

Hence find the maximum and minimum speeds and the corresponding positions of the particle.

ANSWERS

1. $\sqrt{13}$, [redacted] $\sqrt{5}$
2. $\sqrt{17}$
3. 2
4. $\frac{1}{9}\sqrt{730}$
5. $\frac{1}{3}\sqrt{20}$
6. $\frac{1}{2}\sqrt{17}$
7. Max of 4 at (0,3) and (10,–3). Min of 3 at (4,0) and (–4,0).

Differentiation
Motion in a Plane

QUESTION

(1) A particle moving in the xy plane is governed by the laws

$$x = t^2 + 3\sin t \qquad y = 2t^2 - \cos t$$

Find, when $t = 10\pi$,

(a) the speed of the particle

(b) the direction of motion relative to the x direction.

(2) As it takes off, a light aircraft's movement can be modelled by the equations

$$x = 3 + \ln(t+2) \qquad y = 4 + \ln(t+3)$$

Find, when $t = 10$,

(a) the magnitude and direction of its velocity

(b) the magnitude and direction of its acceleration.

ANSWERS

(1) At given time, speed of particle is 141.86 and it makes an angle of $62.35°$ with positive direction of x axis (measured anti-clockwise).

(2) (a) speed = 0.11 and direction is $42.7°$ anti-clockwise from positive direction of x axis.

(b) acc. has magnitude 0.009 and direction is $220.43°$ anti-clockwise from positive direction of x axis.

Differentiation
Related Rates of Change

QUESTION

1. If $P = (2m + 3)^4$, find $\dfrac{dm}{dt}$ when $m = 1$, given that $\dfrac{dP}{dt} = 2$

2. If $r = \dfrac{1+p}{1+p^2}$, find $\dfrac{dp}{dt}$ when $p = 2$, given that $\dfrac{dr}{dt} = 14$

3. The radius of a circular oil slick is increasing at a rate of 0·2 metres/second. Find the rate at which the area is increasing when the radius is 10 m.

4. A rectangle has dimensions x cm and y cm. Both x and y are changing but in such a way that the area of the rectangle remains constant at 40cm^2.

   (a) Show that $\dfrac{dy}{dt} = -\dfrac{40}{x^2}\cdot\dfrac{dx}{dt}$

   (b) If x increases at a rate of 0·2 cm/sec, find the rate at which y is changing when $x = 8$.

5. The volume of a cylinder is constant at 50 cm^3, but both height h and radius r are changing.

   (a) Show that $\dfrac{dh}{dt} = -\dfrac{100}{\pi r^3}\cdot\dfrac{dr}{dt}$

   (b) At an instant when the radius is 5 cm, the height is decreasing by 3 cm/sec. Find the rate of change of radius at this instant.

6. A particle moves along a straight line so that its velocity, v m/sec, when it is s metres from a fixed point, is given by $v = s^2 + 3$.

   Find an expression for its acceleration, a metres/sec^2, in terms of s.

7. A spherical balloon is inflated so that its volume increases at a constant rate of 200 cm^3/second. Find the rate of increase of the surface area of the balloon when the radius is 100 cm.

8. The volume of a sphere is increasing at a rate of 6 cm^3/sec. Find the rate at which the surface area is increasing when the radius is 3 cm.

ANSWERS

1. ~~2000~~ $\frac{1}{500}$	2. -50	3. 4π
4. (a) Proof	(b) $-0{\cdot}125$ cm/sec	5. 11·8 cm/sec
6. $2s(s^2 + 3)$	7. 4 ~~02~~ cm/sec^2	8. 4 cm/sec^2.

Differentiation
Related Rates of Change

QUESTIONS

1. The radius of a cone of constant height 30cm is increasing at a rate of 1cm/sec. Calculate the rate of change of the volume when $r = 2$ cm.

2. A spherical snowball is melting so that it remains spherical at all times. When its radius is equal to 2cm, the radius is decreasing at a rate of $\frac{1}{16}$ cm/s. Find the rate at which the volume of the snowball is decreasing at that moment.

   Hint: Remember the negative as the radius is decreasing.

3. A plane flies directly overhead at a height of 1 km. How fast is the plane moving away from the origin when it is 1.25 km away if the speed of the plane at that moment is 300km/h?

4. A particle moves along the curve with equation $x^2 - 3xy + 2y^2 = 4$. If at the point $(2, 3)$ the rate of change of the y co-ordinate is 3 units per second, calculate the rate of change of the x co-ordinate with respect to time.

5. The dimensions of a rectangle vary so that its area remains constant at 24 cm^2. When one side is 6cm long that side is decreasing by 10cm/sec. At what rate is the other side changing at that moment?

6. Grain is being ejected from a chute at a rate of 1.5m^3 / minute and is forming a conical heap whose height is half of its base radius. Find the rate of increase of the height of this cone when the base has a radius of 3 metres.

   Hint: You first of all need to express r in terms of h in the formula for the volume of a cone.

ANSWERS

1. 40π cm^3/sec i.e. volume is increasing at 40π cm^3/sec.

2. $-\pi$ cm^3/sec i.e. volume is decreasing at π cm^3/sec.

3. 180km/h

4. $x = 18/5$ units per second.

5. increasing at a rate of $20/3$ cm/sec.

6. $1/6\pi$ m^3/min.