Integration
Revision

Integrate the following functions :-

1. $f(x) = 8x^3$ 2. $f(x) = \dfrac{6}{x^3}$ 3. $f(x) = \sqrt{x}$

4. $f(x) = \dfrac{1}{\sqrt{x}}$ 5. $f(x) = \dfrac{3x^4 + 6}{x^2}$ 6. $f(x) = \dfrac{1 - 3x}{\sqrt{x}}$

7. $f(x) = (3x + 4)^4$ 8. $f(x) = (1 - 2x)^4$ 9. $f(x) = \dfrac{1}{(2x - 3)^2}$

10. $f(x) = \dfrac{1}{\sqrt{(4x + 1)}}$ 11. $f(x) = \dfrac{3}{(3x + 1)^{\frac{3}{2}}}$ 12. $f(x) = \sin 3x$

13. $f(x) = \cos\left(\dfrac{1}{3}x\right)$ 14. $f(x) = \sin^2 x$ 15. $f(x) = \cos^2\left(\dfrac{1}{2}x\right)$

ANSWER

1. $2x^4 + c$ 2. $-\dfrac{3}{x^2} + c$ 3. $\dfrac{2}{3}x^{\frac{3}{2}} + c$ (etc.)

4. $2\sqrt{x}$ 5. $x^3 - \dfrac{6}{x}$ 6. $2x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$

7. $\dfrac{(3x + 4)^5}{15}$ 8. $-\dfrac{(1 - 2x)^5}{10}$ 9. $-\dfrac{1}{2(2x - 3)}$

10. $\dfrac{\sqrt{4x + 1}}{2}$ 11. $-\dfrac{2}{\sqrt{3x + 1}}$ 12. $-\dfrac{1}{3}\cos 3x$

13. $3\sin\dfrac{1}{3}x$ 14. $\dfrac{1}{2}x - \dfrac{1}{4}\sin 2x$ 15. $\dfrac{1}{2}x + \dfrac{1}{2}\sin x$

1. Find

(a) $\displaystyle\int 4x^6 \, dx$ (b) $\displaystyle\int_0^2 3x^2 - 7x \, dx$ (c) $\displaystyle\int (2x - 3)^2 \, dx$ (d) $\displaystyle\int \dfrac{x^3 - 7}{\sqrt{x}} \, dx$

(e) $\displaystyle\int_0^\pi \cos 2x \, dx$ (f) $\displaystyle\int (4x - 5)^3 \, dx$ (g) $\displaystyle\int_2^5 (3x + 1)^{-2} \, dx$

ANSWER

1.   (a) $\dfrac{4x^7}{7}$ (b) -6 (c) $\dfrac{(2x - 3)^3}{6}$ (d) $\dfrac{2x^{7/2}}{7} - 14x^{1/2}$

(e) 0 (f) $\dfrac{(4x - 5)^4}{16}$ (g) $\dfrac{3}{112}$

Integration
Revision

QUESTIONS

Integrate:

(1) $\int 3x \, dx$ (2) $\int 5x^2 \, dx$ (3) $\int \frac{dx}{2}$ (4) $\int d\theta$ (5) $\int \left(4x^2 - 5x + 1\right) dx$

(6) $\int x\left(8x - \frac{1}{2}\right) dx$ (7) $\int (2x-3)(x+4) \, dx$ (8) $\int \sqrt[3]{x} \, dx$ (9) $\int \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx$

(10) $\int a \, dt$ (11) $\int \left(\frac{1}{x^3} - \frac{1}{x^2} - 1\right) dx$ (12) $\int \left(\frac{x^2 + 3x}{x}\right) dx$

(13) $\int \frac{x^3 - x^2 + 1}{x^2} \, dx$ (14) $\int \sqrt{2x+3} \, dx$ (15) $\int \frac{dx}{(3-2x)^2}$

(16) $\int \sqrt{ax+b} \, dx$ (17) $\int (4x+5)^2 \, dx$

(18) $\int x(1+x)(1+x^2) \, dx$ (19) $\int \left(4x^3 - 6x^2 + 5\right) dx$ (20) $\int \frac{x^4 + 1}{x^2} \, dx$

(21) $\int \frac{dx}{(2x-1)^4}$ (22) $\int \frac{dx}{(3-2x)^3}$ (23) $\int \left(\frac{x^4 + 3x + 1}{x^3}\right) dx$

Answers

(1) $\frac{3x^2}{2} + C$ (2) $\frac{5x^3}{3} + C$ (3) $\frac{x}{2} + C$ (4) $\theta + C$

(5) $\frac{4x^3}{3} - \frac{5x^2}{2} + x + C$ (6) $\frac{8x^3}{3} - \frac{x^2}{4} + C$ (7) $\frac{2x^3}{3} + \frac{5x^2}{2} - 12x + C$

(8) $\frac{3x^{\frac{4}{3}}}{4} + C$ (9) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (10) $at + C$ (11) $-\frac{1}{2x^2} + \frac{1}{x} - x + C$

(12) $\frac{x^2}{2} + 3x + C$ (13) $\frac{x^2}{2} - x - \frac{1}{x} + C$ (14) $\frac{1}{3}(2x+3)^{\frac{3}{2}} + C$

(15) $\frac{1}{2(3-2x)} + C$ (16) $\frac{2(ax+b)^{\frac{3}{2}}}{3a} + C$ (17) $\frac{1}{12}(4x+5)^3 + C$

(18) $\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + C$ (19) $x^4 - 2x^3 + 5x + C$ (20) $\frac{x^3}{3} - \frac{1}{x} + C$

(21) $-\frac{1}{6(2x-1)^3} + C$ (22) $\frac{1}{4(3-2x)^2} + C$ (23) $\frac{x^2}{2} - \frac{3}{x} - \frac{1}{2x^2} + C$

Integration
Basic Substitution

QUESTION

Use the method of substitution to find

(a) $\int 5x^2(x^3-4)^4\,dx$ (b) $\int(3x^2-5x)^5(6x-5)\,dx$ (c) $\int e^{5x}\,dx$

(d) $\int_2^6 2xe^{x^2}\,dx$ (e) $\int 4\sin x\cos^3 x\,dx$ (f) $\int \cos x\sin^4 x\,dx$

(g) $\int e^x\sin(e^x)\,dx$ (h) $\int_1^8 \dfrac{2x}{x^2+8}\,dx$ (i) $\int \dfrac{5\cos x}{1+\sin x}\,dx$

(j) $\int_0^{\pi/2} \dfrac{1-2\sin x}{x+2\cos x}\,dx$ (k) $\int \sec^2 x\tan^4 x\,dx$ (l) $\int 5x^2(x^3-4)^{-4}\,dx$

ANSWER

(a) $\dfrac{1}{3}(x^3-4)^5$ (b) $\dfrac{1}{6}(3x^2-5x)^6$ (c) $\dfrac{1}{5}e^{5x}$ (d) $e^{36}-e^4\approx 4.31\times 10^{15}$

(e) $-\cos^4 x$ (f) $\dfrac{1}{5}\sin^5 x$ (g) $-\cos e^x$ (h) $3\ln 2\approx 2.0794$

(i) $5\ln|1+\sin x|$ (j) $-2\ln 2+\ln\pi\approx -0.2416$ (k) $\dfrac{1}{5}\tan^5 x$

(l) $-\dfrac{5}{9(x^3-4)^3}$

QUESTION

1. $\int x(x^2-3)^5\,dx$ & 2. $\int x^2(x^3-1)^2\,dx$

3. $\int x\sqrt{(1-x^2)}\,dx$ & 4. $\int \cos x\sin^4 x\,dx$

5. $\int \dfrac{x}{(1-x^2)^3}\,dx$ & 6. $\int \dfrac{x^3}{1+x^4}\,dx$

7. $\int \dfrac{\sin x}{\cos^3 x}\,dx$ & 8. $\int \dfrac{\cos x}{\sin^6 x}\,dx$

9. $\int \dfrac{e^x}{3e^x-1}\,dx$ & 10. $\int \dfrac{\sec^2 x}{\tan x}\,dx$

ANSWER

1. $\dfrac{1}{12}(x^2-3)^6$ 2. $\dfrac{1}{9}(x^3-1)^3$ 3. $-\dfrac{1}{3}(1-x^2)^{\frac{3}{2}}$

4. $-\dfrac{1}{5}\sin^5 x$ 5. $\dfrac{1}{4(1-x^2)^2}$ 6. $\dfrac{1}{4}\ln(1+x^4)$

7. $\dfrac{1}{2}\sec^2 x$ 8. $-\dfrac{1}{5\sin^5 x}$ 9. $\dfrac{1}{3}\ln(3e^x-1)$

10. $\ln(\tan x)$

Integration
Basic Substitution

Integrate the following functions :-

1. $f(x) = e^{5x}$

2. $f(x) = e^{-2x}$

3. $f(x) = 3e^{\frac{1}{2}x}$

4. $f(x) = \dfrac{1}{3x}$

5. $f(x) = \dfrac{1}{x+5}$

6. $f(x) = \dfrac{1}{2x-3}$

7. $f(x) = \left(e^x + e^{-x}\right)^2$

8. $f(x) = \dfrac{1+e^x}{e^x}$

9. $f(x) = e^{2x} + \dfrac{1}{e^{2x}}$

10. $f(x) = \dfrac{6}{3x+2}$

11. $f(x) = \dfrac{3}{1-2x}$

12. $f(x) = \dfrac{5}{6-7x}$

13. $f(x) = \sec^2 4x$

14. $f(x) = \sec^2(\pi + 2x)$

15. $f(x) = 3\sec^2 2x$

ANSWER :

1. $\dfrac{1}{5}e^{5x}$

2. $-\dfrac{1}{2}e^{-2x}$

3. $6e^{\frac{1}{2}x}$

4. $\dfrac{1}{3}\ln 3x$

5. $\ln(x+5)$

6. $\dfrac{1}{2}\ln(2x-3)$

7. $\dfrac{1}{2}e^{2x} + 2x - \dfrac{1}{2}e^{-2x}$

8. $x - e^{-x}$

9. $\dfrac{1}{2}e^{2x} - \dfrac{1}{2}e^{-2x}$

10. $2\ln(3x+2)$

11. $-\dfrac{3}{2}\ln(1-2x)$

12. $-\dfrac{5}{7}\ln(6-7x)$

13. $\dfrac{1}{4}\tan 4x$

14. $\dfrac{1}{2}\tan(\pi + 2x)$

15. $\dfrac{3}{2}\tan 2x$

QUESTION

(1) $\displaystyle\int \frac{6x+5}{3x^2+5x+1}\,dx$

(2) $\displaystyle\int \frac{x}{2x^2+3}\,dx$

(3) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

(4) $\displaystyle\int \frac{e^{3t}}{e^{3t}+6}\,dt$

(5) $\displaystyle\int \frac{x}{x^2-1}\,dx$

(6) $\displaystyle\int \frac{1+\cos 2x}{2x+\sin 2x}\,dx$

(7) $\displaystyle\int \tan x\,dx$

(8) $\displaystyle\int \cot x\,dx$

Answers

(1) $\ln(3x^2+5x+1) + C$

(2) $\dfrac{1}{4}\ln(2x^2+3) + C$

(3) $\ln(e^x+1) + C$

(4) $\dfrac{1}{3}\ln(e^{3t}+6) + C$

(5) $\dfrac{1}{2}\ln(x^2-1) + C$

(6) $\dfrac{1}{2}\ln(2x+\sin 2x) + C$

(7) $-\ln(\cos x) + C$

(8) $\ln(\sin x) + C$

Integration
Basic Substitution

QUESTION

(1) $\int \dfrac{\sin x}{1+\cos x}\,dx$

(2) $\int \dfrac{1-2\sin x}{x+2\cos x}\,dx$

(3) $\int \sin x . \cos^4 x \, dx$

(4) $\int \sin^3 x . \cos^2 x \, dx$

(5) $\int \dfrac{\sin 2x}{a+b\cos^2 x}\,dx$

(6) $\int \dfrac{2x^3}{1+x^4}\,dx$

(7) $\int x^2 e^{x^3}\,dx$

(8) $\int 3x e^{x^2}\,dx$

(9) $\int x e^{(x^2+1)}\,dx$

Answers

(1) $-\ln(\cos x) + C$

(2) $\ln(x+\cos x) + C$

(3) $-\dfrac{1}{5}\cos^5 x + C$

(4) $\dfrac{1}{3}\cos^3 x - \dfrac{2}{5}\cos^5 x + C$ (using the substitution $u = \cos^2 x$)

(5) $-\dfrac{1}{b}\ln(a+b\cos^2 x) + C$

(6) $\dfrac{1}{2}\ln(a^4+1) + C$

(7) $\dfrac{e^x}{3} + C$

(8) $3e^{x^2/2} + C$

(9) $\dfrac{1}{2}e^{(x^2+1)} + C$

ANSWER QUESTION

(1) $\int \cos 5x \, dx$

(2) $\int \sin 3x \, dx$

(3) $\int \cos(2x+a)\,dx$

(4) $\int \sin\left(\dfrac{1}{3}x\right)dx$

(5) $\int \sec^2\left(\dfrac{x}{3}\right)dx$

(6) $\int \left(\cos 3x - \sin\left(\dfrac{x}{3}\right)\right)dx$

(7) $\int \sin 2x \, dx$

(8) $\int \cos(1-3x)\,dx$

(9) $\int \sec^2(2x+1)\,dx$

Answers

(1) $\dfrac{1}{5}\sin 5x + C$

(2) $-\dfrac{1}{3}\cos 3x + C$

(3) $\dfrac{1}{2}\sin(2x+a) + C$

(4) $-3\cos\dfrac{1}{3}x + C$

(5) $3\tan\dfrac{x}{3} + C$

(6) $\dfrac{1}{3}\sin 3x + 3\cos\dfrac{x}{3} + C$

(7) $-\dfrac{1}{2}\sin 2x + C$

(8) $-\dfrac{1}{3}\sin(1-3x) + C$

(9) $\dfrac{1}{2}\tan(2x+1) + C$

Integration
Basic Substitution

QUESTION

(1) $\int \dfrac{1.4}{x}\,dx$

(2) $\int \dfrac{dx}{x+3}$

(3) $\int \left(\dfrac{3}{x-1}-\dfrac{4}{x-2}\right)dx$

(4) $\int \dfrac{dx}{3-2x}$

(5) $\int 3e^{2x}\,dx$

(6) $\int e^{3x-1}\,dx$

(7) $\int \left(e^{x}+e^{-x}\right)^{2}dx$

(8) $\int e^{3x}\,dx$

(9) $\int e^{\frac{x}{2}}\,dx$

(10) $\int \left[\dfrac{1+e^{x}}{e^{x}}\right]dx$

(11) $\int \dfrac{dx}{1-x}$

(12) $\int \left(-e^{-\frac{2}{3}x}\right)dx$

(13) $\int 3e^{-\frac{2}{3}(1-x)}\,dx$

(14) $\int \dfrac{dx}{3x+4}$

(15) $\int \dfrac{dx}{-2x-3}$

Answers

(1) $1.4\ln x + C$

(2) $\ln(x+3) + C$

(3) $3\ln(x-1) - 4\ln(x-2) + C$

(4) $-\dfrac{1}{2}\ln(3-2x) + C$

(5) $\dfrac{3}{2}e^{2x} + C$

(6) $\dfrac{1}{3}e^{3x-1} + C$

(7) $\dfrac{1}{2}e^{2x} + 2x - \dfrac{1}{2}e^{-2x} + C$

(8) $\dfrac{1}{3}e^{3x} + C$

(9) $2e^{\frac{x}{2}} + C$

(10) $-e^{-x} + x + C$

(11) $-\ln(1-x) + C$

(12) $x + \dfrac{3}{2}e^{-\frac{2}{3}x} + C$

(13) $\dfrac{9}{4}e^{-\frac{2}{3}(1-x)} + C$

(14) $\dfrac{1}{3}\ln(3x+4) + C$

(15) $-\dfrac{1}{2}\ln(-2x-3) + C$

Integration
Basic Substitution

1. $\int_1^2\left(\dfrac{2}{x^2}-\dfrac{5}{x^4}\right)dx$

2. $\int_{\frac{1}{4}}^1\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)dx$

3. $\int_1^4\dfrac{2x^2+3}{\sqrt{x}}dx$

4. $\int_1^6\sqrt{(x+3)}\,dx$

5. $\int_0^1\dfrac{1}{(4+5x)^2}dx$

6. $\int_3^{12}(x-4)^{\frac{1}{3}}dx$

7. $\int_0^{\frac{\pi}{4}}\cos 2x\,dx$

8. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos ec^2x\,dx$

9. $\int_0^{2\pi}\sin^2 x\,dx$

10. $\int_0^2 e^{-3x}\,dx$

11. $\int_0^1 e^{1-x}\,dx$

12. $\int_0^1 e^{\frac{x}{2}}\,dx$

13. $\int_5^9\dfrac{1}{x-3}dx$

14. $\int_0^1\dfrac{1}{3x+2}dx$

15. $\int_{-4}^0\dfrac{1}{1-2x}dx$

ANSWER

1. $-\dfrac{11}{24}$

2. $-\dfrac{5}{12}$

3. $\dfrac{154}{5}$

4. $\dfrac{38}{3}$

5. $\dfrac{1}{36}$

6. 9

7. $\dfrac{1}{2}$

8. 1

9. π

10. $\dfrac{2}{5}\left(1-\dfrac{1}{e}\right)$

11. $e-1$

12. $2e-2$

13. $\ln 3$

14. $\dfrac{1}{3}\ln\dfrac{5}{2}$

15. $\ln 3$

QUESTION

Evaluate:

(1) $\int_0^2\dfrac{2x^3}{1+x^4}dx$

(2) $\int_1^3 x^2 e^{x^3}dx$

(3) $\int_0^2 xe^{(x^2+1)}dx$

(4) $\int_0^1\dfrac{x^2}{1+x^3}dx$

(5) $\int_1^e\dfrac{\ln x}{x^2}dx$

(6) $\int_0^1\dfrac{x}{\sqrt{3+x^2}}dx$

(7) $\int_0^{\frac{\pi}{4}}\tan^3 x\,dx$

Answers

(1) $\dfrac{1}{2}\ln 17$

(2) $\dfrac{1}{3}e(e^{26}-1)$

(3) $\dfrac{1}{2}e(e^4-1)$

(4) $\dfrac{1}{3}\ln 2$

(5) $\dfrac{e-2}{e}$

(6) $2-\sqrt{3}$

(7) $\dfrac{1}{2}-\dfrac{1}{2}\ln 2$

Integration
"Given" Substitutions

Work out the following using the given substitution

(a) $\int \dfrac{x+1}{x^2+2x-7}\,dx,$ $u = x^2+2x-7$

(b) $\displaystyle\int_e^{e^2} \dfrac{1}{x \ln x}\,dx,$ $u = \ln x$

(c) $\int \dfrac{x^3}{(1+x^2)^{3/2}}\,dx,$ $x = \sqrt{u^2-1}$

(d) $\int \dfrac{x}{\sqrt{2+4x}}\,dx,$ $u^2 = 2+4x$

(e) $\displaystyle\int_0^2 \dfrac{x+1}{\sqrt{16-x^2}}\,dx,$ $x = 4\sin u$

(f) $\int \dfrac{x^2+2}{(x+1)^2}\,dx,$ $u = x+1$

ANSWER

(a) $\dfrac{\ln\left|x^2+2x-7\right|}{2}$ (b) 0.6931 (c) $\dfrac{x^2+2}{\sqrt{1+x^2}}$ (d) $\dfrac{1}{6}(2+4x)^{1/2}(x-1)$

(e) $\dfrac{\pi}{6}-2\sqrt{3}+4 \approx 1.0595$ (f) $x - \dfrac{3}{x+1} - 2\ln|x+1|$

Integrate the following functions (substitution given) :-

1. $\int 9x(3x+2)^3\,dx$ where $u = 3x+2$ 2. $\int 7x(2x+3)^5\,dx$ where $u = 2x+3$

3. $\int 3x\sqrt{(1+x^2)}\,dx$ where $u = 1+x^2$ 4. $\int \dfrac{3x}{\sqrt{(2x+3)}}\,dx$ where $u = 2x+3$

ANSWER

1. $\dfrac{1}{5}(3x+2)^5 - \dfrac{1}{2}(3x+2)^4 + c$ 2. $\dfrac{1}{4}(2x+3)^7 - \dfrac{7}{8}(2x+3)^6 + c$

3. $(1+x)^{3/2} + c$ 4. $\dfrac{1}{2}(2x+3)^{3/2} - \dfrac{9}{2}(2x+3)^{1/2} + c$

Integration
"Given" Substitutions

Integrate the following functions (substitution given) :-

1. $\int \dfrac{x}{\sqrt{(1-x^2)}} dx$ given $x = \sin\theta$

2. $\int \sqrt{(4-x^2)} \, dx$ given $x = 2\sin\theta$

3. $\int \dfrac{x}{\sqrt{(9-x^2)}} dx$ given $x = 3\sin\theta$

4. $\int \dfrac{x^2}{\sqrt{(4-x^2)}} dx$ given $x = 2\sin\theta$

ANSWERS

1. $-\sqrt{(1-x^2)} + c$

2. $2\sin^{-1}(x/2) + x\sqrt{(4-x^2)} + c$

3. $-\sqrt{(9-x^2)} + c$

4. $2\sin^{-1}(x/2) - \dfrac{x}{2}\sqrt{(4-x^2)} + c$

Integration
Type 1 & Type 2 Partial Fractions Without Long Division

QUESTION

Integrate

1. $\int \dfrac{x+8}{(x+2)(x+4)} dx$

2. $\int \dfrac{x^2 - 6x - 7}{(x-1)(x-2)(x+3)} dx$

ANSWER

1. $3\ln(x+2) - 2\ln(x+4) + c$

2. $3\ln(x-1) - 3\ln(x-2) + \ln(x+1) + c$

QUESTION

Integrate

1. $\int \dfrac{3x^2 + x + 1}{x(x+1)^2} dx$

2. $\int \dfrac{x^2 - 2x + 10}{(x+2)(x-1)^2} dx$

3. $\int \dfrac{25}{(2x-1)^2(x+2)} dx$

4. $\int \dfrac{5x+2}{(x-2)^2(x+1)} dx$

ANSWER

1. $\ln x + 2\ln(x+1) + \dfrac{3}{x+1} + c$

2. $2\ln(x+2) - \ln(x-1) - \dfrac{3}{x-1} + c$

3. $\ln(x+2) - \ln(2x-1) - \dfrac{5}{2x-1} + c$

4. $\dfrac{1}{3}\ln(x-2) - \dfrac{1}{3}\ln(x+1) - \dfrac{4}{x-2} + c$

QUESTION

(1) $\int \dfrac{dx}{x^2 - 1}$

(2) $\int \dfrac{3x-1}{x^2 + x - 6} dx$

(3) $\int \dfrac{x+8}{x^2 + 6x + 8} dx$

Answers

(1) $\dfrac{1}{2}\ln(x-1) - \dfrac{1}{2}\ln(x+1) + C$

(2) $2\ln(x+3) + \ln(x-2) + C$

(3) $-2\ln(x+4) + 3\ln(x+2) + C$

Integration
Type 1 & Type 2 Partial Fractions Without Long Division

QUESTION

1.   Use partial fractions to help find

(a) $\int \dfrac{3x+4}{x(x+2)}\,dx$ (b) $\int \dfrac{dx}{x^2-3x+2}$ (c) $\int \dfrac{x}{x^2-6x+9}\,dx$

(d) $\int_1^2 \dfrac{x+3}{(x-9)(x-3)}\,dx$ (e) $\int \dfrac{3x-2}{(x-1)^2}\,dx$ (f) $\int_0^1 \dfrac{dx}{(x+1)(x+2)^2}$

(g) $\int \dfrac{6x}{(x+1)(x-1)(x-2)}\,dx$ (h) $\int_2^3 \dfrac{8x}{(x-1)(x+1)^2}\,dx$

ANSWER   (+C OMITTED)

1.   (a) $2\ln|x|+\ln|x+2|$ (b) $-\ln|x-1|+\ln|x-2|$ (c) $\ln|x-3|-\dfrac{3}{x-3}$

(d) $2\ln 7-5\ln 2 \approx 0.4261$ (e) $3\ln|x-1|-\dfrac{1}{x-1}$ (f) $2\ln 2-\ln 3-\dfrac{1}{6}\approx 0.121$

(g) $4\ln|x-2|-3\ln|x-1|-\ln|x+1|$ (h) $\dfrac{1}{3}-2\ln 2+2\ln 3 \approx 1.443$

QUESTION

(1) $\int \dfrac{2x-1}{(x+2)^2}\,dx$ (2) $\int \dfrac{2x+1}{(x+2)(x-3)^2}\,dx$

(3) $\int \dfrac{x}{(x+1)^2(x-1)}\,dx$ (4) $\int \dfrac{x^3+1}{x(x-1)^3}\,dx$

Answers

(1) $2\ln(x+2)+\dfrac{5}{x+2}$

(2) $-\dfrac{3}{25}\ln(x+2)-\dfrac{19}{175}\ln(x-3)-\dfrac{5}{7}(x-3)^{-1}+C$

(3) $-\dfrac{1}{4}\ln(x+1)+\dfrac{1}{4}\ln(x-1)-\dfrac{1}{2(x+1)}+C$

(4) $\ln x+\dfrac{1}{3}\ln(x-1)-\dfrac{5}{3(x-1)}-\dfrac{1}{(x-1)^2}+C$

Integration
Long Division (Maybe Including Type 1 & Type 2 P.Fs)

QUESTION

Use long division then partial fractions, if required, to find

(a) $\int \dfrac{x^2}{x^2-4}\,dx$ 　(b) $\int \dfrac{2x^3+3x^2-3}{x^2-1}\,dx$ 　(c) $\int \dfrac{x^4+5x^3+6x^2+x-1}{(x-1)(x+2)(x+3)}\,dx$

(d) $\int \dfrac{x^3}{x-2}\,dx$ 　(e) $\int_{0.5}^{2} \dfrac{x^4+1}{x^2+3x+2}\,dx$

ANSWER (+C OMITTED)

(a) $x+\ln|x-2|-\ln|x+2|$ 　(b) $x^2+3x+\ln|x^2-1|$

(c) $\dfrac{x^2}{2}+x+\ln\left|\dfrac{(x-1)(x+2)}{x+3}\right|$ 　(d) $\dfrac{x^3}{3}+x^2+4x+8\ln|x-2|$

(e) $\dfrac{15}{2}-49\ln 2+17\ln 5 \approx 0.8962$

QUESTION

(1) $\int \dfrac{6x^2+1}{2x-1}\,dx$ 　(2) $\int \dfrac{1+x^2}{1-x}\,dx$ 　(3) $\int \dfrac{6x^2-x+7}{2x+1}\,dx$

(4) $\int \dfrac{x^2}{1-2x}\,dx$ 　(5) $\int \dfrac{3-4x^2}{2-3x}\,dx$ 　(6) $\int \dfrac{x-2}{x+1}\,dx$

(7) $\int \dfrac{x+5}{x+2}\,dx$ 　(8) $\int \dfrac{x}{3x+1}\,dx$

Answers

(1) $\dfrac{3x^2}{2}+\dfrac{3x}{2}+\dfrac{5}{4}\ln(2x-1)+C$ 　(2) $-\dfrac{x^2}{2}-x-2\ln(1-x)+C$

(3) $\dfrac{3x^2}{2}-2x+\dfrac{9}{2}\ln(2x+1)+C$ 　(4) $-\dfrac{x^2}{4}-\dfrac{x}{4}-\dfrac{1}{8}\ln(1-2x)+C$

(5) $\dfrac{2x^2}{3}+\dfrac{8x}{9}-\dfrac{11}{27}\ln(2-3x)+C$ 　(6) $x-3\ln(x+1)+C$

(7) $x+3\ln(x+2)+C$ 　(8) $\dfrac{1}{3}x-\dfrac{1}{9}\ln(3x+1)+C$

Integration
Long Division (Maybe Including Type 1 & Type 2 P.Fs)

QUESTION

(1) $\int \dfrac{x^3 - 2x - 13}{x^2 - 2x - 3} \, dx$

(2) $\int \dfrac{2x^3 - 2x^2 - 11x - 8}{x^2 - x - 6} \, dx$

ANSWER:

(1) $\dfrac{1}{2}x^2 + 2x + 2\ln(x - 3) + 3\ln(x + 1) + c$

(2) $x^2 - \ln(x - 3) + 2\ln(x + 2) + C$

Integration
Inverse Trig. Functions

QUESTION

1. $\int \dfrac{1}{\sqrt{49-x^2}}\,dx$

2. $\int \dfrac{1}{49+x^2}\,dx$

3. $\int \dfrac{1}{\sqrt{9-x^2}}\,dx$

4. $\int \dfrac{1}{100+x^2}\,dx$

5. $\int \dfrac{1}{\sqrt{36-25x^2}}\,dx$

6. $\int \dfrac{1}{36+25x^2}\,dx$

7. $\int \dfrac{2}{25+4x^2}\,dx$

8. $\int \dfrac{3}{\sqrt{36-9x^2}}\,dx$

Evaluate

9. $\int_1^{\sqrt{3}} \dfrac{2}{1+x^2}\,dx$

10. $\int_0^{\sqrt{2}} \dfrac{1}{\sqrt{4-x^2}}\,dx$

11. $\int_{\frac{1}{2}}^{1} \dfrac{3}{\sqrt{1-x^2}}\,dx$

12. $\int_0^3 \dfrac{1}{9+x^2}\,dx$

ANSWERS

1. $\sin^{-1}\left(\dfrac{x}{7}\right)+c$

2. $\dfrac{1}{7}\tan^{-1}\left(\dfrac{x}{7}\right)+c$

3. $\sin^{-1}\left(\dfrac{x}{3}\right)+c$

4. $\dfrac{1}{10}\tan^{-1}\left(\dfrac{x}{10}\right)+c$

5. $\dfrac{1}{5}\sin^{-1}\left(\dfrac{5x}{6}\right)+c$

6. $\dfrac{1}{30}\tan^{-1}\left(\dfrac{5x}{6}\right)+c$

7. $\dfrac{1}{5}\tan^{-1}\left(\dfrac{2x}{5}\right)+c$

8. $\sin^{-1}\left(\dfrac{x}{2}\right)+c$

9. $\dfrac{\pi}{6}$

10. $\dfrac{\pi}{4}$

11. π

12. $\dfrac{\pi}{12}$

Integration
Inverse Trig. Functions

QUESTION

(1) $\displaystyle\int \frac{dx}{9+x^2}$

(2) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

(3) $\displaystyle\int \frac{dx}{\sqrt{25-9x^2}}$

(4) $\displaystyle\int \frac{dx}{\sqrt{16-x^2}}$

(5) $\displaystyle\int \frac{dx}{x^2+16}$

(6) $\displaystyle\int \frac{dx}{\sqrt{5-x^2}}$

(7) $\displaystyle\int \frac{dx}{4x^2+9}$

(8) $\displaystyle\int \frac{dx}{9x^2+4}$

ANSWER

(1) $\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}+C$

(2) $\sin^{-1}\dfrac{x}{2}+C$

(3) $\dfrac{1}{3}\sin^{-1}\dfrac{3x}{5}+C$

(4) $\sin^{-1}\dfrac{x}{4}+C$

(5) $\dfrac{1}{4}\tan^{-1}\dfrac{x}{4}+C$

(6) $\sin^{-1}\dfrac{x}{\sqrt{5}}+C$

(7) $\dfrac{1}{6}\tan^{-1}\dfrac{2x}{3}+C$

(8) $\dfrac{1}{6}\tan^{-1}\dfrac{3x}{2}+C$

Integration
Type 3 Partial Fractions (Maybe Including Long Division)

QUESTKM

Integrate :-

1. $\displaystyle \int \frac{3x+1}{(x-1)(x^2+1)}\, dx$

2. $\displaystyle \int \frac{3x^2+92x}{(x+6)(x^2+1)}\, dx$

3. $\displaystyle \int \frac{x}{x^4-1}\, dx$

4. $\displaystyle \int \frac{x}{(x+1)(x^2+4)}\, dx$

ANSWERS

1. $2\ln(x-1) - \ln(x^2+1) + \tan^{-1}x + c$

2. $\dfrac{15}{2}\ln(x^2+1) - 12\ln(x+6) + 2\tan^{-1}x + c$

3. $\dfrac{1}{4}\ln(x+1) + \dfrac{1}{4}\ln(x-1) - \dfrac{1}{4}\ln(x^2+1) + c$

4. $\dfrac{1}{10}\ln(x^2+4) - \dfrac{1}{5}\ln(x+1) + \dfrac{2}{5}\tan^{-1}\left(\dfrac{x}{2}\right) + c$

QUESTKM

(1) $\displaystyle \int \frac{dx}{x(x^2+1)}$

(2) $\displaystyle \int \frac{x\,dx}{x^4-1}$

(3) $\displaystyle \int \frac{x\,dx}{(x+1)(x^2+4)}$

(4) $\displaystyle \int \frac{(x+1)^2\,dx}{x^3+x}$

(5) $\displaystyle \int \frac{dx}{(x^2+1)(x-2)}$

(6) $\displaystyle \int \frac{(x+2)\,dx}{(x^2+4)(1-x)}$

Answers

(1) $\ln x - \dfrac{1}{2}\ln(x^2+1) + C$

(2) $\dfrac{1}{4}\ln(x-1) + \dfrac{1}{4}\ln(x+1) - \dfrac{1}{4}\ln(x^2+1) + C$

(3) $-\dfrac{1}{5}\ln(x+1) + \dfrac{1}{10}\ln(x^2+4) + \dfrac{2}{5}\tan^{-1}\dfrac{x}{2} + C$

(4) $\ln x + 2\tan^{-1}x + C$

(5) $\dfrac{1}{5}\ln(x-2) - \dfrac{1}{10}\ln(x^2+1) - \dfrac{2}{5}\tan^{-1}x + C$

(6) $-\dfrac{3}{5}\ln(1-x) + \dfrac{3}{10}\ln(x^2+4) - \dfrac{2}{10}\tan^{-1}\dfrac{x}{2} + C$

Integration
"Easy" Integration by Parts

QUESTION

Integrate :-

1. $\int x \sin x \, dx$

2. $\int x \sin 3x \, dx$

3. $\int x \cos 4x \, dx$

4. $\int x e^x \, dx$

ANSWER 1

1. $-x\cos x + \sin x + c$

2. $-\dfrac{x}{3}\cos 3x + \dfrac{1}{9}\sin 3x + c$

3. $\dfrac{1}{4}x\sin 4x + \dfrac{1}{16}\cos 4x + c$

4. $xe^x - e^x + c$

QUESTION

① Use integration by parts to find

(a) $\int x \cos x \, dx$ (b) $\int x e^{-x} \, dx$ (c) $\int (3x+2)\sin x \, dx$

(d) $\int_0^{\pi/12} x \sin 2x \, dx$ (e) $\int_0^1 x e^{2x} \, dx$

② Using the fact that $\cos 2x = 1 - 2\sin^2 x$, evaluate $\int_0^{\pi} x \sin^2 x \, dx$.

ANSWER (+ C OMITTED)

① (a) $x\sin x + \cos x$ (b) $-e^{-x}(x+1)$ (c) $3\sin x - 3x\cos x - 2\cos x$

(d) $\dfrac{1}{8} - \dfrac{\pi\sqrt{3}}{48} \approx 0.0116$ (e) $\dfrac{1}{4}(e^2 + 1) \approx 2.0973$

② $\dfrac{\pi^2}{4} \approx 2.4674$

Integration
"Hard" Integration by Parts

Integrate :-

1. $\int x^2 \cos x \, dx$

2. $\int x^2 \sin 3x \, dx$

3. $\int x^2 e^{2x} \, dx$

4. $\int x^2 \cos 2x \, dx$

5. $\int x^2 e^{-x} \, dx$

6. $\int x^3 e^x \, dx$

ANSWER

1. $x^2 \sin x + 2x \cos x - 2 \sin x + c$

2. $-\frac{1}{3} x^2 \cos 3x + \frac{2}{9} x \sin 3x + \frac{2}{27} \cos 3x + c$

3. $\frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + c$

4. $\frac{1}{2} x^2 \sin 2x + \frac{1}{2} x \cos 2x - \frac{1}{4} \sin 2x + c$

5. $-x^2 e^{-x} - 2x e^{-x} - 2 e^{-x} + c$

6. $x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x + c$

QUESTION

Integrate :-

1. $\int \tan^{-1} x \, dx$

2. $\int \sin^{-1} 3x \, dx$

3. $\int \tan^{-1} 2x \, dx$

4. $\int \sin^{-1} \frac{1}{2} x \, dx$

5. $\int \cos^{-1} x \, dx$

6. $\int \tan^{-1} \frac{1}{2} x \, dx$

7. $\int \ln 2x \, dx$

8. $\int (\ln x)^2 \, dx$

ANSWER

1. $x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + c$

2. $x \sin^{-1} 3x - \frac{1}{3} \sqrt{(1 - 9x^2)} + c$

3. $x \tan^{-1} 2x - \frac{1}{4} \ln(1 + 4x^2) + c$

4. $x \sin^{-1} \frac{1}{2} x + \sqrt{(4 - x^2)} + c$

5. $x \cos^{-1} x - \sqrt{(1 - x^2)} + c$

6. $x \tan^{-1} \frac{1}{2} x - \ln(4 + x^2) + c$

7. $x \ln 2x - x + c$

8. $x (\ln x)^2 - 2x \ln x + 2x + c$

Integration
"Hard" Integration by Parts

QUESTION

Integrate

1. $\int e^x \sin 2x \, dx$ 2. $\int e^{-x} \sin x \, dx$

3. $\int e^{-2x} \cos 3x \, dx$ 4. $\int e^x \cos^2 x \, dx$

ANSWER:

1. $\frac{1}{5} e^x \sin 2x - \frac{2}{5} e^x \cos 2x + c$ 2. $-\frac{1}{2} e^{-x} \cos x - \frac{1}{2} e^{-x} \sin x + c$

3. $\frac{3}{13} e^{-2x} \sin 3x - \frac{2}{13} e^{-2x} \cos 3x + c$ 4. $e^x \cos^2 x + \frac{1}{5} e^x \sin 2x - \frac{2}{5} e^x \cos 2x + c$

QUESTION

(1) $\int x^3 \ln x \, dx$

(2) $\int \sqrt{x} \ln x \, dx$

(3) $\int \frac{1}{x^3} \ln x \, dx$

ANSWER:

(1) $-\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + c$ (2) $\frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{4}{9} x^{\frac{3}{2}} + c$ (3) $\frac{x^4}{4} \ln|x| - \frac{x^4}{16}$

QUESTION

Use more than one application of integration by parts to find

(a) $\int x^2 \cos x \, dx$ (b) $\int x^2 e^x \, dx$ (c) $\int_0^{\pi/2} x^2 \sin x \, dx$

(d) $\int e^x \cos x \, dx$

ANSWER: (+C OMITTED)

(a) $x^2 \sin x - 2 \sin x + 2x \cos x$ (b) $e^x (x^2 - 2x + 2)$

(c) $\pi - 2 \approx 1.1416$ (d) $\frac{1}{2} e^x (\cos x + \sin x)$

Integration
"Hard" Integration by Parts

QUESTION

(1) $\int x^2 e^x \, dx$ (2) $\int x^2 \cos x \, dx$ (3) $\int x \ln x \, dx$ (4) $\int x^2 \ln x \, dx$ (5) $\int \sqrt{x} \, \ln x \, dx$

(6) $\int e^x \cos 2x \, dx$ (7) $\int \tan^{-1} x \, dx$ (8) $\int x \tan^{-1} x \, dx$ (9) $\int e^x \sin x \, dx$

ANSWER

(1) $x^2 e^x - 2xe^x + 2e^x + C$ (2) $x^2 \sin x + 2x\cos x - 2\sin x + C$ (3) $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

(4) $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$ (5) $\frac{2}{3}x^{3/2} \ln x - \frac{4}{9}x^{3/2} + C$ (6) $\frac{1}{5}e^x \cos 2x + \frac{2}{5}e^x \sin 2x + C$

(7) $x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$ (8) $x^2 \tan^{-1} x - \frac{1}{2}\ln(1+x^2) - x\tan^{-1} x + \frac{1}{2}\ln(1+x^2) + C$

(9) $\frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x + C$

QUESTION

(1) $\int_1^e \frac{1}{x^2} \ln x \, dx$ (2) $\int_0^\infty x^3 e^{-x} \, dx$ (3) $\int_0^\infty e^{-x} \sin 2x \, dx$

(4) $\int_0^1 \tan^{-1} x \, dx$ (5) $\int_{\frac{1}{2}}^1 x \sin^{-1} x \, dx$

Answers

(1) $1 - 2e^{-1}$ (2) 6 (3) $\frac{2}{5}$ (4) $\frac{\pi}{4} - \frac{1}{2}\ln 2$ (5) $\frac{\pi}{24} - \frac{3\sqrt{3}}{4}$

Integration
Area Bounded By x Axis

1. Find the area enclosed by the following curves and the x-axis between the lines given.

   (a) $y = 3x^2 + 2$, $\quad x = 0, x = 2$

   (b) $y = x^3 - x$, $\quad x = 1, x = 2$

2. Find the area enclosed by the following curves and the x-axis.
   (A rough sketch could help)

   (a) $y = 6 + x - x^2$

   (b) $y = x(x - 2)(x - 3)$

ANSWER

1. (a) 12   (b) $2^1/_4$ 2. (a) $205/_6$ (b) $31/_{12}$

Find the areas enclosed by the following curves.

1. $y = x(10 - x)$ and $y = 4x$.

2. $y = 4x - x^2$ and $y = x^2 - 4x + 6$.

3. $y = 2\sqrt{x}$ and $y = \dfrac{x^2}{4}$

4. $y = x^3 + x^2 - 5x$ and $y = x^2 - x$.

5. $y = \sin x$ and $y = \cos x$

6. $y^2 = 4ax$ and $x^2 = 4ay$
   (both are parabolas)

ANSWER

1. 36   2. $2^2/_3$ 3. $5^1/_3$

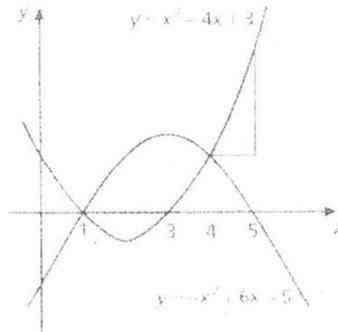4. 4   5. $\sqrt{2}$ 6. $\dfrac{16a^2}{3}$

Integration
Area Bounded By x Axis

1. The diagram below shows a sketch of the curve with equation $y = (x+2)(x-1)^2$. Find the total shaded area.

2. Calculate the shaded area in the following diagram.

ANSWER

1. 40.75 units^2

2. $13\dfrac{2}{3}$ units^2

Integration
Area Bounded By y Axis

QUESTION

Find the area enclosed by the following curves and the y -axis between the lines given.

1. $x = y^2$, $y = 3$.

2. $y = x^3$, $y = 1$, $y = 8$.

3. $x = \dfrac{1}{\sqrt{y}}$, $y = 2$, $y = 3$.

4. $y^2 = 1 - x$, $y = 0$, $y = 1$.

5. $y = {}^1/x^3$, $y = 8$, $y = 27$.

6. $y = \ln x$, $y = 2$, $y = 5$

ANSWERS

1.   9

2. $11^1/4$

3. $2\sqrt{3} - 2\sqrt{2}$

4. $^2/3$

5. $7^1/2$

6. $e^2(e^3 - 1)$

Integration
Volume Generated By Rotation Around An Axis

QUESTION

1. Find the volumes of solids of revolution formed when the regions bounded by the following curves and the x – axis are rotated through one revolution about the x – axis.
   Sketch the curves.

   (a) $y = \dfrac{4}{x}$, $x = 1$ and $x = 4$

   (b) $y = \sqrt{x}$, $x = 0$ and $x = 4$

   (c) $x + 2y = 2$, $x = 0$ and $x = 2$

   (d) $y = \dfrac{1}{x^2}$, $x = \dfrac{1}{3}$ and $x = \dfrac{1}{2}$

   (e) $y = x(x - 1)$

   (f) $y = \sqrt{(9 - x^2)}$

   (g) $y^2 = 8x$, $x = 0$ and $x = 4$

   (h) $y = \sin x$, $x = 0$ and $x = \pi$

2. Find the volumes of solids of revolution formed when the regions in the first quadrant bounded by the following curves and the y – axis are rotated through one revolution about the y – axis.
   Sketch the curves.

   (a) $x = \sqrt{y}$ and $y = 4$

   (b) $x = y^2$ and $y = 1$

   (c) $xy = 1$, $y = 3$ and $y = 6$

   (d) $y = 4 - x^2$, $y = -4$ and $y = 4$

   (e) $xy^2 = 2$, $y = 2$ and $y = 4$

   (f) $x = y^2 + 1$, $y = -1$ and $y = 1$

   (g) $y = \ln x$, $y = 2$ and $y = 5$

   (h) $y = \cos x$

ANSWER:

1. (a) 12π (b) 8π (c) $\dfrac{2\pi}{3}$ (d) $\dfrac{19\pi}{3}$

   (e) $\dfrac{1\pi}{30}$ (f) 18π (g) 64π (h) $\dfrac{1\pi^2}{2}$

2. (a) 8π (b) $\dfrac{1\pi}{5}$ (c) $\dfrac{1\pi}{6}$ (d) 32π

   (e) $\dfrac{7\pi}{48}$ (f) $\dfrac{56\pi}{15}$ (g) $\dfrac{1}{2}(e^{10} - e^4)\pi$ (h) $\pi(\pi - 2)$

Integration

Volume Generated By Rotation Around An Axis

QUESTIONS:

1. The volume the solid obtained by rotating a curve $y = f(x)$ about the x axis
    between the points $x = a$ and $x = b$ is given by

    $$V = \pi \int_a^b y^2 dx.$$

    Calculate the volume of the solid obtained by rotating the curve $y = \sqrt{9 - x^2}$
    about the x axis between the points $x = -3$ and $x = 3$.

2. The section of the curve $y = x^3$ between the points $x = 0$ and $x = 2$ is rotated
    about the y axis to form a volume of revolution. Calculate this volume V,
    given that

    $$V = 2\pi \int_a^b xf(x)\, dx$$

    gives the volume obtained by rotating a curve $y = f(x)$ about the y axis
    between the points $x = a$ and $x = b$.

(Note that in question 2 the area being rotated around the y axis is the area bounded by
the curve and the x axis. Consequently it is easier to use the "method of shells" here,
for which the relevant formula is given. The question could still be done using the
method of discs/washers but it would be a bit more awkward e.g. you would have to
find the volume generated when the area between the curves $x = 2$ and $x = y^{1/3}$
between $y = 0$ and $y = 8$ is rotated around the y axis.)

ANSWERS:

1. 36π units^3
2. $64\pi/5$ units^3

Integration
Rectilinear Motion

Questions

1. The performance of a prototype surface-to-air missile was measured on a horizontal test bed at the firing range and it was found that, until its fuel was exhausted, its acceleration (measured in m/s^2) t seconds after firing was given by

$$a = 8 + 10t - \frac{3}{4}t^2.$$

(a) Obtain a formula for its velocity, t seconds after firing.

(b) The missile contained enough fuel for 10 seconds. What horizontal distance did it cover in this time?

N.B. Assume that at $t = 0$, $v = 0$ and $s = 0$.

2. A particle proceeds in a straight line with acceleration given by $a = 3\cos^2 t$ and, when $t = 0$, $x = 0$ units and $v = 10$ units / sec. Find expressions for the velocity v and distance travelled x as functions of t.

Hint: $\cos^2 t = \frac{1}{2}(\cos 2t + 1)$

Answers

1. (a) $v = 8t + 5t^2 - \dfrac{t^3}{4}$ (b) 1441.67 metres

2. $v = \dfrac{3}{4}\sin 2t + \dfrac{3t}{2} + 10$ $x = -\dfrac{3}{8}\cos 2t + \dfrac{3t^2}{4} + 10t + \dfrac{3}{8}$