Matrices & Systems of Equations
Order & Elements

QUESTIONS

1. State the order of each of the following matrices:-

(a) $\begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ 4 & 8 \\ 1 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$

2. List the order of these matrices and say if any pairs are equal.

$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

$D = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $E = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $F = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $G = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $J = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ $K = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $L = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

3. Determine the values of x and y in each of the following:-

(a) $\begin{pmatrix} 3x & -y \end{pmatrix} = \begin{pmatrix} 12 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} x+3 \\ 4-y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} x+2y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$ (d) $\begin{pmatrix} x^2 & y^2 \\ y^3 & x^3 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ -27 & 8 \end{pmatrix}$

ANSWER.

1. (a) 2 x 4 (b) 3 x 2 (c) 4 x 2

2. A = C (1 x 3), F = G (2 x 1), H = L (2 x 2)

3. (a) $x = 4, y = -3$ (b) $x = 4, y = -1$ (c) $x = 5, y = 2$ (d) $x = 2, y = -3$

Matrices & Systems of Equations
Addition

QUESTION

1. Find the sum of the following matrices:-

(a) $\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}4\\5\end{pmatrix}$ (b) $\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}-1\\3\end{pmatrix}$ (c) $\begin{pmatrix}2a\\b\end{pmatrix}+\begin{pmatrix}7a\\-3b\end{pmatrix}$

(d) $\begin{pmatrix}2u\\-3v\end{pmatrix}+\begin{pmatrix}-2u\\3v\end{pmatrix}$ (e) $\begin{pmatrix}2 & 5\end{pmatrix}+\begin{pmatrix}1 & 4\end{pmatrix}$ (f) $\begin{pmatrix}2 & -3\end{pmatrix}+\begin{pmatrix}-5 & 8\end{pmatrix}$

(g) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}$ (h) $\begin{pmatrix}2 & 3 & 1\\5 & 1 & 0\end{pmatrix}+\begin{pmatrix}1 & -2 & 3\\-7 & 1 & -4\end{pmatrix}$

2. $A=\begin{pmatrix}3 & 2\\1 & -1\end{pmatrix}$, $B=\begin{pmatrix}1 & 4\\5 & 2\end{pmatrix}$ and $C=\begin{pmatrix}-2 & 3\\1 & -4\end{pmatrix}$

Find the matrices:-

(a) A + B (b) B + C (c) (A + B) + C (d) A + (B + C)

Is it true that (A + B) + C = A + (B + C) ?

3. Given that $A=\begin{pmatrix}3 & -4\\-5 & 1\end{pmatrix}$ and $B=\begin{pmatrix}-3 & 4\\5 & -1\end{pmatrix}$, find the matrices:-

(a) A + B (b) B + A

Comment on your results.

ANSWER.

1. (a) $\begin{pmatrix}7\\6\end{pmatrix}$ (b) $\begin{pmatrix}1\\7\end{pmatrix}$ (c) $\begin{pmatrix}9a\\-2b\end{pmatrix}$ (d) $\begin{pmatrix}0\\0\end{pmatrix}$

(e) $\begin{pmatrix}3 & 9\end{pmatrix}$ (f) $\begin{pmatrix}-3 & 5\end{pmatrix}$ (g) $\begin{pmatrix}3 & 3\\4 & 6\end{pmatrix}$ (h) $\begin{pmatrix}3 & 1 & 4\\-2 & 2 & -4\end{pmatrix}$

2. (a) $\begin{pmatrix}4 & 6\\6 & 1\end{pmatrix}$ (b) $\begin{pmatrix}-1 & 7\\6 & -2\end{pmatrix}$ (c) $\begin{pmatrix}2 & 9\\7 & -3\end{pmatrix}$ (d) $\begin{pmatrix}2 & 9\\7 & -3\end{pmatrix}$, true

3. A = –B

Matrices & Systems of Equations
Subtraction

QUESTION

1. Subtract the following matrices.

(a) $\begin{pmatrix}3\\1\end{pmatrix}-\begin{pmatrix}4\\5\end{pmatrix}$ (b) $\begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}-1\\3\end{pmatrix}$ (c) $\begin{pmatrix}2a\\b\end{pmatrix}-\begin{pmatrix}7a\\-3b\end{pmatrix}$

(d) $\begin{pmatrix}2u\\-3v\end{pmatrix}-\begin{pmatrix}-2u\\3v\end{pmatrix}$ (e) $(2\ \ 5)-(1\ \ 4)$ (f) $(2\ \ -3)-(-5\ \ 8)$

(g) $\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}2&3\\4&5\end{pmatrix}$ (h) $\begin{pmatrix}4&7\\2&3\end{pmatrix}-\begin{pmatrix}8&0\\3&9\end{pmatrix}$ (i) $\begin{pmatrix}2&3&1\\5&1&0\end{pmatrix}-\begin{pmatrix}1&-2&3\\-7&1&-4\end{pmatrix}$

2. $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$, $B=\begin{pmatrix}-2&3\\0&1\end{pmatrix}$ and $C=\begin{pmatrix}5&2\\-1&0\end{pmatrix}$. Find in simplest form :-

(a) A + B (b) A + C (c) A + B + C
(d) A – B (e) C – B (f) C – A
(g) (A + C) + (A + B) (h) (A + C) – (A + B)

3. Solve each of the following equations for the 2 x 2 matrix X:-

(a) $X+\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&2\\2&1\end{pmatrix}$ (b) $X+\begin{pmatrix}2&3\\4&5\end{pmatrix}=\begin{pmatrix}4&-1\\3&2\end{pmatrix}$

4. If $A=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}$, $B=\begin{pmatrix}3&-2\\1&5\\4&6\end{pmatrix}$ and $C=\begin{pmatrix}4&2\\1&0\\-3&5\end{pmatrix}$, simplify:-

(a) A + B (b) B – C (c) (A + B) – C (d) A + (B – C)

ANSWER.

1. (a) $\begin{pmatrix}-1\\-4\end{pmatrix}$ (b) $\begin{pmatrix}3\\1\end{pmatrix}$ (c) $\begin{pmatrix}-5a\\4b\end{pmatrix}$ (d) $\begin{pmatrix}4u\\-6v\end{pmatrix}$

(e) $(1\ \ 1)$ (f) $(7\ \ -11)$ (g) $\begin{pmatrix}-1&-3\\-4&-4\end{pmatrix}$ (h) $\begin{pmatrix}-4&7\\-1&-6\end{pmatrix}$

(i) $\begin{pmatrix}1&5&-2\\12&0&4\end{pmatrix}$

2. (a) $\begin{pmatrix}-1&5\\3&5\end{pmatrix}$ (b) $\begin{pmatrix}6&4\\2&4\end{pmatrix}$ (c) $\begin{pmatrix}4&7\\2&5\end{pmatrix}$ (d) $\begin{pmatrix}3&-1\\3&3\end{pmatrix}$

(e) $\begin{pmatrix}7&-1\\-1&-1\end{pmatrix}$ (f) $\begin{pmatrix}4&0\\-4&-4\end{pmatrix}$ (g) $\begin{pmatrix}5&9\\5&9\end{pmatrix}$ (h) $\begin{pmatrix}7&-1\\-1&-1\end{pmatrix}$

3. (a) $\begin{pmatrix}-1&2\\2&0\end{pmatrix}$ (b) $\begin{pmatrix}2&-4\\-1&-3\end{pmatrix}$

4. (a) $\begin{pmatrix}4&0\\4&9\\9&12\end{pmatrix}$ (b) $\begin{pmatrix}-1&-4\\0&5\\7&1\end{pmatrix}$ (c) $\begin{pmatrix}0&-2\\3&9\\12&7\end{pmatrix}$ (d) $\begin{pmatrix}0&-2\\3&9\\12&7\end{pmatrix}$

Matrices & Systems of Equations
Scalar Multiplication

QUESTION 1

1. If $A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$, find (a) 2A (b) 3A (c) –5A (d) –A

2. If $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 5 & 0 \end{pmatrix}$, find in their simplest form:-

   (a) A – B (b) 2(A + B) (c) 2A

   (d) 2B (e) 2A + 2B (f) 6A

   (g) 3(2A) (h) 8B (i) 4(2B)

   This question demonstrates the following properties:_

   (i) $k(A \pm B) = kA \pm kB$ and (ii) $k(tA) = (kt)A$

   where k and t are real numbers.

3. If $A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$, simplify:-

   (a) 3A + 2B (b) 4A – 3B (c) 5A – 4B (d) 2(A – 5B)

4. Solve each of the following equations for the matrix X :-

   (a) $3X = \begin{pmatrix} 6 & -3 \\ 12 & 9 \end{pmatrix}$ (b) $2X + \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ 2 & 8 \end{pmatrix}$

   (c) $4X - \begin{pmatrix} 3 & 1 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 0 & 13 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} - 3X = \begin{pmatrix} -5 & 10 \\ 8 & 9 \end{pmatrix}$

5. Find the matrix X in each of the following:-

   (a) $2\begin{pmatrix} 1 & -1 & 3 \\ 2 & -7 & 5 \end{pmatrix} + X = 3\begin{pmatrix} 1 & 2 & -4 \\ 3 & -5 & 1 \end{pmatrix}$ (b) $5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 3X = 4\begin{pmatrix} -4 & 7 \\ 3 & 8 \end{pmatrix}$

6. Given that $2\begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} 7 & -2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$, find p, q, r and s.

Matrices & Systems of Equations
Scalar Multiplication

ANSWERS

1. (a) $\begin{pmatrix} 6 & 8 \\ -2 & -4 \end{pmatrix}$ (b) $\begin{pmatrix} 9 & 12 \\ -3 & -6 \end{pmatrix}$ (c) $\begin{pmatrix} -15 & -20 \\ 5 & 10 \end{pmatrix}$ (d) $\begin{pmatrix} -3 & -4 \\ 1 & 2 \end{pmatrix}$

2. (a) $\begin{pmatrix} 1 & 5 & -2 \\ -2 & -5 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 10 & 6 & 8 \\ 12 & 10 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 8 & 2 \\ 4 & 0 & 6 \end{pmatrix}$

   (d) $\begin{pmatrix} 4 & -2 & 6 \\ 8 & 10 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 10 & 6 & 8 \\ 12 & 10 & 6 \end{pmatrix}$ (f) $\begin{pmatrix} 18 & 24 & 6 \\ 12 & 0 & 18 \end{pmatrix}$

   (g) $\begin{pmatrix} 18 & 24 & 6 \\ 12 & 0 & 18 \end{pmatrix}$ (h) $\begin{pmatrix} 16 & -8 & 24 \\ 32 & 40 & 0 \end{pmatrix}$ (i) $\begin{pmatrix} 16 & -8 & 24 \\ 32 & 40 & 0 \end{pmatrix}$

3. (a) $\begin{pmatrix} -2 & -7 \\ 18 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 20 & -15 \\ 7 & 10 \end{pmatrix}$ (c) $\begin{pmatrix} 26 & -19 \\ 8 & 13 \end{pmatrix}$ (d) $\begin{pmatrix} 44 & -16 \\ -22 & 22 \end{pmatrix}$

4. (a) $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 \\ -1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & -3 \\ -4 & -2 \end{pmatrix}$

5. (a) $\begin{pmatrix} 1 & 8 & -18 \\ 5 & -1 & -7 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & -6 \\ 1 & -4 \end{pmatrix}$

6. $p = -1, q = 4, r = 3, s = -2$

Matrices & Systems of Equations
Multiplication & Identities

QUESTION

1. Find the following matrix products by first considering the order of the product.

(a) $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}$ (b) $\begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix}$ (c) $\begin{pmatrix}5 & -2\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}$

(d) $\begin{pmatrix}3 & 1 & 2\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}$ (e) $\begin{pmatrix}2 & -3 & 4\end{pmatrix}\begin{pmatrix}5\\1\\2\end{pmatrix}$ (f) $\begin{pmatrix}8 & -5 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

(g) $\begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}$ (h) $\begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}$ (i) $\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}$

(j) $\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}5\\-2\end{pmatrix}$ (k) $\begin{pmatrix}2 & 3\\1 & -2\end{pmatrix}\begin{pmatrix}1\\-3\end{pmatrix}$ (l) $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}6\\-4\end{pmatrix}$

(m) $\begin{pmatrix}1 & -1\\2 & 3\end{pmatrix}\begin{pmatrix}5\\4\end{pmatrix}$ (n) $\begin{pmatrix}5\\4\end{pmatrix}\begin{pmatrix}1 & -1\\2 & 3\end{pmatrix}$ (o) $\begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}$

(p) $\begin{pmatrix}2 & 1 & 3\\3 & 0 & 1\\1 & 2 & 3\end{pmatrix}\begin{pmatrix}4\\0\\1\end{pmatrix}$ (q) $\begin{pmatrix}2 & 1\\3 & 0\\1 & 2\end{pmatrix}\begin{pmatrix}4\\0\\1\end{pmatrix}$ (r) $\begin{pmatrix}1 & -2 & 3\\-1 & 4 & 2\\3 & 1 & 0\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}$

(s) $\begin{pmatrix}\cos a & \sin a\end{pmatrix}\begin{pmatrix}\cos a & -\sin a\\ \sin a & \cos a\end{pmatrix}$

2. By forming a system of simultaneous equations, find x and y.

(a) $\begin{pmatrix}3 & 0\\0 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}12\\8\end{pmatrix}$ (b) $\begin{pmatrix}x & 0\\1 & y\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}6\\-1\end{pmatrix}$

(c) $\begin{pmatrix}2 & -1\\1 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}11\\-7\end{pmatrix}$ (d) $\begin{pmatrix}x & y\\y & x\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}5\\-1\end{pmatrix}$

3. If $A=\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}, B=\begin{pmatrix}4 & 5\\2 & 0\end{pmatrix}, C=\begin{pmatrix}4\\-3\end{pmatrix}$

(a) find (i) AB (ii) BA and comment on the result.

(b) find (i) A(BC) (ii) (AB)C and comment on the result.

4. Given that $A=\begin{pmatrix}3 & -1\\5 & 2\end{pmatrix}$, find A^2 and A^3. (Note :- $A^2 \neq \begin{pmatrix}9 & 1\\25 & 4\end{pmatrix}$)

5. If $P=\begin{pmatrix}1 & -2\\-3 & 6\end{pmatrix}$ and $Q=\begin{pmatrix}2 & 2\\1 & 1\end{pmatrix}$, find PQ and QP.

Matrices & Systems of Equations
Multiplication & Identities

6. Find the following products.

(a) $\begin{pmatrix}1 & -2\\ -3 & 6\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (b) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ -3 & 6\end{pmatrix}$ (c) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

(d) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ (e) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (f) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

The 2×2 matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is called the **unit matrix** of order 2 and is denoted by **I**.

It behaves like unity in the real number system. If A is a 2×2 matrix, then IA = AI = A.

7. If $\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, find a, b, c and d.

8. If $\begin{pmatrix}p & q\\ r & s\end{pmatrix}\begin{pmatrix}2 & 1\\ 4 & 3\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, find p, q, r and s.

9. If $A = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, find p and q such that $A^2 = pA + qI$.

10. If $A = \begin{pmatrix}3 & -1\\ 2 & -5\end{pmatrix}$, find p and q such that $A^2 = pA + qI$.

11. If $A = \begin{pmatrix}1 & 1\\ 3 & 2\\ 1 & 2\end{pmatrix}$ and $B = \begin{pmatrix}2 & 1 & 3\\ 3 & -1 & 2\end{pmatrix}$, find AB and BA.

12. If $A = \begin{pmatrix}1 & 0 & 2\\ 1 & -1 & 1\\ 2 & 1 & 3\end{pmatrix}$ and $B = \begin{pmatrix}2 & -1 & 3\\ 0 & 1 & -1\\ 1 & 1 & 1\end{pmatrix}$, find AB and BA

13. Calculate M^2 and M^3 when $\theta = 60°$, given that $M = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$

14. A matrix B is such that $B^2 = 6B - 9I$, where I is the 2×2 unit matrix
Find integers p and q such that $B^3 = pB + qI$.

15. A matrix A is such that $A^2 = 3A - 4I$, where I is the 2×2 unit matrix.
Find rational numbers p and q such $A^3 = pA + qI$.

Matrices & Systems of Equations
Multiplication & Identities

ANSWERS

1. (a) (11) (b) (26) (c) (9) (d) (13)

   (e) (15) (f) $(8x-5y-z)$ (g) $\begin{pmatrix}9\\8\end{pmatrix}$ (h) $\begin{pmatrix}10\\4\end{pmatrix}$

   (i) $\begin{pmatrix}1\\8\end{pmatrix}$ (j) $\begin{pmatrix}7\\5\end{pmatrix}$ (k) $\begin{pmatrix}-7\\7\end{pmatrix}$ (l) $\begin{pmatrix}4\\6\end{pmatrix}$

   (m) $\begin{pmatrix}1\\22\end{pmatrix}$ (n) not possible (o) (26) (p) $\begin{pmatrix}11\\13\\7\end{pmatrix}$

   (q) not possible (r) $\begin{pmatrix}9\\8\\7\end{pmatrix}$ (s) $\begin{pmatrix}1\\0\end{pmatrix}$

2. (a) $x=4,\ y=-4$ (b) $x=3, y=-1$ (c) $x=8, y=5$ (d) $x=2, y=-1$

3. (a) (i) $\begin{pmatrix}8 & 5\\14 & 15\end{pmatrix}$ (ii) $\begin{pmatrix}19 & 13\\2 & 4\end{pmatrix}$ $AB \neq BA$

   (b) (i) $\begin{pmatrix}17\\11\end{pmatrix}$ (ii) $\begin{pmatrix}17\\11\end{pmatrix}$ $A(BC) = (AB)C$

4. $\begin{pmatrix}4 & -5\\25 & -1\end{pmatrix}, \begin{pmatrix}-13 & -14\\70 & -27\end{pmatrix}$

5. $\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}-4 & 8\\-2 & 4\end{pmatrix}$

6. (a) $\begin{pmatrix}1 & -2\\-3 & 6\end{pmatrix}$ (b) $\begin{pmatrix}1 & -2\\-3 & 6\end{pmatrix}$ (c) $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ (d) $\begin{pmatrix}a & b\\c & d\end{pmatrix}$

   (e) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (f) $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

7. $a=1, b=-1, c=-2, d=3$

8. $p=\frac{3}{2}, q=-\frac{1}{2}, r=-2, s=1$

9. $p=5, q=2$

10. $p=-2, q=13$

11. $\begin{pmatrix}5 & 0 & 5\\12 & 1 & 13\\8 & -1 & 7\end{pmatrix}, \begin{pmatrix}8 & 10\\2 & 5\end{pmatrix}$

Matrices & Systems of Equations
Multiplication & Identities

12. $\begin{pmatrix} 4 & 1 & 5 \\ 3 & -1 & 5 \\ 7 & 2 & 8 \end{pmatrix}, \begin{pmatrix} 7 & 4 & 12 \\ -1 & -2 & -2 \\ 4 & 0 & 6 \end{pmatrix}$

13. $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

14. $p = 27, q = -54$

15. $p = 5, q = -12$

QUESTION

① Calculate $\begin{bmatrix} 3 & -4 & 1 & 7 \\ 2 & 5 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 4 & 3 & 7 \\ -2 & 4 & 9 \\ 2 & -4 & 1 \\ 0 & 2 & 8 \end{bmatrix}$

② If $A = \begin{bmatrix} -3 & 2 & 9 \\ 4 & 1 & 8 \\ 3 & 6 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 & 3 \\ 5 & 7 & 2 \\ 5 & 6 & 4 \end{bmatrix}$ find $A^2 + 3B - 2I$ where I is the 3×3 identity matrix.

ANSWER

① $\begin{bmatrix} 22 & 3 & 42 \\ -4 & 30 & 58 \end{bmatrix}$

② $\begin{bmatrix} 54 & 47 & -47 \\ 31 & 76 & 10 \\ 15 & 0 & 110 \end{bmatrix}$

Matrices & Systems of Equations
Determinants & Inverses

QUESTION1

1. Evaluate

(a) $\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & -1 \\ -4 & -1 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$

2. If $A = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}$,

find (i) AB and show that det(AB) = detAdetB.

(ii) BA and show that det(AB) = det(BA).

3. Evaluate

(a) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$ (b) $\begin{vmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{vmatrix}$ (c) $\begin{vmatrix} \ln 2 & \ln 4 \\ \ln 5 & \ln 6 \end{vmatrix}$

4. Find the determinants of the following matrices.

(a) $\begin{pmatrix} -2 & 1 & 4 \\ 3 & -2 & 5 \\ 0 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 3 \\ 0 & -1 & 4 \\ 2 & 6 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 0 & 1 \\ 3 & -4 & 5 \\ -7 & -3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ (e) $\begin{pmatrix} -7 & 14 & 7 \\ 2 & -8 & 6 \\ 9 & -3 & 12 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 8 & -10 \\ 2 & 4 & 15 \\ 1 & 12 & 5 \end{pmatrix}$

5. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}$

find (i) AB and show that det(AB) = detAdetB.

(ii) BA and show that det(AB) = det(BA).

6. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x).$

Matrices & Systems of Equations
Determinants & Inverses

ANSWERS

1. (a) 13 (b) – 5 (c) – 6 (d) 0

2. Proofs

3. (a) 1 (b) $\cos 4\theta$ (c) $\ln 2 . \ln \frac{6}{25}$ (from $\ln 2 \ln 6 - \ln 5 \ln 4$)

4. (a) 25 (b) 14 (c) 51 (d) 0

   (e) 1428 (f) – 320

5. Proofs

6. Proof

QUESTIONS

1. Find the inverse of the following 2×2 matrices, if they exist :

   (a) $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ (b) $B = \begin{pmatrix} 7 & 4 \\ 16 & 9 \end{pmatrix}$ (c) $C = \begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$

   (d) $D = \begin{pmatrix} 5 & 7 \\ 6 & 9 \end{pmatrix}$ (e) $E = \begin{pmatrix} -2 & 4 \\ 1 & -1 \end{pmatrix}$ (f) $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

2. Given that $P = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, find

   (a) P^{-1} (b) Q^{-1} (c) $(PQ)^{-1}$

   (d) $P^{-1}Q^{-1}$ (e) $Q^{-1}P^{-1}$ (f) $(QP)^{-1}$

ANSWERS

1. (a) $\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -9 & 4 \\ 16 & -7 \end{pmatrix}$ (c) does not exist (d) $\begin{pmatrix} 3 & -\frac{7}{3} \\ -2 & \frac{5}{3} \end{pmatrix}$

   (e) $\begin{pmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

2. (a) $\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{3}{2} & -\frac{19}{2} \\ -\frac{1}{2} & \frac{7}{2} \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -\frac{11}{2} \\ -1 & 2 \end{pmatrix}$

   (e) $\begin{pmatrix} \frac{3}{2} & -\frac{19}{2} \\ -\frac{1}{2} & \frac{7}{2} \end{pmatrix}$ (f) $\begin{pmatrix} 3 & -\frac{11}{2} \\ -1 & 2 \end{pmatrix}$

Matrices & Systems of Equations
Determinants & Inverses

QUESTION

Find the inverses of the following 3×3 matrices, if they exist :–

1. $\begin{pmatrix} 3 & 4 & 5 \\ 4 & 3 & 11 \\ 1 & 0 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 4 & 8 & 3 \\ 3 & 5 & 1 \\ 1 & 4 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 4 & 2 & 1 \\ 3 & 1 & 2 \\ 3 & 5 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 8 & 5 \\ 2 & 10 & 7 \\ 9 & 7 & 3 \end{pmatrix}$

ANSWERS

1. $\begin{pmatrix} \frac{9}{8} & -\frac{3}{2} & \frac{29}{8} \\ -\frac{1}{8} & \frac{1}{2} & -\frac{13}{8} \\ -\frac{3}{8} & \frac{1}{2} & -\frac{7}{8} \end{pmatrix}$

2. $\begin{pmatrix} 11 & -12 & -7 \\ -8 & 9 & 5 \\ 7 & -8 & -4 \end{pmatrix}$

3. $\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

4. $\begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{18} & \frac{5}{18} \\ -\frac{2}{3} & \frac{7}{9} & \frac{1}{9} \end{pmatrix}$

5. $\begin{pmatrix} -\frac{1}{12} & -\frac{1}{3} & \frac{7}{12} \\ -\frac{1}{12} & \frac{2}{3} & -\frac{5}{12} \\ \frac{5}{12} & -\frac{1}{3} & \frac{1}{12} \end{pmatrix}$

6. $\begin{pmatrix} -\frac{1}{3} & \frac{11}{57} & \frac{2}{19} \\ 1 & -\frac{14}{19} & \frac{1}{19} \\ -\frac{4}{3} & \frac{65}{57} & -\frac{2}{19} \end{pmatrix}$

Matrices & Systems of Equations
Determinants & Inverses

QUESTION

1. Find the determinant for each of the following matrices

(a) $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 1 & 3 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

2. For what values of k are the following matrices singular?

(a) $\begin{bmatrix} 1 & 2k \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 1 & 3 \\ 1 & k & 2 \\ 2 & -1 & 1 \end{bmatrix}$

3. Let $M = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $N = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$, where a, b, c and $d \in \Re$. Find MN and, by considering determinants, derive the identity

$$(a^2 - b^2)(c^2 - d^2) = (ac + bd)^2 - (ad + bc)^2.$$

HINT: Recall that $\det(AB) = \det A \times \det B$ for two matrices A and B.

4. Find the inverses, where they exist, for the following matrices

(a) $\begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3x & 5x \\ 2x & 4x \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 10 & 2 \\ 1 & 5 & 2 \\ 4 & 3 & -1 \end{bmatrix}$

5. The $n \times n$ matrix A satisfies the equation $A^2 = 5A + 3I$ where I is the $n \times n$ identity matrix.

(a) Show that A is invertible and hence express A^{-1} in the form $pA + qI$.

(b) Find an expression for A^4.

Matrices & Systems of Equations
Determinants & Inverses

ANSWERS

1. (a) 14 (b) 2

2. (a) det = $6-6k$ so singular when $k=1$

   (b) det = $8-2k$ so singular when $k=4$

3. Proof. (Find *MN* and then det(*MN*). Also find det(*MN*) via det*M*det*N*.)

4. (a) $\begin{bmatrix} -3 & 4 \\ 4 & 5 \end{bmatrix}$ (b) no inverse exists. (c) $\frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$

   (d) $\begin{bmatrix} -11 & 16 & 10 \\ 9 & -13 & -8 \\ -17 & 25 & 15 \end{bmatrix}$

5. (a) A is invertible with inverse $\frac{A-5I}{3} = \frac{A}{3} - \frac{5I}{3}$.

   (b) $A^4 = 25A^2 + 30A + 9I$

Matrices & Systems of Equations
Transpose & Symmetric

N.B. There are not really any symmetric matrix questions so refer to notes for revision on them.

QUESTION

① For the matrices $A = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 3 & -2 \\ 3 & 8 & 1 & 0 \end{pmatrix}$

prove that $(A+B)' = A' + B'$?

② For the matrix $A = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$, show that $(A')' = A$

③ $P = \begin{pmatrix} x & 9 \\ -3 & y \end{pmatrix}$ and $Q = \begin{pmatrix} 5 & -3 \\ 9 & -4 \end{pmatrix}$. Find x and y, given that $P' = Q$

ANSWER

① Proof

② Proof

③ $x = 5, y = -4$

Matrices & Systems of Equations
Transformations

QUESTION

1. Find the matrices associated with the following transformations.

   (a) Reflection in the y – axis.

   (b) Reflection in the line $y = x$.

   (c) Reflection in the line $y = -x$.

   (d) A rotation of $\frac{\pi}{2}$ radians clockwise.

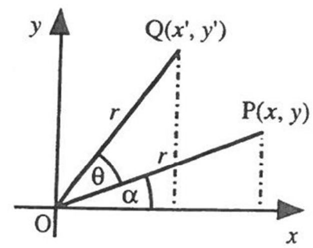
   (e) A rotation of $\frac{\pi}{2}$ radians anti–clockwise.

   (f) A dilatation, about O, where the scale factor is k.

2. Prove that the matrix associated with a general rotation of θ radians about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

   The diagram opposite may be helpful.

   OP, which makes an angle of α radians with the x–axis, is rotated through θ radians to OQ.



   Hence, show that the matrix associated with a rotation of:-

   (i) π radians is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

   (ii) $\frac{\pi}{2}$ radians anti–clockwise is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

   (iii) $\frac{\pi}{2}$ radians clockwise is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

ANSWER

1. (a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

   (d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

2. Proofs

Matrices & Systems of Equations
Solving Linear Systems via Inverse Matrices

QUESTIONS

1. Rewrite the following systems of equations in matrix form (do not solve)

(a) $\begin{matrix} x+2y=4 \\ 3x-y=5 \end{matrix}$ (b) $\begin{matrix} 3x=-6 \\ 4x+y=-7 \end{matrix}$ (c) $\begin{matrix} x+2y+z=8 \\ 3x+y-2z=-1 \\ x+5y-z=8 \end{matrix}$ (d) $\begin{matrix} 3x+y=5 \\ x+2y-3z=-12 \\ x+2z=10 \end{matrix}$

2. Use the inverse matrix method to solve

(a) $\begin{matrix} 2x+y=9 \\ 3x+2y=16 \end{matrix}$ (b) $\begin{matrix} 5x-2y=34 \\ 3x-2y=18 \end{matrix}$ (c) $\begin{matrix} x+y+z=6 \\ 2x+3y+7z=28 \\ x-y+8z=25 \end{matrix}$

3. Solve the following matrix equations for the 2×2 matrix X

(a) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix} X = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}$ (c) $X\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 13 \\ -4 & 13 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} X - \begin{bmatrix} 6 & 20 \\ 4 & 10 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -2 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix} + X\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$

ANSWERS

1. (a) $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 1 & 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 8 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & -3 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 10 \end{bmatrix}$

2. (a) $x=2 \quad y=5$ (b) $x=8 \quad y=3$ (c) $x=2 \quad y=1 \quad z=3$

3. (a) $X = \begin{bmatrix} -3 & -1 \\ 10 & 3 \end{bmatrix}$ (b) $X = \begin{bmatrix} 2 & 9/4 \\ 1 & -3/2 \end{bmatrix}$ (c) $X = \begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}$

(d) $X = \begin{bmatrix} 2 & 8 \\ -1 & 1 \end{bmatrix}$ (e) $X = \begin{bmatrix} 6 & -9 \\ 4 & -3 \end{bmatrix}$

Matrices & Systems of Equations
Solving Linear Systems via Inverse Matrices

QUESTION:

Solve the following systems of equations :

1. $x - y = 5$, $x + y = 11$
2. $3x + y = 7$, $3x + 2y = 5$
3. $2x + y = 5$, $2x + 3y = -1$
4. $3x - 4y = 18$, $5x + y = 7$
5. $2x + 3y = 5$, $4x - 5y = 21$
6. $5x - 3y = 9$, $7x - 6y = 9$

ANSWER:

1. $x = 8, y = 3$
2. $x = 3, y = -2$
3. $x = 4, y = -3$
4. $x = 2, y = -3$
5. $x = 4, y = -1$
6. $x = 3, y = 2$

QUESTION

Solve the following systems of equations :

1. $x - 2y + z = 6$, $3x + y - 2z = 4$, $7x - 6y - z = 10$
2. $5x - y + 2z = 25$, $3x + 2y - 3z = 16$, $2x - y + z = 9$
3. $x + y + z = 2$, $3x - y + 2z = 4$, $2x + 3y + z = 7$
4. $2x + 4y + 5z = -3$, $4x - y - 7z = 6$, $6x + 3y - z = 3$

ANSWER:

1. $x = 5, y = 3, z = 7$
2. $x = 5, y = 2, z = 1$
3. $x = 3, y = 1, z = -2$
4. $x = \frac{7}{6}, y = -\frac{4}{3}, z = 0$

Matrices & Systems of Equations
Solving Linear Systems via Gaussian Elimination

QUESTION

Use Gaussian Elimination to solve the following systems of equations

(a) $\begin{aligned} 2x+3y-z&=-1 \\ x-3y-2z&=4 \\ 5x+y+3z&=4 \end{aligned}$ (b) $\begin{aligned} 4x+2y+z&=3 \\ x+3y+5z&=3 \\ 2x+3z&=5 \end{aligned}$ (c) $\begin{aligned} x+y+z&=0 \\ 2x-y+z&=-1.1 \\ x+3y+2z&=0.9 \end{aligned}$ (d) $\begin{aligned} x-z&=2 \\ 2y-3z&=6 \\ 2x+y+z&=1 \end{aligned}$

ANSWER

(a) $x=1 \quad y=-1 \quad z=0$ (b) $x=1 \quad y=-1 \quad z=1$

(c) $x=-0.7 \quad y=0.2 \quad z=0.5$ (d) $x=2/3 \quad y=1 \quad z=-4/3$

QUESTION

Solve the following equations using Gaussian Elimination
(You may have to interchange rows before you start) :-

1. $\begin{aligned} x+y+z&=1 \\ 3x+3y+z&=4 \\ 3x+2y+2z&=7 \end{aligned}$

2. $\begin{aligned} x-2y+z&=6 \\ 3x+y-2z&=4 \\ 7x-6y-z&=10 \end{aligned}$

3. $\begin{aligned} 5x-y+2z&=25 \\ 3x+2y-3z&=16 \\ 2x-y+z&=9 \end{aligned}$

4. $\begin{aligned} x+y+z&=2 \\ 3x-y+2z&=4 \\ 2x+3y+z&=7 \end{aligned}$

5. $\begin{aligned} 5x-3y+6z&=0 \\ x+5y+2z&=0 \\ -x+2y+5z&=0 \end{aligned}$

ANSWER

1. $x=5,\ y=-7/2,\ z=-1/2.$

2. $x=5,\ y=3,\ z=7.$

3. $x=5,\ y=2,\ z=1.$

4. $x=3,\ y=1,\ z=-2.$

5. $x=0,\ y=0,\ z=0.$

Matrices & Systems of Equations
Linear Systems that are Inconsistent or Infinite

Questions

1. Attempt to use Gaussian Elimination on each system of equations. Where there are an infinite number of solutions write this down in terms of a constant. Where there are no solutions state this.

(a)
$$\begin{aligned} 3x + 2y + 5z &= 0 \\ 2x + y - 2z &= 5 \\ 7x + 4y + z &= 10 \end{aligned}$$

(b)
$$\begin{aligned} 2x - y + 3z &= 6 \\ x + y + 2z &= 7 \\ 4x + y + 7z &= 9 \end{aligned}$$

(c)
$$\begin{aligned} x + 2y - z &= 3 \\ x + 3z &= 5 \\ 4x + 2y + 8z &= 10 \end{aligned}$$

(d)
$$\begin{aligned} 5x - 3y - z &= -12 \\ 2x + y + 3z &= 3 \\ 20x - y + 13z &= -9 \end{aligned}$$

2. The system of equations

$$\begin{aligned} x + 2y - z &= 8 \\ 3x + y + 2z &= -1 \\ x + y + kz &= -6 \end{aligned}$$

has no solutions. What is the value of k?

3. The system of equations

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y - 2z &= -1 \\ x - y + kz &= 7 \end{aligned}$$

has infinitely many solutions. What is the value of k?

4. For what values of d and e will the three equations

$$\begin{aligned} x + 3y - 2z &= 8 \\ 2x + y - 3z &= 5 \\ 7x - 4y + dz &= e \end{aligned}$$

have

(a) no solutions
(b) infinitely many solutions
(c) a unique solution

Matrices & Systems of Equations
Linear Systems that are Inconsistent or Infinite

Answers

1. (a) infinite, $z = k \quad y = -16k - 15 \quad x = 9k + 10$
   (b) inconsistent so no solutions
   (c) inconsistent so no solutions
   (d) infinite, $z = k \quad y = \dfrac{39 - 17k}{11} \quad x = \dfrac{-8k - 3}{11}$

2. $k = 0$

3. $k = 9$

4. (a) $d = -9$ and $e \neq 1$
   (b) $d = -9$ and $e = 1$
   (c) $d \neq -9$

Matrices & Systems of Equations
Ill Conditioning

Questions

1. Show, by calculating the exact values, that the following system demonstrates ill conditioning.

$$\begin{aligned} x + 0.99y &= 1.99 \\ 0.99x + 0.98y &= 1.9699 \end{aligned} \qquad \begin{aligned} x + 0.99y &= 2 \\ 0.99x + 0.98y &= 1.97 \end{aligned}$$

2. (*a*) Use Gaussian elimination to obtain the solution of the following system of equations in terms of the parameter λ.

$$4x + 6z = 1$$
$$2x - 2y + 4z = -1$$
$$-x + y + \lambda z = 2$$

(*b*) Describe what happens when $\lambda = -2$.

(*c*) When $\lambda = -1{\cdot}9$ the solution is $x = -22{\cdot}25$, $y = 8{\cdot}25$, $z = 15$.

Find the solution when $\lambda = -2{\cdot}1$.

Comment on these solutions.

3. Which of the following systems of equations have the potential to be ill conditioned?

(a) $\begin{aligned} 2x + 9y &= 17 \\ 3x - 5y &= 7 \end{aligned}$ (b) $\begin{aligned} 9x + 8y &= 1 \\ 8x + 7y &= 1 \end{aligned}$

(c) $\begin{aligned} 8x + 13y &= 18 \\ 3x + 5y &= 7 \end{aligned}$ (d) $\begin{aligned} 9x - 10y &= 91 \\ 8x - 9y &= 82 \end{aligned}$

4. At a cross roads, traffic lights automatically control the flow of two streams of traffic.
A sad observer with nothing better to do times the changes of the lights.
10 turns for the eastbound traffic plus 10 turns for the northbound traffic took 1700 seconds.
10 turns for the eastbound traffic plus 11 turns for the northbound traffic took 1790 seconds.
(a) Form a system of equations using x and y seconds to represent the time it takes for 1 turn of the eastbound and northbound traffic respectively.
(b) Solve the system to obtain the values of x and y.
(c) Assuming the original equations were made to the nearest second, explore the confidence you can place on your solutions.

Matrices & Systems of Equations
Ill Conditioning

Answers

1. In the first system we get $x = 0.01$ and $y = 2$.
   In the second system we get $x = -97$ and $y = 100$.
   Despite only changing 1.99 to 2 and 1.9699 to 1.97 (both tiny alterations) there has been a relatively large change in the solutions. Therefore the system of equations are ill conditioned.

2. (a) $z = \dfrac{3}{2(2+\lambda)}$ $\quad y = \dfrac{3\lambda+9}{4(2+\lambda)}$ $\quad x = \dfrac{\lambda-7}{4(2+\lambda)}$
   (b) there are no solutions.
   (c) $z = -15$ $\quad y = -6.75$ $\quad x = 22.75$. Not a big change in λ but a relatively big change in the solutions so equations are ill conditioned.

3. (a) no (b) yes (c) yes (d) yes

4. (a)
$$\begin{aligned} 10x + 10y &= 1700 \\ 10x + 11y &= 1790 \end{aligned}$$
   (b) $x = 80$ $\quad y = 90$
   (c) Rather than 1700 we could have anything between 1699.5 and 1700.49
   Rather than 1790 we could have anything between 1789.5 and 1790.49
   However, when we solve for x and y with different values there isn't a big change in the solutions so equations are not really ill conditioned.