Integration by Parts

To reverse the chain rule we have the method of u-substitution. To reverse the product rule we also have a method, called *Integration by Parts*. The formula is given by:

Theorem (Integration by Parts Formula)

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$$

where $F(x)$ is an anti-derivative of $f(x)$.

Remember, all of the techniques that we talk about are supposed to make integrating easier! Even though this formula expresses one integral in terms of a second integral, the idea is that the second integral, $\int F(x)g'(x)\,dx$, is easier to evaluate. The key to integration by parts is making the right choice for $f(x)$ and $g(x)$. Sometimes we may need to try multiple options before we can apply the formula. Let's see it in action.

Example 1 Find

$$\int x\cos(x)\,dx.$$

We have to decide what to assign to $f(x)$ and what to assign to $g(x)$. Our goal is to make the integral *easier*. One thing to bear in mind is that whichever term we let equal $g(x)$ we need to differentiate - so if differentiating makes a part of the integrand simpler that's probably what we want! In this cases differentiating $\cos(x)$ gives $-\sin(x)$, which is no easier to deal with. But differentiating x gives 1 which *is* simpler. So we have,

$$\begin{aligned} g(x) &= x & f(x) &= \cos(x) \\ g'(x) &= 1 & F(x) &= \sin(x) \end{aligned}$$

$$\begin{aligned} \int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx \\ &= \boxed{x\sin(x) + \cos(x) + C} \end{aligned}$$

Example 2 Evaluate

$$\int_0^4 xe^{-x}\,dx.$$

$$\begin{aligned} g(x) &= x & f(x) &= e^{-x} \\ g'(x) &= 1 & F(x) &= -e^{-x} \end{aligned}$$

$$\begin{aligned} \int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big|_0^4 - \int_0^4 -e^{-x}\,dx \\ &= -xe^{-x} - e^{-x}\Big|_0^4 \\ &= \left[-4e^{-4} - e^{-4}\right] - \left[0 - e^{-0}\right] \\ &= -5e^{-4} + 1 \\ &= \boxed{1 - 5e^{-4}} \end{aligned}$$

Example 3 Evaluate

$$\int x^2 e^x \, dx.$$

$$\begin{aligned} g(x) &= x^2 & f(x) &= e^x \\ g'(x) &= 2x & F(x) &= e^x \end{aligned}$$

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx.$$

It's at this point we see that we still cannot integrate the integral on the write easily. This is okay. **Sometimes we may have to apply the integration by parts formula more than once!**

$$\begin{aligned} g_1(x) &= x & f_1(x) &= e^x \\ g_1'(x) &= 1 & F_1(x) &= e^x \end{aligned}$$

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2\int x e^x \, dx \\ &= x^2 e^x - 2\left[x e^x - \int e^x \, dx\right] \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= \boxed{\left(x^2 - 2x + 2\right) e^x + C} \end{aligned}$$

The previous technique works for any integral of the form $\int x^n e^{mx} \, dx$, where n is any *positive* integer and m is any integer. What if n was negative? Then this case we would set $g(x) = e^x$.

Example 4 In Example 3 we have to apply the Integration by Parts Formula multiple times. There is a convenient way to "book-keep" our work. This is done by creating a table. Let's see how by examining Example 3 again.

$$\int x^2 e^x \, dx.$$

Let $g(x) = x^2$ and $f(x) = e^x$. Then,

Differentiate $g(x)$		Integrate $f(x)$
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

Then the integral is,

$$\begin{aligned} \int x^2 e^x \, dx &= +x^2 \cdot e^x - 2x \cdot e^x + 2 \cdot e^x + C \\ &= \boxed{\left(x^2 - 2x + 2\right) e^x + C} \end{aligned}$$

We *have* actually used the integration by parts formula, but we have just made our lives easier by condensing the work into a neat table. This method is extremely useful when Integration by Parts needs to be used over and over again.

The next example exposes a potential flaw in always using the tabular method above. Sometimes applying the integration by parts formula may never terminate, thus your table will get awfully big.

Example 5 Find the integral

$$\int e^x \sin(x)\, dx.$$

We need to apply Integration by Parts twice before we see something:

(1)

$$u = e^x \qquad dv = \sin(x)$$
$$du = e^x\, dx \qquad v = -\cos(x)$$

(2)

$$u = e^x \qquad dv = \cos(x)$$
$$du = e^x\, dx \qquad v = \sin(x)$$

$$\begin{aligned}\int e^x \sin(x)\, dx &= -e^x \cos(x) + \int e^x \cos(x)\, dx \\ &= -e^x \cos(x) + \left(e^x \sin(x) - \int e^x \sin(x)\, dx\right) \\ &= -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\, dx\end{aligned}$$

Notice that now the integral we are interested in, $\int e^x \sin(x)\, dx$, appears on both the left and right hand side of the equation. So, if we add this integral to both sides we get

$$\begin{aligned}\implies \quad 2\int e^x \sin(x)\, dx &= e^x\left(-\cos(x) + \sin(x)\right) \\ \implies \quad \int e^x \sin(x)\, dx &= \boxed{\frac{e^x\left(\sin(x) - \cos(x)\right)}{2}}\end{aligned}$$

This "trick" comes up often when we are dealing with the product of two functions with "non-terminating" derivatives. By this we mean that you can keep differentiating functions like e^x and trig functions indefinitely and never reach 0. Polynomials on the other hand will eventually "terminate" and their n^{th} derivative (where n is the degree of the polynomial) is identically 0.

Practice Problems

Try some of the problems below. If you get stuck, don't worry! There are hints on the next page! But do try without looking at them first, chances are you won't get hints on your exam.

1. $\displaystyle\int t\sin(2t)\,dt$
2. $\displaystyle\int x^2\cos(3x)\,dx$
3. $\displaystyle\int \sin^{-1}(x)\,dx$
4. $\displaystyle\int p^5\ln(p)\,dp$
5. $\displaystyle\int_0^1 (x^2+1)e^{-x}\,dx$
6. $\displaystyle\int_4^9 \frac{\ln(y)}{\sqrt{y}}\,dy$
7. $\displaystyle\int_0^{\pi} x^3\cos(x)\,dx$
8. $\displaystyle\int_1^{\sqrt{3}} \tan^{-1}(1/x)\,dx$
9. $\displaystyle\int_1^2 \frac{(\ln(x))^2}{x^3}\,dx$
10. $\displaystyle\int (\ln(x))^2\,dx$
11. $\displaystyle\int 4x\cos(2-3x)\,dx$
12. $\displaystyle\int_6^0 (2+5x)e^{x/3}\,dx$
13. $\displaystyle\int (t^2+3t)\sin(2t)\,dt$
14. $\displaystyle\int_0^{\pi} x^2\cos(4x)\,dx$
15. $\displaystyle\int (4z^3-9z^2+7z+3)e^{-z}\,dz$
16. $\displaystyle\int 8te^{7t}\,dt$
17. $\displaystyle\int \sqrt{x^3}\ln(\sqrt[3]{x})\,dx$
18. $\displaystyle\int t\sec^2(2t)\,dt$
19. $\displaystyle\int e^{-\theta}\cos(2\theta)\,d\theta$
20. $\displaystyle\int e^{2z}\cos(z/4)\,dz$
21. $\displaystyle\int_1^2 \frac{\ln(x)}{x^2}\,dx$
22. $\displaystyle\int_0^1 \frac{y}{e^{2y}}\,dy$
23. $\displaystyle\int_0^{1/2} \cos(x)\ln(\sin(x))\,dx$
24. $\displaystyle\int x^4(\ln(x))^2\,dx$

Challenge Problems

1. $\displaystyle\int \ln(x)\,dx$
2. $\displaystyle\int t^7\sin(2t^4)\,dt$
3. $\displaystyle\int_1^4 (2-x)^2\ln(4x)\,dx$
4. $\displaystyle\int \tan^{-1}(x)\,dx$
5. $\displaystyle\int \sin^{-1}(x)\,dx$
6. $\displaystyle\int \cos(\sqrt{x})\,dx$
7. $\displaystyle\int t^3e^{-t^2}\,dt$
8. $\displaystyle\int x\ln(1+x)\,dx$
9. $\displaystyle\int \sin(\ln(x))\,dx$

Hints to Practice Problems

1. $g(t) = t$
2. Apply twice,
   start with $g(x) = x^2$
3. $g(x) = \sin^{-1}(x)$
4. $g(p) = \ln(p)$
5. Apply twice,
   start with $g(x) = x^2 + 1$
6. $g(y) = \ln(y)$
7. Apply three times,
   start with $g(x) = x^3$
8. $g(x) = \tan^{-1}(1/x)$
9. Apply twice,
   start with $g(x) = (\ln(x))^2$
10. $g(x) = (\ln(x))^2$
11. $g(x) = 4x$
12. $g(x) = 2 + 5x$
13. Apply twice,
    start with $g(t) = t^2 + 3t$
14. Apply twice,
    start with $g(x) = x^2$
15. Apply three times,
    start with $g(z) = 4z^3 - 9z^2 + 7z + 3$
16. $g(t) = 8t$
17. $g(x) = \ln(x)$
18. $g(t) = t$
19. Think Example 5.
20. Think Example 5.
21. $g(x) = \ln(x)$
22. $g(y) = y$
23. $g(x) = \ln(\sin(x))$
24. Apply twice,
    start with $g(x) = (\ln(x))^2$

Hints to Challenge Problems

1. $g(x) = \ln(x),\ f(x) = 1$
2. $g(t) = t^4$
3. $g(x) = \ln(4x)$
4. $g(x) = \tan^{-1}(x)$
5. $g(x) = \sin^{-1}(x)$
6. $u = \sqrt{x}$, then parts.
7. $u = -t^2$, then parts.
8. $u = 1 + x$, then parts.
9. $e^u = x$, then parts.

Answers to Practice Problems

1. $\frac{1}{4}\left(\sin(2t) - 2t\cos(2t)\right) + C$
2. $\frac{1}{27}\left((9x^2 - 2)\sin(3x) + 6x\cos(3x)\right) + C$
3. $x\sin^{-1}(x) + \sqrt{1-x^2} + C$
4. $\frac{1}{36}p^6\left(6\ln(p) - 1\right) + C$
5. $\frac{3e-6}{e}$
6. $4\left(\ln\left(\frac{27}{4}\right) - 1\right)$
7. $3\left(4 - \pi^2\right)$
8. $\frac{1}{12}\left(\left(2\sqrt{3} - 3\right)\pi + \ln(64)\right)$
9. $\frac{1}{16}\left(3 - \ln(2)(\ln(2) + 1)\right)$
10. $x\left(\ln(x)^2 - 2\ln(x) + 2\right) + C$
11. $\frac{1}{9}\left(4\cos(2-3x) - 12x\sin(2-3x)\right) + C$
12. $-39 - 51e^2$
13. $\frac{1}{4}\left(\left(-2t^2 - 6t + 1\right)\cos(2t) + (2t+3)\sin(2t)\right) + C$
14. $\frac{\pi}{8}$
15. $-(4z^3 + 3z^+13z + 16)e^{-z} + C$
16. $\frac{1}{49}e^{7t}\left(56t - 8\right) + C$
17. $\frac{2}{75}x^{5/2}\left(5\ln(x) - 2\right) + C$
18. $\frac{1}{4}\left(2t\tan(2t) + \ln|\cos(2t|\right) + C$
19.
20. $\frac{4}{65}e^{2z}\left(8\cos\left(\frac{z}{4}\right) + \sin\left(\frac{z}{4}\right)\right) + C$
21. $\frac{1-\ln(2)}{2}$
22. $\frac{1}{4}\left(1 - \frac{3}{e^2}\right)$
23. $\sin\left(\frac{1}{2}\right)\left(\ln\left(\sin\left(\frac{1}{2}\right)\right) - 1\right)$
24. $\frac{1}{125}x^5\left(25\ln(x)^2 - 10\ln(x) + 2\right) + C$

Answers to Challenge Problems

1. $x\left(\ln(x) - 1\right) + C$
2. $\frac{1}{16}\left(\sin(2t^4) - 2t^4\cos(2t^4)\right) + C$
3. $\ln\left(65,536 \cdot 2^{2/3}\right) - 4$
4. $x\tan^{-1}(x) - \ln(\sqrt{1+x^2}) + C$
5. $x\sin^{-1}(x) + \sqrt{1-x^2} + C$
6. $2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$
7. $-\frac{1}{2}e^{t^2}\left(t^2 + 1\right) + C$
8. $\frac{1}{4}\left(2(x^2 - 1)\ln(1+x) - x + 2\right) + C$
9. $\frac{1}{2}x\left(\sin(\ln(x)) - \cos(\ln(x))\right) + C$

Solutions to Practice Problems

1. Let $g(t) = t$ and $f(t) = \sin(2t)$. So $g'(t) = 1$ and $F(t) = -\dfrac{1}{2}\cos(2t)$. Then,

$$\int t\sin(2t)\,dt = -\frac{1}{2}t\cos(2t) + \frac{1}{2}\int \cos(2t)\,dt = -\frac{1}{2}t\cos(2t) + \frac{1}{4}\sin(2t) + C = \boxed{\frac{1}{4}\left(\sin(2t) - 2t\cos(2t)\right) + C}$$

2. Let $g(x) = x^2$ and $f(x) = \cos(3x)$. Then,

Differentiate $g(x)$		Integrate $f(x)$
x^2	$+$	$\cos(3x)$
$2x$	$-$	$\frac{1}{3}\sin(3x)$
2	$+$	$-\frac{1}{9}\cos(3x)$
0		$-\frac{1}{27}\sin(3x)$

$$\begin{aligned}\int x^2\cos(3x)\,dx &= \frac{1}{3}x^2\sin(3x) + \frac{2}{9}x\cos(3x) - \frac{2}{27}\sin(3x) + C \\ &= \boxed{\frac{1}{27}\left(\left(9x^2 - 2\right)\sin(3x) + 6x\cos(3x)\right) + C}\end{aligned}$$

3. Let $g(x) = \sin^{-1}(x)$ and $f(x) = 1$. So $g'(x) = \dfrac{1}{\sqrt{1-x^2}}$ and $F(x) = x$.

   Let $u = 1 - x^2$. So $-\dfrac{1}{2}\,du = x\,dx$. Then,

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}(x) + \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = x\sin^{-1}(x) + \sqrt{u} + C = \boxed{x\sin^{-1}(x) + \sqrt{1-x^2} + C}$$

4. Let $g(p) = \ln(p)$ and $f(p) = p^5$. So $g'(p) = \dfrac{1}{p}$ and $F(p) = \dfrac{1}{6}p^6$. Then,

$$\int p^5\ln(p)\,dp = \frac{1}{6}p^6\ln(p) - \frac{1}{6}\int p^5\,dp = \frac{1}{6}p^6\ln(p) - \frac{1}{36}p^6 + C = \boxed{\frac{1}{36}p^6\left(6\ln(p) - 1\right) + C}$$

5. Let $g(x) = x^2 + 1$ and $f(x) = e^{-x}$. Then,

Differentiate $g(x)$		Integrate $f(x)$
x^2+1	$+$	e^{-x}
$2x$	$-$	$-e^{-x}$
2	$+$	e^{-x}
0		$-e^{-x}$

$$\begin{aligned}\int_0^1 (x^2+1)e^{-x}\,dx &= -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_0^1 \\ &= -e^{-x}\left(x^2 + 2x + 3\right)\Big|_0^1 \\ &= -6e^{-1} + 3 \\ &= \boxed{\frac{3e-6}{e}}\end{aligned}$$

6. Let $g(y) = \ln(y)$ and $f(y) = y^{-1/2}$. So $g'(y) = y^{-1}$ and $F(y) = 2y^{1/2}$. Then,

$$\begin{aligned}\int_4^9 \frac{\ln(y)}{\sqrt{y}}\,dy &= 2\sqrt{y}\ln(y)\Big|_4^9 - 2\int_4^9 y^{-1/2}\,dy = 2\sqrt{y}\ln(y) - 4\sqrt{y}\Big|_4^9 = 6\ln(9) - 12 - [4\ln(4) - 8] \\ &= 4\ln(27) - 4 - 4\ln(4) = \boxed{4\left(\ln\left(\frac{27}{4}\right) - 1\right)}\end{aligned}$$

7. Let $g(x) = x^3$ and $f(x) = \cos(x)$. Then,

Differentiate $g(x)$		Integrate $f(x)$
x^3	$+$	$\cos(x)$
$3x^2$	$-$	$\sin(x)$
$6x$	$+$	$-\cos(x)$
6	$-$	$-\sin(x)$
0		$\cos(x)$

$$\begin{aligned}\int_0^\pi x^3\cos(x)\,dx &= x^3\sin(x) + 3x^2\cos(x) - 6x\sin(x) - 6\cos(x)\Big|_0^\pi \\ &= x\left(x^2-6\right)\sin(x) + 3\left(x^2-2\right)\cos(x)\Big|_0^\pi \\ &= 0 - 3\left(\pi^2-2\right) - [0-6] \\ &= \boxed{3\left(4-\pi^2\right)}\end{aligned}$$

8. Let $g(x) = \tan^{-1}(1/x)$ and $f(x) = 1$. So $g'(x) = -\dfrac{1}{x^2+1}$ and $F(x) = x$.

Let $u = x^2+1$. So $\dfrac{1}{2}\,du = x\,dx$. Then,

$$\begin{aligned}\int_1^{\sqrt{3}} \tan^{-1}\left(\frac{1}{x}\right)\,dx &= x\tan^{-1}(x)\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2+1}\,dx = x\tan^{-1}\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \frac{1}{2}\int_{x=1}^{x=\sqrt{3}} \frac{1}{u}\,du \\ &= x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln|u|\Big|_{x=1}^{x=\sqrt{3}} = x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln(x^2+1)\Big|_1^{\sqrt{3}} \\ &= \sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) + \frac{1}{2}\ln(4) - \left[\tan^{-1}(1) + \frac{1}{2}\ln(2)\right] \\ &= \sqrt{3}\frac{\pi}{6} + \ln(2) - \frac{\pi}{4} - \frac{1}{2}\ln(2) \\ &= \boxed{\frac{1}{12}\left((2\sqrt{3}-3)\pi + \ln(64)\right)}\end{aligned}$$

9. Let $g_1(x) = (\ln(x))^2$ and $f_1(x) = \dfrac{1}{x^3}$. So $g_1'(x) = \dfrac{2\ln(x)}{x}$ and $F_1(x) = -\dfrac{1}{2x^2}$.

Let $g_2(x) = \ln(x)$ and $f_2(x) = \dfrac{1}{x^3}$. So $g_2'(x) = \dfrac{1}{x}$ and $F_2(x) = -\dfrac{1}{2x^2}$. Then,

$$\int \frac{(\ln(x))^2}{x}\,dx = -\frac{(\ln(x))^2}{2x^2} + \int \frac{\ln(x)}{x^3}\,dx = -\frac{(\ln(x))^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}\,dx = -\frac{(\ln(x))^2 + \ln(x)}{2x^2} - \frac{1}{4x^2} + C.$$

$$\int_1^2 \frac{(\ln(x))^2}{x}\,dx = -\frac{2(\ln(x))^2 + 2\ln(x) + 1}{4x^2}\Big|_1^2 = -\frac{2(\ln(2))^2 + 2\ln(2) + 1}{16} - \left[-\frac{0+0+1}{4}\right] = \boxed{\frac{1}{16}\left(3 - 2(\ln(2))^2 - 2\ln(2)\right)}$$

10. Let $g_1(x) = (\ln(x))^2$ and $f_1(x) = 1$. So $g_1'(x) = \dfrac{2\ln(x)}{x}$ and $F_1(x) = x$.

Let $g_2(x) = \ln(x)$ and $f_2(x) = 1$. So $g_2'(x) = \dfrac{1}{x}$ and $F_2(x) = x$. Then,

$$\begin{aligned}\int (\ln(x))^2\,dx &= x(\ln(x))^2 - 2\int \ln(x)\,dx = x(\ln(x))^2 - 2\left[x\ln(x) - \int 1\,dx\right] \\ &= x(\ln(x))^2 - 2\left[x\ln(x) - x\right] + C = \boxed{x\left((\ln(x))^2 - 2\ln(x) + 2\right) + C}\end{aligned}$$