Advanced Highers Maths – Matrices

1. For matrix $A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$
   (a) Write down A^T
   (b) Find A^2
   (c) Find A^{-1}
   (d) Verify that $AA^{-1} = I$

2. For matrices $A = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$

   (a) Find matrices AB and BA. Are they equal?
   (b) Find $|A|$ and $|B|$
   (c) Find $|AB|$ and $|BA|$ Are they equal?
   (d) Comment on the relationship between $|A|$, $|B|$ and $|AB|$

3. For matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$
   (a) Find A^2
   (b) Make up a suggestion of what the matrix A could represent

4. Matrix C is defined such that $C^2 = 5C - 3I$

   (a) Find an expression for C^{-1} in terms of C and I
   (b) Find an expression for C^4 in terms of C and I

5. A Junior School has 40 teachers and 300 pupils
   A Senior School has 100 teachers and 700 pupils

   (a) Represent this as a matrix

   Each teacher has one sandwiches and five biscuits a day.
   Each pupil has two sandwiches and three biscuits s day

   (b) Represent this as a matrix
   (c) Multiply the two matrices to find how many sandwiches and biscuits are eaten in total at the Junior School, and at the Senior School

Revision

6. Express in partial fractions:
$$\frac{x^2 + 18x + 1}{(x-1)(x-3)^2}$$

7. (a) Write down an simplify the general term in the expansion of $\left(\frac{3}{x} + \frac{x^2}{2}\right)^{12}$
   (b) Hence determine the coefficient of *unity*.

Advanced Higher Maths – Matrices Solutions

1. For matrix $A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$

   (a) Write down A^T

   $$A^T = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}$$

   (b) Find A^2

   $$A^2 = \begin{pmatrix} 21 & 30 \\ 6 & 9 \end{pmatrix}$$

   (c) Find A^{-1}

   $$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}$$

   (d) Verify that $AA^{-1} = I$

   $$AA^{-1} = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. For matrices $A = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$

   (a) Find matrices AB and BA Are they equal?

   $$AB = \begin{pmatrix} 0 & 12 \\ 1 & 7 \end{pmatrix}, \qquad BA = \begin{pmatrix} -2 & -1 \\ 6 & 9 \end{pmatrix}, \qquad AB \neq BA$$

   (b) Find $|A|$ and $|B|$

   $$|A| = 3, \qquad |B| = -4$$

   (c) Find $|AB|$ and $|BA|$ Are they equal?

   $$|AB| = -12, \qquad |BA| = -12, \qquad |AB| = |BA|$$

   (d) Comment on the relationship between $|A|$, $|B|$ and $|AB|$

   $$|A| \times |B| = |AB|$$

3. For matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$

   (a) Find A^2

   $$A^2 = \begin{pmatrix} 10 & 7 & 7 \\ 6 & 3 & 2 \\ 3 & 7 & 11 \end{pmatrix}$$

   (b) Make up a suggestion of what the matrix A could represent

   Perhaps the number of brother, sisters, and pets owned by three different people (the first person has one brother, two sisters and three goldfish, the second has no brothers, one sister, two dogs etc.)

4. Matrix C is defined such that $C^2 = 5C - 3I$

   (a) Find an expression for C^{-1} in terms of C and I

   $$C^2 = 5C - 3I$$
   $$C^{-1}C^2 = C^{-1}(5C - 3I)$$
   $$C = 5I - 3C^{-1}$$
   $$3C^{-1} = 5I - C$$
   $$C^{-1} = \frac{5}{3}I - \frac{1}{3}C$$

(b) Find an expression for C^4 in terms of C and I

$$C^4$$
$$= (C^2)$$
$$= (5C - 3I)^2$$
$$= 25C^2 - 30C + 9I$$
$$= 25(5C - 3I) - 30C + 9I$$
$$= 95C - 66I$$

5. A Junior School has 40 teachers and 300 pupils
   A Senior School has 100 teachers and 700 pupils

   (a) Represent this as a matrix

   $$\begin{pmatrix} 40 & 300 \\ 100 & 700 \end{pmatrix}$$

   This matrix is school (rows) by people (columns)

   Each teacher has one sandwiches and five biscuits a day
   Each pupil has two sandwiches and three biscuits s day

   (b) Represent this as a matrix

   $$\begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}$$

   This matrix is people (rows) by food (columns)

   (c) Multiply the two matrices to find how many sandwiches and biscuits are eaten in total at the Junior School, and at the Senior School

   $$\begin{pmatrix} 40 & 300 \\ 100 & 700 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 640 & 1100 \\ 1500 & 2600 \end{pmatrix}$$

   This resulting matrix is schools (rows) by food (columns).
   So at the Junior School each day 640 sandwiches and 1100 biscuits are eaten.
   So at the Senior School each day 1500 sandwiches and 2600 biscuits are eaten.

Revision

6. Express in partial fractions:

   $$\frac{x^2 + 18x + 1}{(x - 1)(x - 3)^2} = \frac{5}{x - 1} - \frac{4}{x - 3} + \frac{32}{(x - 3)^2}$$

7. (a) Write down an simplify the general term in the expansion of

   $$\left(\frac{3}{x} + \frac{x^2}{2}\right)^{12}$$

   $$= \binom{12}{r}\left(\frac{3}{x}\right)^r\left(\frac{x^2}{2}\right)^{12-r}$$

   $$= \binom{12}{r} 3^r x^{-r}(x^2)^{12-r}\left(\frac{1}{2}\right)^{12-r}$$

   $$= \binom{12}{r} 3^r x^{-r} x^{24-2r} 2^{r-12}$$

$$= \binom{12}{r} 3^r 2^{r-12} x^{24-3r}$$

(b) Hence determine the *coefficient of unity*.

For the power of x to be zero we need $24 - 3r = 0$ hence $r = 8$
In which case

$$\binom{12}{r} 3^r 2^{r-12} x^{24-3r}$$
$$= \binom{12}{8} 3^8 2^{8-12} x^0$$
$$= \frac{3\,247\,695}{16}$$