2001

A10. A chemical plant food loses effectiveness at a rate proportional to the amount present in the soil. The amount M grams of plant food effective after t days satisfies the differential equation

$$\frac{dM}{dt} = kM, \quad \text{where } k \text{ is a constant.}$$

(a) Find the general solution for M in terms of t where the initial amount of plant food is M_0 grams. **(3)**

(b) Find the value of k if, after 30 days, only half the initial amount of plant food is effective. **(3)**

(c) What percentage of the original amount of plant food is effective after 35 days? **(2)**

(d) The plant food has to be renewed when its effectiveness falls below 25%. Is the manufacturer of the plant food justified in calling its product "sixty day super food"? **(2)**

B2. Find the general solution of the following differential equation $\frac{dy}{dx} + \frac{y}{x} = x,\ x > 0.$ **(4)**

B5. Find the general solution of the following differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x - 1.$ **(5)**

2002

B5. Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 4\cos x.$ **(6)**

Hence determine the solution which satisfies $y(0) = 0$ and $y'(0) = 1$. **(4)**

2003

A11. The volume $V(t)$ of a cell at time t changes according to the law

$$\frac{dV}{dt} = V(10 - V) \qquad \text{for } 0 < V < 10.$$

Show that $\frac{1}{10}\ln V - \frac{1}{10}\ln(10 - V) = t + C$ for some constant C. **(4)**

Given that $V(0)=5$, show that $V(t) = \frac{10e^{10t}}{1 + e^{10t}}$. **(3)**

Obtain the limiting value of $V(t)$ as $t \to \infty$ **(2)**

B6. Solve the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$,

given that $y = 2$ and $\frac{dy}{dx} = 1$, when $x = 0$ **(10)**

2004

15a) A mathematical biologist believes that the differential equation $x\dfrac{dy}{dx} - 3y = x^4$ models a process. Find the general solution of the differential equation. **(5)**

Given that $y = 2$ when $x = 1$, find the particular solution, expressing y in terms of x. **(2)**

b) The biologist subsequently decides that a better model is given by the equation $y\dfrac{dy}{dx} - 3x = x^4$
Given that $y = 2$ when $x = 1$, obtain y in terms of x. **(4)**

2005

14. Obtain the general solution of the differential equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 20\sin x$. **(7)**

Hence find the particular solution for which $y = 0$ and $\dfrac{dy}{dx} = 0$ when $x = 0$. **(3)**

2006

8. Solve the differential equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$ given that when $x = 0$, $y = 0$ and $\dfrac{dy}{dx}$. **(6)**

2007

8. Obtain the general solution of the equation $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = e^{2x}$. **(6)**

14. A garden centre advertises young plants to be used as hedging.
After planting, the growth G metres (ie the increase in height) after t years is modelled by the differential equation **(2)**

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

where k is a constant and $G = 0$ when $t = 0$.

(a) Express G in terms of t and k. **(4)**

(b) Given that a plant grows 0.6metres by the end of 5 years, find the value of k correct to 3 decimal places. **(2)**

(c) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified? **(2)**

(d) Given that the initial height of the plants was 0.3m, what is the likely long term height of the plants? **(2)**

2008

13. Obtain the general solution of the differential equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2x^2$. **(7)**

Given that $y = \dfrac{1}{2}$ and $\dfrac{dy}{dx} = 1$, when $x = 0$, find the particular solution. **(3)**

2009

3. Given that $x^2e^y\dfrac{dy}{dx}=1$

and $y = 0$ when $x = 1$, find y in terms of x. **(4)**

15. (a) Solve the differential equation $(x+1)\dfrac{dy}{dx}-3y=(x+1)^4$

given that $y = 16$ when $x = 1$, expressing the answer in the form $y = f(x)$. **(6)**

(b) Hence find the area enclosed by the graphs of $y = f(x)$, $y = (1 - x)^4$ and the x-axis. **(4)**

2010

11. Obtain the general solution of the equation $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+5y=0.$ **(4)**

Hence obtain the solution for which $y = 3$ when $x = 0$ and $y=e^{-\pi}$ when $x=\dfrac{\pi}{2}$. **(3)**

2011

9. Given that $y > -1$ and $x > -1$, obtain the general solution of the differential equation

$$\frac{dy}{dx}=3(1+y)\sqrt{1+x}$$

Expressing your answer in the form $y = f(x)$. **(5)**

14. Find the general solution of the differential equation $\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-2y=e^x+12$ **(7)**

Find the particular solution for which $y=-\dfrac{3}{2}$ and $\dfrac{dy}{dx}=\dfrac{1}{2}$ when $x = 0$. **(3)**

2012

15. (a) Express $\dfrac{1}{(x-1)(x+2)^2}$ in partial fractions **(4)**

(b) Obtain the general solution of the differential equation

$$(x-1)\frac{dy}{dx}-y=\frac{x-1}{(x+2)^2},$$

expressing your answer in the form $y=f(x)$. **(7)**

2013

14. Solve the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}, \text{ given that } y = 1 \text{ and } \frac{dy}{dx} = -1 \text{ when } x = 0.$$ **(11)**

16. In an environment without enough resources to support a population greater than 1000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{dP}{dt} = P(1000 - P).$$

Show that $\ln\frac{P}{1000-P} = 1000t + C$ for some constant C. **(4)**

Hence show that $P(t) = \frac{1000K}{K + e^{-1000t}}$ for some constant K. **(3)**

Given that $P(0) = 200$, determine at what time t, $P(t) = 900$. **(3)**

2014

8. Find the solution $y = f(x)$ to the differential equation $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

given that $y = 4$ and $\frac{dy}{dx} = 3$ when $x = 0$. **(6)**

2015

16. Solve the second order differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 3e^{2x}$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 0$. **(10)**

18. Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli's Law states that the rate of change of volume, V, of water in the tank is proportional to the square root of the height, h, of the water above the hole.

This is given by the differential equation: $\frac{dV}{dt} = -k\sqrt{h}, \quad k > 0.$

a) For a cylindrical tank with constant cross-sectional area, A, show that the rate of change of the height of the water in the tank is given by $\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}.$ **(2)**

b) Initially, when the height of the water is 144cm, the rate at which the height is changing is –0·3cm/hr.
By solving the differential equation in part (a), show that $h = \left(12 - \frac{1}{80}t\right)^2.$ **(4)**

c) How many days will it take for the tank to empty? **(2)**

d) Given that the tank has radius 20cm, find the rate at which the water was being delivered to the vegetation (in cm^3 /hr) at the end of the fourth day. **(3)**

2016

15. Solve the differential equation $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12x^2+2x-5$

given $y=-6$ and $\frac{dy}{dx}=3$, when $x=0$. **10 marks**

2017

9. Solve $\frac{dy}{dx}=e^{2x}\left(1+y^2\right)$ given that when $x=0$, $y=1$.

Express y in terms of x. **5 marks**

14. Find the particular solution of the differential equation $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=8\sin x+19\cos x$

given that $y=7$ and $\frac{dy}{dx}=\frac{1}{2}$ when $x=0$. **10 marks**

2019

8. Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2}+11\frac{dy}{dx}+28y=0$$

given that $y=0$ and $\frac{dy}{dx}=9$, when $x=0$. **5**

13. An electronic device contains a timer circuit that switches off when the voltage, V, reaches a set value.

The rate of change of the voltage is given by

$$\frac{dV}{dt}=k\left(12-V\right),$$

where k is a constant, t is the time in seconds, and $0\leq V<12$.

Given that $V=2$ when $t=0$, express V in terms of k and t. **5**

Answers

2001

A10 a) $M = M_0 e^{kt}$ b) $k = -0.231$ B2 $y = \frac{x^2}{3} + \frac{C}{x}$ B5 $y = Ae^x + Be^{-3x} - 2x - 1$

c) 44.5% d) Yes

2002 B5. $y(x) = \frac{e^{-x}}{10}(-8\cos 2x - \sin 2x) + \frac{2}{5}(2\cos x + \sin x)$

2003 A11 a) Proof b) Proof c) 10 as $t \to \infty$ B6 $y = (1-2x)e^{2x} + e^x$

2004 Q15 $y = (x+c)x^3$ $y = (x+1)x^3$ $y = \sqrt{2\left(\frac{x^5}{5} + \frac{3x^2}{2} + \frac{3}{10}\right)}$

2005 Q14 $y = Ae^x + Be^{2x} + 2\sin x + 6\cos x$ $y = -10e^x + 4e^{2x} + 2\sin x + 6\cos x$

2006 Q8 $y = 2e^{-x}\sin x$

2007 Q8 $y = (A + Bx)e^{-3x} + \frac{1}{25}e^{2x}$

Q14.a) $G = 25k\left(1 - e^{-t/25}\right)$ b) 0.132 c) Yes as G is approx 1.09 d) Limit = 3.6 metres

2008 Q13 $y = Ae^x + Be^{2x} + x^2 + 3x + \frac{7}{2}$ $y = -4e^x + e^{2x} + x^2 + 3x + \frac{7}{2}$

2009 Q3 $y = \ln\left(2 - \frac{1}{x}\right)$ Q15 $y = (x+1)^4$ $\frac{2}{5}$

2010 Q11 $y = e^{-2x}(A\cos x + B\sin x)$

$y = e^{-2x}(3\cos x + \sin x)$

2011 Q9 $y = Ae^{2(1+x)^{3/2}} - 1$ Q14. $y = Ae^{2x} + Be^{-y} - \frac{1}{2}e^x - 6$ $y = 2e^{2x} + 3e^{-y} - \frac{1}{2}e^x - 6$

2012 Q15 $\frac{1}{9}\left(\frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}\right)$ $\frac{x-1}{9}\left(\ln\left|\frac{x-1}{x+2}\right| + \frac{3}{x+2}\right) + c(x-1)$

2013 Q14. $y = e^{3x} - 4xe^{3x} + 2x^2e^{3x}$ Q16. $t = \frac{1}{1000}\ln 36$

[or 0.003584 (4sf)]

2014 Q8. $y = 4e^{\frac{1}{2}x} + xe^{\frac{1}{2}x}$

2015 16. $y = \frac{5}{6}e^{-x}\cos 3x + \frac{1}{6}e^{-x}\sin 3x + \frac{1}{6}e^{2x}$ 18 c) 40 days d) 108π cm^3 / hr

2016 15. $y = 8e^{-3x} - 15e^{-2x} + 2x^2 - 3x + 1$

2017 9. $y = \tan\left(\frac{1}{2}e^{2x} + \frac{\pi}{4} - \frac{1}{2}\right)$ **(5)** 14. $y = 5e^{3x} - 14xe^{3x} - \frac{1}{2}\sin x + 2\cos x$

(10)

2019

8. $y = 3e^{-4x} - 3e^{-7x}$ 13. $V = 12 - 10e^{-kt}$