Maclaurin Series Expansions

Worksheet 1

***Answer Key** : scan the **QR Code** with your phone, or tablet, to see the answer key.*
*Or click on the **link in the header** of this page.*
*File Name: **Maclaurin Series Expansions - Worksheet 1 - Answer Key with Working***

Exercise 1

Write the first four, non-zero, terms in the Maclaurin series expansions of each of the following functions:

1. $f(x) = e^{3x}$
2. $f(x) = sin(x)$
3. $f(x) = \dfrac{1}{1+x}$
4. $f(x) = ln\,(1+x)$
5. $f(x) = cos(2x)$
6. $f(x) = \sqrt{1+3x}$
7. $f(x) = e^{x^2}$ (use the expansion of e^x)
8. $f(x) = \dfrac{1}{1-x}$

Exercise 2

Using the first three, non-zero, terms of the Maclaurin expansions of $sin(x)$ and $cos(x)$ as well as algebraic long division of polynomials find the first three non-zero terms of the Maclaurin expansion of $tan(x)$.

Exercise 3

Find an upper bound to the error in approximating each of the following functions by their Maclaurin polynomials, $p_n(x)$, to the degree stated, for $x = 0.5$.

1. $e^x, p_3(x)$
2. $sin(2x), p_5(x)$
3. $\dfrac{1}{1+x}, p_3(x)$s
4. $\sqrt{1+x}, p_4(x)$

Exercise 1

> **Maclaurin Series Expansions**
>
> $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

1) $f(x) = e^{3x}$

Working:

$f'(x) = 3.e^{3x}$ ⋮ $f'(0) = 3$

$f''(x) = 3^2.e^{3x}$ ⋮ $f''(0) = 3^2 = 9$

$f^{(3)}(x) = 3^3.e^{3x}$ ⋮ $f^{(3)}(0) = 3^3 = 27$

since $f(0) = 1$ we can stop at the 3^{rd} derivative.

$$e^{3x} = 1 + 3x + \frac{9}{2!}.x^2 + \frac{27}{3!}x^3 + \cdots$$

$$\boxed{e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots}$$

2) $f(x) = \sin(x)$

$f'(x) = \cos(x)$ ⋮ $f'(0) = 1$

$f''(x) = -\sin(x)$ ⋮ $f''(0) = 0$

$f^{(3)}(x) = -\cos(x)$ ⋮ $f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin(x)$ ⋮ $f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos(x)$ ⋮ $f^{(5)}(0) = 1$

$f^{(6)}(x) = -\sin(x)$ ⋮ $f^{(6)}(0) = 0$

$f^{(7)}(x) = -\cos(x)$ ⋮ $f^{(7)}(0) = -1$

$f(0) = 0$.

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

3) $f(x) = \dfrac{1}{1+x}$

Write $f(x)$ in the form:

$f(x) = (1+x)^{-1}$ ⋮ $f(0) = 1$

$f'(x) = -(1+x)^{-2}$ ⋮ $f'(0) = -1$

$f''(x) = 2(1+x)^{-3}$ ⋮ $f''(0) = 2$

$$f^{(3)}(x) = -3\times 2(1+x)^{-4} \quad ; \quad f^{(3)}(0) = -6$$

$$\frac{1}{1+x} = 1 - x + \frac{2}{2!}\cdot x^2 - \frac{6}{3!}x^3 + \cdots$$

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots}$$

4) $f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x}$$

$$f'(x) = (1+x)^{-1} \qquad f'(0) = 1$$

$$f''(x) = -(1+x)^{-2} \qquad f''(0) = -1$$

$$f^{(3)}(x) = 2(1+x)^{-3} \qquad f^{(3)}(0) = 2$$

$$f^{(4)}(x) = -3\times 2(1+x)^{-4} \qquad f^{(4)}(0) = -6$$

$$f(0) = \ln(1) = 0$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 + \cdots$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$$

Notice that we could have obtained this by integrating our answer to question 3.

5) $f(x) = \cos(2x)$

$$f'(x) = -2\cdot\sin(2x) \qquad f'(0) = 0$$

$$f''(x) = -2^2\cdot\cos(2x) \qquad f''(0) = -4$$

$$f^{(3)}(x) = 2^3\cdot\sin(2x) \qquad f^{(3)}(0) = 0$$

$$f^{(4)}(x) = 2^4\cdot\cos(2x) \qquad f^{(4)}(0) = 16$$

$$f^{(5)}(x) = -2^5\cdot\sin(2x) \qquad f^{(5)}(0) = 0$$

$$f^{(6)}(x) = -2^6\cdot\cos(2x) \qquad f^{(6)}(0) = -64$$

$$f(0) = \cos(0) = 1$$

$$\cos(2x) = 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 - \frac{64}{6!}x^6 + \cdots$$

$$\boxed{\cos(2x) = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots}$$

$$\frac{16}{4\times3\times2} =$$

$$\frac{\cancel{64}\ \cancel{8}4}{3\,\cancel{8}\times5\times\cancel{4}\times3\times\cancel{2}} = \frac{4}{45}$$

6) $f(x)=\sqrt{1+3x}$

$f(x)=(1+3x)^{1/2}$ | $f(0)=1$

$f'(x)=\frac{3}{2}(1+3x)^{-1/2}$ | $f'(0)=\frac{3}{2}$

$f''(x)=-\frac{3^2}{2^2}\cdot(1+3x)^{-3/2}$ | $f''(0)=-\frac{9}{4}$

$f^{(3)}(x)=\frac{3^3}{2^3}(1+3x)^{-5/2}$ | $f^{(3)}(0)=\frac{27}{8}$

$$\sqrt{1+3x}=1+\frac{3x}{2}-\frac{9}{4}\cdot\frac{x^2}{2!}+\frac{27}{8}\cdot\frac{x^3}{3!}-\cdots$$

$$\boxed{\sqrt{1+3x}=1+\frac{3}{2}x-\frac{9}{8}x^2+\frac{9}{16}x^3-\cdots}$$

7) $f(x)=e^{x^2}$

Consider $g(x)=e^x$

$g'(x)=e^x$ | $g'(0)=1$

$g''(x)=e^x$ | $g''(0)=1$

$g^{(3)}(x)=e^x$ | $g^{(2)}(0)=1$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$f(x)=g(x^2)$:

$$\boxed{e^{x^2}=1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\cdots}$$

8) $f(x)=\frac{1}{1-x}$

$f(x)=(1-x)^{-1}$ | $f(0)=1$

$f'(x)=(1-x)^{-2}$ | $f'(0)=1$

$f''(x)=2.(1-x)^{-3}$ | $f''(0)=2=2!$

$f^{(3)}(x)=3\times2.(1-x)^{-4}$ | $f^{(3)}(0)=3\times2=3!$

$$\frac{1}{1-x}=1+x+\frac{2!}{2!}x^2+\frac{3!}{3!}x^3+\cdots$$

$$\boxed{\frac{1}{1-x}=1+x+x^2+x^3+\cdots}$$

Notice that $\frac{1}{1-x}$ is the infinite sum of the geometric series with first term 1 and common ratio x ; $|x|<1$.

4×3×2

$4!$

Exercise 2

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \dots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots}$$

Algebraic Long Division :

$$\begin{array}{r|l}
 & x + \frac{x^3}{3} + \frac{2}{15}.x^5 + \dots \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots & x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \\
 & -\left(x - \frac{x^3}{2} + \frac{x^5}{24} - \dots\right) \\
\cline{2-2}
 & \frac{x^3}{3} - \frac{x^5}{30} - \dots \\
 & -\left(\frac{x^3}{3} - \frac{x^5}{6} + \dots\right) \\
\cline{2-2}
 & \frac{2}{15}.x^5 + \dots \\
 & -\left(\frac{2}{15}.x^5 - \dots\right) \\
\cline{2-2}
 & 0 + \dots
\end{array}$$

$$\boxed{\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}.x^5 + \dots}$$

Exercise 3

1) $$e^x = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}_{P_3(x)} + \underbrace{R_3(x)}_{\text{error term}}$$

$$\boxed{\begin{gathered} R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \\ c \in]0, x[\end{gathered}}$$

Lagrange form of the error term

$$R_3(0.5) = \frac{e^c}{4!} \times 0.5^4 \quad , \ c \in]0, 0.5[$$

$$0 < R < \frac{e^{0.5}}{24} . 0.5^4 = 0.004294 \quad \text{(using calculator)}$$

So $\boxed{R_3(x) < 0.004294}$

2) $$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + R_5(x)$$

$$\sin(2x) = \underbrace{2x - \frac{4}{3}x^3 + \frac{4}{15}x^5}_{P_5(x)} + \underbrace{R_5(x)}_{\text{error term}}$$

$$R_5(x) = \frac{f^{(6)}(c)}{5!} \cdot x^6 \quad , c \in]0, x[$$

since $f(x) = \sin(2x)$: $f^{(6)}(x) = -2^6 \cdot \sin(2x) = -64 \cdot \sin(2x)$

$$R_5(0.5) = -\frac{64 \cdot \sin(2c)}{5!} \cdot (0.5)^6 \quad , c \in]0, 0.5[$$

Note: work with absolute value of $R_5(x)$!

$$0 \leq |R_5(0.5)| = \left| \frac{-64 \cdot \sin(2c)}{120} \cdot (0.5)^6 \right| \leq \frac{64}{120} \times \frac{1}{2^6} = \frac{1}{120}$$

$$\boxed{0 \leq |R_5(0.5)| \leq \frac{1}{120}}$$

3) using our results from ex.1 question 3:

$$\frac{1}{1+x} = \underbrace{1 - x + x^2 - x^3}_{P_3(x)} + \underbrace{R_3(x)}_{\text{error term}}$$

$$R_3(x) = \frac{f^{(4)}(c)}{4!} x^4 \quad , c \in]0, x[$$

$$f^{(4)}(x) = 4!\,(1+x)^{-5} = \frac{4!}{(1+x)^5}$$

so $\quad R_3(x) = \dfrac{x^4}{(1+c)^5} \quad , c \in]0, x[$

$$R_3(0.5) = \frac{0.5^4}{(1+c)^5} \quad , c \in]0, 0.5[$$

$$\boxed{R_3(0.5) < 0.5^4 = \frac{1}{16}}$$

4) $\sqrt{1+x}$

let $f(x) = (1+x)^{1/2}$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = -\frac{1}{2^2} \cdot (1+x)^{-3/2}$

$f^{(3)}(x) = \frac{3}{2^3} \cdot (1+x)^{-5/2}$

$f^{(4)}(x) = -\frac{5 \times 3}{2^4} \cdot (1+x)^{-7/2}$

$f^{(5)}(x) = \frac{7 \times 5 \times 3}{2^5} (1+x)^{-9/2}$

$f(0) = 1$

$f'(0) = \frac{1}{2}$

$f''(0) = -\frac{1}{4}$

$f^{(3)}(0) = \frac{3}{8}$

$f^{(4)}(0) = -\frac{15}{16}$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{4}\cdot\frac{x^2}{2!} + \frac{3}{8}\cdot\frac{x^3}{3!} - \frac{15}{16}\cdot\frac{x^4}{4!} + R_4(x)$$

$$\sqrt{1+x} = \underbrace{1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}\cdot x^4}_{P_4(x)} + \underbrace{R_4(x)}_{\text{error term}}$$

$$R_4(x) = \frac{f^{(5)}(c)}{5!}x^5 \quad , \quad c \in]0,x[$$

since $f^{(5)}(x) = \dfrac{105}{32(1+x)^{9/2}}$

$$\frac{105}{32}$$

$$R_4(x) = \frac{105}{32\times 120(1+c)^{9/2}}\cdot x^5 \quad , \quad c \in]0,x[$$

$$R_4(x) = \frac{7}{256(1+c)^{9/2}}\cdot x^5$$

$$105 = \frac{\cancel{3}\times\cancel{5}\times 7}{32\times\cancel{3}\times 4\times\cancel{5}\times 2} = \frac{7}{256}$$

$$R_4(x) < \frac{7}{256}\cdot x^5$$

$$R_4(0.5) = R_4\left(\frac{1}{2}\right) < \frac{7}{256}\times\frac{1}{2^5} = \frac{7}{8192}$$

$$\boxed{R_4(0.5) < \frac{7}{8192}}$$