Vectors

Revision of Vectors from Higher Maths

If $P(x, y, z)$ then relative to an origin, O, $\vec{OP} = \underline{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

and the magnitude of $\underline{p}$ is $|\underline{p}| = \sqrt{x^2 + y^2 + z^2}$

A unit vector is a vector with a magnitude $= 1$. Unit vector in direction of $\underline{p} = \frac{1}{|\underline{p}|}\underline{p}$.

There are three mutually perpendicular unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$

where $\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This allows us to write $\underline{p}$ as $x\underline{i} + y\underline{j} + z\underline{k}$.

Laws

Commutative: $\underline{a} + \underline{b} = \underline{b} + \underline{a}$

Associative: $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$

Zero vector (or Identity vector) $\underline{a} + \underline{0} = \underline{0} + \underline{a} = \underline{a}$

Negative of a vector $\underline{a} + -\underline{a} = \underline{0}$

Multiplication by a scalar $|k\underline{a}| = k|\underline{a}|$

If $k > 0$, $k\underline{a}$ is parallel to $\underline{a}$ and is in the same direction.

If $k < 0$, $k\underline{a}$ is parallel to $\underline{a}$ and is in the opposite direction.

Position vectors If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are coordinates relative to an origin, O, then

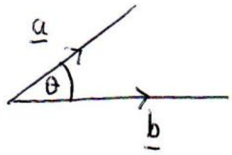
$$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

and the distance between A and B is $|\vec{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$.

The midpoint of $AB = \frac{\underline{a} + \underline{b}}{2}$.

Scalar Product (or Dot Product)

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$



where $\mathbf{a}$ and $\mathbf{b}$ are tail-to-tail.

(and $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ since $\cos 0 = 1$)

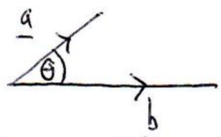
$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

is the component form of the scalar (or dot) product

Note: • $\mathbf{a}\mathbf{b}$ doesn't mean anything! You must write $\mathbf{a} \cdot \mathbf{b}$ if you mean the scalar (or dot) product. We are about to move on to the Vector (or Cross) Product.

• The answer to $\mathbf{a} \cdot \mathbf{b}$ is a scalar (i.e. a constant not a vector!)

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$



for $\mathbf{a}$ and $\mathbf{b}$ as tail-to-tail vectors.

If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a} \cdot \mathbf{b} = 0$.
If $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

provided that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$.

so $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ (since unit vectors)

and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$ (since perpendicular vectors)

Distributive Law: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

Q1 Calculate the angle, θ, between vectors $\mathbf{p} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{q} = 4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

Ans1

$$\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \text{ and } \mathbf{q} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \Rightarrow |\mathbf{p}| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38} \text{ and } |\mathbf{q}| = \sqrt{4^2 + 1^2 + 3^2} = \sqrt{26}$$

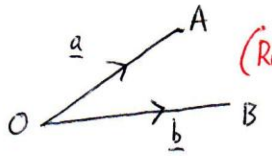
$$\mathbf{p} \cdot \mathbf{q} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = (3)(4) + (2)(1) + (5)(3) = 12 + 2 + 15 = 29$$

$\left[\text{NOT } \begin{pmatrix} 12 \\ 2 \\ 15 \end{pmatrix} !\right]$

$$\cos\theta = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{29}{\sqrt{38}\sqrt{26}} = \frac{29}{\sqrt{988}} \Rightarrow \theta = \cos^{-1}\left(\frac{29}{\sqrt{988}}\right) = 22.7^\circ$$

Q2 A is $(1,-2,5)$ and B is $(3,4,-1)$. Calculate angle AOB.

Ans 2 Notice that in this question, the vertex is the origin, O (since O is the middle letter).



(Randomly positioned, but to show tail-to-tail vectors $\overrightarrow{OA}$ ($\underline{a}$) & $\overrightarrow{OB}$ ($\underline{b}$).

$$\cos A\hat{O}B = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$$

Not θ since no θ in this question.

$$\underline{a}.\underline{b} = \begin{pmatrix}1\\-2\\5\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-1\end{pmatrix} = (1)(3)+(-2)(4)+(5)(-1)$$

$$= 3-8-5 = -10$$

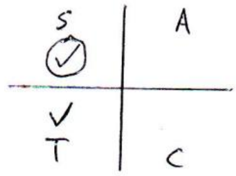
$$|\underline{a}| = \sqrt{1^2+(-2)^2+5^2} = \sqrt{1+4+25} = \sqrt{30}$$

notice brackets when squaring a negative!

$$|\underline{b}| = \sqrt{3^2+4^2+(-1)^2} = \sqrt{9+16+1} = \sqrt{26}$$

$$\cos A\hat{O}B = \frac{-10}{\sqrt{30}\sqrt{26}} = \frac{-10}{\sqrt{780}}$$

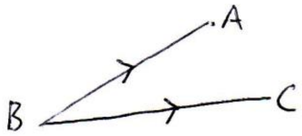
$$\left[\text{Consider } \cos\alpha = +\frac{10}{\sqrt{780}} \Rightarrow \alpha = 69.0°\right]$$



Angle AOB $= 180° - 69.0° = \underline{\underline{111.0°}}$ (not $180°+69.0° = 249.0°$ since an angle between two lines will either be acute or obtuse, not reflex.)

Q3 A is $(1,7,13)$, B is $(1,3,7)$ and C is $(2,-3,9)$. Calculate angle ABC.

Ans3 Notice that in this question, the vertex is <u>NOT</u> the origin (it is B) so we <u>must</u> modify the formula to suit.



(Randomly positioned, but it shows that the tail-to-tail vectors are $\overrightarrow{BA}$ (<u>not</u> $\overrightarrow{AB}$) and $\overrightarrow{BC}$.)

$$\cos A\hat{B}C = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$$

NOT $\frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$

$\overrightarrow{BA} = \underline{a} - \underline{b}$ (position vectors)

$$= \begin{pmatrix}1\\7\\13\end{pmatrix} - \begin{pmatrix}1\\3\\7\end{pmatrix} = \begin{pmatrix}0\\4\\6\end{pmatrix}$$

$$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix}2\\-3\\9\end{pmatrix} - \begin{pmatrix}1\\3\\7\end{pmatrix} = \begin{pmatrix}1\\-6\\2\end{pmatrix}$$

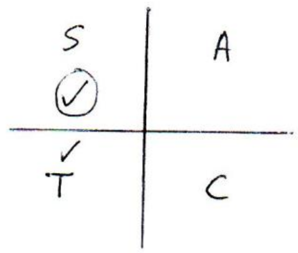
$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix} = (0)(1) + (4)(-6) + (6)(2) = -12$$

$$|\overrightarrow{BA}| = \sqrt{0^2 + 4^2 + 6^2} = \sqrt{52} \qquad |\overrightarrow{BC}| = \sqrt{1^2 + (-6)^2 + 2^2} = \sqrt{41}$$

Show 0^2

$$\cos A\hat{B}C = \frac{-12}{\sqrt{52}\sqrt{41}} = \frac{-12}{\sqrt{2132}}$$

$$\left[\text{Consider } \cos\alpha = +\frac{12}{\sqrt{2132}} \Rightarrow \alpha = 74 \cdot 9^\circ \right]$$



Angle ABC $= 180^\circ - 74 \cdot 9^\circ = \underline{\underline{105 \cdot 1^\circ}}$ (not $180^\circ + 74 \cdot 9^\circ = 254 \cdot 9^\circ$ since two lines don't cross with a reflex angle)

New Vectors Content for the Advanced Higher Maths course

The Vector Product (or Cross Product)

$\underline{a} \times \underline{b}$ is either called the Vector Product (because the answer is a Vector!) or the Cross Product (because we use a 'cross' not a 'dot').

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\theta$$

magnitude of $\underline{a} \times \underline{b}$

Notice: $\underline{a} \times \underline{b}$ is a vector which is perpendicular to both $\underline{a}$ and $\underline{b}$! provided $\underline{a} \neq \underline{0}$ and $\underline{b} \neq \underline{0}$.

so to find a vector which is perpendicular to $\underline{a}$ and $\underline{b}$, find $\underline{a} \times \underline{b}$ (which will be explained in a moment).

Properties of the Vector Product

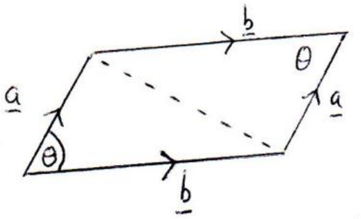
(a)(i) If $\underline{a} \times \underline{b} = \underline{0}$, then $\underline{a}$ and $\underline{b}$ are parallel, provided $\underline{a} \neq \underline{0}$ and $\underline{b} \neq \underline{0}$.

(ii) If $\underline{a} \neq \underline{0}$ and $\underline{b} \neq 0$, and if $\underline{a}$ and $\underline{b}$ are parallel, then $\underline{a} \times \underline{b} = \underline{0}$.

$$\underline{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

underlined

(b) $|\underline{a} \times \underline{b}|$ is the area of a parallelogram with sides $\underline{a}$ and $\underline{b}$



(Since area of a triangle $= \frac{1}{2} ab \sin C$

so area of $\Delta = \frac{1}{2} |\underline{a}||\underline{b}| \sin\theta$

and parallelogram has two identical Δs).

(c) The following statements are equivalent (if $\underline{a} \neq \underline{0}$ and $\underline{b} \neq 0$):

(i) $\underline{a} \times \underline{b} = \underline{0}$

(ii) $\underline{a}$ is parallel to $\underline{b}$ } as mentioned in (a).

(iii) $\underline{a} = k\underline{b}$ ($\underline{a}$ is a linear multiple of $\underline{b}$)

(d) $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$. In fact, $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$.

(e) (i) $k\underline{a} \times \underline{b} = k(\underline{a} \times \underline{b})$ and

(ii) $k\underline{a} \times l\underline{b} = kl(\underline{a} \times \underline{b})$

(f) The distributive law: $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$

$(\underline{b} + \underline{c}) \times \underline{a} = \underline{b} \times \underline{a} + \underline{c} \times \underline{a}$

But how do we calculate $\underline{a} \times \underline{b}$? There is the official (SQA) method, but there are various shortcuts too.

The Component Form of $\underline{a} \times \underline{b}$

If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, we find the value of the 3x3 determinant

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ and remembering the pattern $$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= \underline{i}(a_2b_3 - a_3b_2) - \underline{j}(a_1b_3 - a_3b_1) + \underline{k}(a_1b_2 - a_2b_1)$$

$$= (a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

As for a shortcut, at university, I was shown this method and (rightly or wrongly) I used it throughout the SQA past paper questions on our pbworks site. It's fine if you use this shortcut and don't make a careless error, but if you use the official 3×3 determinant approach but make a careless mistake, you are likely to lose 1 mark only — the first mark tends to be showing the appropriate 3×3 determinant.

If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, write down the components of each horizontally and write the first column that you get again at the end:

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_1 \\ b_1 & b_2 & b_3 & b_1 \end{array}$$

Step 3 Step 1 Step 2

Start in the MIDDLE: find the determinant of the 2×2 matrix: $a_2b_3 - a_3b_2$.

Now move to the RIGHT: find the next determinant: $a_3b_1 - a_1b_3$

Now go back to the START and find the first determinant: $a_1b_2 - a_2b_1$

Write your answer as a VECTOR: $\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$

(In other words, starting in the middle, find the product of the <u>main</u> diagonal and <u>subtract</u> the product of the other diagonal.
Move right, and repeat that process.
Move to the front and repeat the process again.)

<u>Q4</u> If $\underline{a} = 3\underline{i} - \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} + 2\underline{j} - 2\underline{k}$, find: (a) $\underline{a}\times\underline{b}$
(b) $|\underline{a}\times\underline{b}|$.

<u>Ans 4(a)</u>

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & -1 \\ 1 & 2 & -2 \end{vmatrix} = \underline{i}\big((-1)(-2)-(-1)(2)\big) - \underline{j}\big((3)(-2)-(-1)(1)\big) + \underline{k}\big((3)(2)-(-1)(1)\big)$$

$$= 4\underline{i} - (-5)\underline{j} + 7\underline{k}$$

$$\underline{\underline{\underline{a}\times\underline{b} = 4\underline{i} + 5\underline{j} + 7\underline{k}}} \quad \text{OR} \quad \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$$

OR

$$\begin{matrix} 3 & & -1 & & -1 & & 3 \\ & \times & & \times & & \times & \\ 1 & & 2 & & -2 & & 1 \end{matrix}$$

③ ① ②

$$\begin{pmatrix} 2-(-2) \\ -1-(-6) \\ 6-(-1) \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} \text{ OR } 4\underline{i} + 5\underline{j} + 7\underline{k}$$

(b) $|\underline{a} \times \underline{b}| = \sqrt{4^2 + 5^2 + 7^2} = \sqrt{16 + 25 + 49} = \sqrt{90}$

$= \sqrt{9 \times 10} = \sqrt{9} \times \sqrt{10}$

$= \underline{\underline{3\sqrt{10}}}$

Q5 Find the unit vector perpendicular to $\underline{p} = \underline{i} + 4\underline{j} + 2\underline{k}$ and $\underline{q} = 3\underline{i} + 6\underline{j} + 5\underline{k}$.

Ans 5 A vector perpendicular to both $\underline{p}$ and $\underline{q}$ is $\underline{p} \times \underline{q}$ (or $\underline{q} \times \underline{p}$) — one being the negative of the other, so there are two possible correct answers.

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 4 & 2 \\ 3 & 6 & 5 \end{vmatrix} = \underline{i}\left((4)(5) - (2)(6)\right) - \underline{j}\left((1)(5) - (2)(3)\right) + \underline{k}\left((1)(6) - (4)(3)\right)$$

$$= 8\underline{i} - (-1)\underline{j} + -6\underline{k}$$

$$= 8\underline{i} + \underline{j} - 6\underline{k} \text{ OR } \begin{pmatrix} 8 \\ 1 \\ -6 \end{pmatrix}$$

OR

$$\begin{matrix} 1 & & 4 & & 2 & & 1 \\ & \times & & \times & & \times & \\ 3 & & 6 & & 5 & & 3 \end{matrix}$$

③ ① ②

$$\begin{pmatrix} 20-12 \\ 6-5 \\ 6-12 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -6 \end{pmatrix} \text{ OR } 8\underline{i} + \underline{j} - 6\underline{k}$$

This is a vector which is perpendicular to $\underline{p}$ and $\underline{q}$. If we had used $\underline{q} \times \underline{p}$ instead, we would have had $-8\underline{i} - \underline{j} + 6\underline{k}$ or $\begin{pmatrix} -8 \\ -1 \\ 6 \end{pmatrix}$. However, we do not yet have a UNIT vector perpendicular to $\underline{p}$ and $\underline{q}$.

$|\underline{p} \times \underline{q}| = \sqrt{8^2 + 1^2 + (-6)^2} = \sqrt{101}$ so the required unit vector is either

$$\frac{1}{\sqrt{101}}\begin{pmatrix} 8 \\ 1 \\ -6 \end{pmatrix} \text{ or } -\frac{1}{\sqrt{101}}\begin{pmatrix} 8 \\ 1 \\ -6 \end{pmatrix} \text{ OR } \begin{pmatrix} 8/\sqrt{101} \\ 1/\sqrt{101} \\ -6/\sqrt{101} \end{pmatrix} \text{ OR } \begin{pmatrix} -8/\sqrt{101} \\ -1/\sqrt{101} \\ 6/\sqrt{101} \end{pmatrix}$$

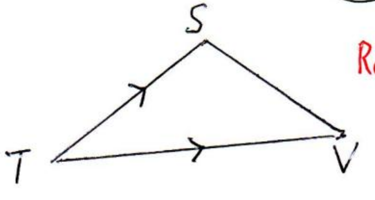
Q6 Find the area of the triangle with vertices $S(1,3,-2)$, $T(4,3,0)$ and $V(2,1,1)$.

Ans6 We can use our new Advanced Higher Maths information or go for a longer solution using Higher and National 5 Maths.

(Method 1) Adv. H. Maths.

$|\underline{a} \times \underline{b}| =$ area of a parallelogram

so $\frac{1}{2}|\underline{a} \times \underline{b}| =$ area of a triangle.



Randomly drawn and random arrows, but both travelling AWAY from the SAME angle.

$$\overrightarrow{TS} = \underline{s} - \underline{t} = \begin{pmatrix}1\\3\\-2\end{pmatrix} - \begin{pmatrix}4\\3\\0\end{pmatrix} = \begin{pmatrix}-3\\0\\-2\end{pmatrix}$$

$$\overrightarrow{TV} = \underline{v} - \underline{t} = \begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}4\\3\\0\end{pmatrix} = \begin{pmatrix}-2\\-2\\1\end{pmatrix}$$

Area of triangle $STV = \frac{1}{2}\left|\overrightarrow{TS} \times \overrightarrow{TV}\right|$

But $\overrightarrow{TS} \times \overrightarrow{TV} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -3 & 0 & -2 \\ -2 & -2 & 1 \end{vmatrix} = \underline{i}(0-4) - \underline{j}(-3-4) + \underline{k}(6-0) = -4\underline{i} + 7\underline{j} + 6\underline{k}$

Area of triangle $STV = \frac{1}{2}\sqrt{(-4)^2 + 7^2 + 6^2} = \frac{1}{2}\sqrt{16+49+36} = \underline{\underline{\frac{1}{2}\sqrt{101} \text{ units}^2}}$

(Method 2) H & N5 Maths Same diagram & position vectors, so $\overrightarrow{TS} = \begin{pmatrix}-3\\0\\-2\end{pmatrix}$ & $\overrightarrow{TV} = \begin{pmatrix}-2\\-2\\1\end{pmatrix}$

$\cos S\hat{T}V = \dfrac{\overrightarrow{TS} \cdot \overrightarrow{TV}}{|\overrightarrow{TS}||\overrightarrow{TV}|}$ $\quad \therefore \overrightarrow{TS} \cdot \overrightarrow{TV} = \begin{pmatrix}-3\\0\\-2\end{pmatrix} \cdot \begin{pmatrix}-2\\-2\\1\end{pmatrix} = 6 + 0 + (-2) = 4$

$|\overrightarrow{TS}| = \sqrt{(-3)^2 + 0^2 + (-2)^2} = \sqrt{13}$ and $|\overrightarrow{TV}| = \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3$

$\cos S\hat{T}V = \dfrac{4}{3\sqrt{13}} \Rightarrow$ Angle $STV = 68 \cdot 297°$

Area of triangle $STV = \frac{1}{2}|\overrightarrow{TS}||\overrightarrow{TV}| \sin S\hat{T}V = \frac{1}{2} \times \sqrt{13} \times 3 \times \sin 68 \cdot 297°$

$= \underline{\underline{5 \cdot 02 \text{ units}^2}} \left(= \frac{\sqrt{101}}{2} \text{ units}^2\right)$

(Beware – method 2 was shortened here since we stole from method 1 at the start.)

There is something called the Vector Triple Product:

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$$

No dot No dot

You could be asked to verify it for particular vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ but you will not be asked to apply it in context and there is no need for you to memorise it.

<u>Q7</u> If $\underline{a} = 3\underline{i} + 5\underline{j} + 7\underline{k}$, $\underline{b} = \underline{i} + \underline{k}$ and $\underline{c} = 2\underline{i} - \underline{j} + 3\underline{k}$, verify that

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}.$$

<u>Ans 7</u> <u>LHS</u> Brackets first: $\underline{b} \times \underline{c}$ $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 1 \\ 2 & -1 & 3 \end{vmatrix}$

$$= \underline{i}(0-(-1)) - \underline{j}(3-2) + \underline{k}(-1-0)$$
$$= \underline{i} - \underline{j} - \underline{k}$$

OR

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 2 & -1 & 3 & 2 \end{array} \quad \begin{array}{c} 0-(-1) \\ 2-3 \\ -1-0 \end{array} \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \underline{i} - \underline{j} - \underline{k}$$

(3) (1) (2)

$$\underline{a} \times (\underline{b} \times \underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 5 & 7 \\ 1 & -1 & -1 \end{vmatrix} = \underline{i}(-5-(-7)) - \underline{j}(-3-7) + \underline{k}(-3-5)$$
$$= 2\underline{i} + 10\underline{j} - 8\underline{k}$$

OR

$$\begin{array}{cccc} 3 & 5 & 7 & 3 \\ 1 & -1 & -1 & 1 \end{array} \quad \begin{pmatrix} -5-(-7) \\ 7-(-3) \\ -3-5 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -8 \end{pmatrix} = 2\underline{i} + 10\underline{j} - 8\underline{k}$$

(3) (1) (2)

Final answer to LHS

<u>RHS</u> $\underline{a}.\underline{c} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 6 + -5 + 21 = 22$ so $(\underline{a}.\underline{c})\underline{b} = 22\underline{b} = 22\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 22 \\ 0 \\ 22 \end{pmatrix}$

$\underline{a}.\underline{b} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 3 + 0 + 7 = 10$ so $(\underline{a}.\underline{b})\underline{c} = 10\underline{c} = 10\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 20 \\ -10 \\ 30 \end{pmatrix}$

$(\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c} = \begin{pmatrix} 22 \\ 0 \\ 22 \end{pmatrix} - \begin{pmatrix} 20 \\ -10 \\ 30 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -8 \end{pmatrix} = 2\underline{i} + 10\underline{j} - 8\underline{k}$ = final answer to LHS (as req^{d})

In general, $(\underline{a} \times \underline{b}) \times \underline{c} \neq \underline{a} \times (\underline{b} \times \underline{c})$. Therefore, $\underline{a} \times \underline{b} \times \underline{c}$ has no meaning! Brackets <u>must</u> be inserted to indicate the order.

In addition, $\underline{a} \cdot (\underline{b} \times \underline{c})$ is known as the Scalar Triple Product (or as the Dot Cross Product) and the answer is a <u>scalar</u> (i.e. a number). Again, we won't be expected to learn this expression or its name, and will not meet it an a context, but we could perhaps meet the following type of question.

<u>Q8</u> Calculate the Scalar Triple Product $\underline{a} \cdot (\underline{b} \times \underline{c})$ when $\underline{a} = \underline{i} + 2\underline{j} - 3\underline{k}$, $\underline{b} = 4\underline{i} - \underline{j} + 2\underline{k}$ and $\underline{c} = 3\underline{i} - 2\underline{j} + 5\underline{k}$.

<u>Ans 8</u> Brackets first: $\underline{b} \times \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & -1 & 2 \\ 3 & -2 & 5 \end{vmatrix}$

$$= \underline{i}\left(-5-(-4)\right) - \underline{j}\left(20-6\right) + \underline{k}\left(-8-(-3)\right)$$
$$= -\underline{i} - 14\underline{j} - 5\underline{k}$$

OR

$$\begin{matrix} 4 & & -1 & & 2 & & 4 \\ & \times & & \times & & \times & \\ 3 & & -2 & & 5 & & 3 \\ & ③ & & ① & & ② & \end{matrix} \qquad \begin{pmatrix} -5-(-4) \\ 6-20 \\ -8-(-3) \end{pmatrix} = \begin{pmatrix} -1 \\ -14 \\ -5 \end{pmatrix} = -\underline{i} - 14\underline{j} - 5\underline{k}$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -14 \\ -5 \end{pmatrix} = -1 + -28 + 15 = \underline{\underline{-14}}$$

[Omit TJ 3, p.4, Ex 1
but try TJ 3, p 9, Ex 2, Q 1 → 7]