Vectors : Equation of a Straight Line

At National 5 & Higher Maths, straight lines (in a plane) were specified by either

- one point and a gradient, or
- two points on the line.

For Advanced Higher Maths, straight lines in space are specified by

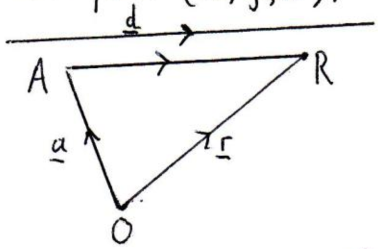
- one point and the direction vector of the line, or
- two points on the line, or
- the intersection of two planes (which will be looked at later).

There are 3 forms in which the equation of a line can be written in 3-D space:

- vector form (less common in SQA exams)
- parametric form
- symmetrical form (also known as Cartesian form).

Vector Form of a Straight Line

Let $\underline{d} \neq \underline{0}$ be a fixed (i.e. known) vector and let A (with position vector $\underline{a}$ from the the origin, O) be a fixed (known) point. Let R (with position vector $\underline{r}$) be a variable point (x, y, z).



$\overrightarrow{AR}$ is parallel to $\underline{d}$, so $\overrightarrow{AR} = t\underline{d}$ (i.e. $\overrightarrow{AR}$ is a multiple of $\underline{d}$) where t is a parameter (and t is a real number).

$$\overrightarrow{AR} = t\underline{d}$$

$$\underline{r} - \underline{a} = t\underline{d}$$

$$\underline{r} = \underline{a} + t\underline{d}$$

is the vector form of the equation of the line through A, parallel to $\underline{d}$ (and $\underline{d}$ is known as the direction vector of the line).

Q9 Find the vector form of the equation of the straight line through $(2, -1, 6)$ and parallel to the vector $\underline{i} + 2\underline{j} - 8\underline{k}$

Ans 9 $\underline{r} = \underline{a} + t\underline{d}$ so $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix}$

Parametric Form of a Straight Line

Let A be the point (a, b, c), $\underline{d}$ be the vector $l\underline{i} + m\underline{j} + n\underline{k}$ and R be the point (x, y, z), then (using the vector form of the straight line)

$$\underline{r} = \underline{a} + t\underline{d}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} tl \\ tm \\ tn \end{pmatrix} = \begin{pmatrix} a+tl \\ b+tm \\ c+tn \end{pmatrix}$$

and so

$$\left.\begin{aligned} x &= a + lt \\ y &= b + mt \\ z &= c + nt \end{aligned}\right\}$$

This is the parametric equation of the straight line passing through (a, b, c) with direction vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ — i.e. parallel to $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$.

Q10 Find the parametric equations of the line through $(3, 4, 5)$ and parallel to $6\underline{i} - 7\underline{j} - \underline{k}$.

Ans 10

$$\begin{aligned} x &= 3 + 6t \\ y &= 4 - 7t \\ z &= 5 - t \end{aligned}$$

Point lying on the line

The direction vector (multiplying the parameter, t)

Symmetrical (or Cartesian) Form of a Straight Line

If we rearrange the three parts of the parametric equation of a straight line, we get:

$$\left.\begin{aligned} x = a + lt &\Rightarrow lt = x - a &\Rightarrow t = \frac{x-a}{l} \\ y = b + mt &\Rightarrow mt = y - b &\Rightarrow t = \frac{y-b}{m} \\ z = c + nt &\Rightarrow nt = z - c &\Rightarrow t = \frac{z-c}{n} \end{aligned}\right\}$$

and since these are all equal to the same value, it follows that:

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

is the symmetrical (Cartesian) form of a straight line through (a, b, c) and parallel to $l\underline{i} + m\underline{j} + n\underline{k}$.

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$ – the Symmetrical/Cartesian form of a straight line passing through a point (a,b,c), parallel to $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ – is the 3-D equivalent to $y-b=m(x-a)$ being the equation of a straight line in 2-D, passing through a point (a,b) with gradient m.

Q11 Write down the symmetrical (or Cartesian) form of the straight line passing through $(1,-2,3)$ which is parallel to $4\underline{i}+5\underline{j}-6\underline{k}$.

Ans 11
$$\frac{x-1}{4} = \frac{y+2}{5} = \frac{z-3}{-6}$$

Sometimes, the direction vector will include a zero (e.g. $\underline{d} = \underline{i}+4\underline{k}$). If a straight line with that direction passed through $(7,8,9)$, the symmetrical equation would have become $\frac{x-7}{1} = \frac{y-8}{0} = \frac{z-9}{4}$, so to avoid the "dividing by zero" notation, we would use the parametric (or vector) form instead:

$$x = 7+t$$
$$y = 8$$
$$z = 9+4t$$

(However, TJ leaves answers with zero on the denominator and doesn't mind that it's not re-written in parametric form.)

Q12 The vector form of a straight line is $\underline{r} = 2\underline{i}-3\underline{j}+\underline{k}+t(-\underline{i}-2\underline{j}+\underline{k})$.
Write down:
(a) the parametric form
(b) the symmetrical form

Ans 12 The vector equation is of the form $\underline{r} = \underline{a}+t\underline{d}$, so the point that we know on this line is $(2,-3,1)$ and the direction vector is $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

(a)
$$x = 2-t$$
$$y = -3-2t$$
$$z = 1+t$$

(b)
$$\frac{x-2}{-1} = \frac{y+3}{-2} = \frac{z-1}{1}$$

Q13 The symmetrical form of a straight line is $\frac{x+4}{6} = \frac{y-2}{1} = \frac{z+5}{-3}$.

Write down:

(a) the parametric form

(b) the vector form.

Ans 13 From the given equation, we can see that one point on the line is $(-4, 2, -5)$ and the direction vector is $\begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix}$.

(a) $x = -4 + 6t$
$y = 2 + t$
$z = -5 - 3t$

(b) $\underline{r} = \begin{pmatrix} -4 \\ 2 \\ -5 \end{pmatrix} + t\begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix}$

OR $\underline{r} = -4\underline{i} + 2\underline{j} - 5\underline{k} + t(6\underline{i} + \underline{j} - 3\underline{k})$

Q14 Find the equation of a straight line joining $A(1, 0, 2)$ and $B(2, 1, 0)$.

Ans 14 In N5 Maths and Higher Maths, we would often have two points, but no given gradient, so the first step was to find the gradient before choosing one of the points to substitute into $y - b = m(x-a)$. This 3-D version is similar; firstly, find the direction vector, and then pick one of the points.
However, in 2-D there was a definite gradient value; in 3-D ANY MULTIPLE of the direction vector is acceptable. For instance, someone might go for $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ but another person may have used $\begin{pmatrix} -l \\ -m \\ -n \end{pmatrix}$. In this question, some people will use $\underline{d} = \overrightarrow{AB}$, but others could use $\underline{d} = \overrightarrow{BA}$. Just pick one and go for it!

$$\underline{d} = \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \qquad \text{whereas } \overrightarrow{BA} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

This question did not state which form was required, so simply pick one (although the parametric form is often the most useful had there been a follow-up question).

$x = 1 + t$, $y = t$, $z = 2 - 2t$ (using A and $\overrightarrow{AB}$)

OR $x = 2 + t$, $y = 1 + t$, $z = -2t$ (using B and $\overrightarrow{AB}$)

OR $x = 1 - t$, $y = -t$, $z = 2 + 2t$ (using A and $\overrightarrow{BA}$)

OR $x = 2 - t$, $y = 1 - t$, $z = 2t$ (using B and $\overrightarrow{BA}$)

SQA would accept any of these, but it does make it awkward when self-marking from a book.

If we had decided to use the symmetrical / Cartesian form:

$$\frac{x-1}{1} = \frac{y}{1} = \frac{z-2}{-2} \quad \text{OR} \quad \frac{x-2}{1} = \frac{y-1}{1} = \frac{z}{-2}$$

using A and $\overrightarrow{AB}$ — using B and $\overrightarrow{AB}$

$$\text{OR} \quad \frac{x-1}{-1} = \frac{y}{-1} = \frac{z-2}{2} \quad \text{OR} \quad \frac{x-2}{-1} = \frac{y-1}{-1} = \frac{z}{2}$$

using A and $\overrightarrow{BA}$ — using B and $\overrightarrow{BA}$

* (If your top expressions are the same as those in the answers, but your denominators are ALL the SAME multiple of the given answer, you're fine!)

If you had chosen the vector form, the answer could be any of:

$\underline{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + t\begin{pmatrix}1\\1\\-2\end{pmatrix}$ OR $\underline{r} = \underline{i} + 2\underline{k} + t(\underline{i} + \underline{j} - 2\underline{k})$ using A and $\overrightarrow{AB}$

$\underline{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\-2\end{pmatrix}$ OR $\underline{r} = 2\underline{i} + \underline{j} + t(\underline{i} + \underline{j} - 2\underline{k})$ using B and $\overrightarrow{AB}$

$\underline{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + t\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ OR $\underline{r} = \underline{i} + 2\underline{k} + t(-\underline{i} - \underline{j} + 2\underline{k})$ using A and $\overrightarrow{BA}$

$\underline{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ OR $\underline{r} = 2\underline{i} + \underline{j} + t(-\underline{i} - \underline{j} + 2\underline{k})$ using B and $\overrightarrow{BA}$

(For the SQA markers' sanity, the question is very likely to be more specific, or avoided.)

** If your top expressions are different from the given answer, but your denominators are the same as (or are multiples of) those in the answers, then take the coordinate that you didn't choose originally and substitute it into your answer and see if you get the same value all three times. For example, suppose you wrote

$\frac{x-2}{-1} = \frac{y-1}{-1} = \frac{z}{2}$ but the book had chosen $\frac{x-1}{1} = \frac{y}{1} = \frac{z-2}{-2}$.

Your denominator is the negative of theirs, so that's OK so far. You chose B, so take A(1,0,2) and substitute into your answer: $\frac{1-2}{-1} = \frac{0-1}{-1} = \frac{2}{2}$. All 3 fractions equal one so your answer is correct.

This check works for parametric too. Suppose you wrote

$$x = 1 - t \qquad\qquad\qquad\qquad x = 2 + t$$
$$y = -t \qquad \text{but the book chose} \qquad y = 1 + t$$
$$z = 2 + 2t \qquad\qquad\qquad\qquad z = -2t$$

The coefficients of t are all the negatives of the other, but that's fine.
You had chosen $A(1,0,2)$ but the book chose $B(2,1,0)$.
Substitute the other point, in this case B, into your answer:

$$\left.\begin{aligned} 2 &= 1 - t && \Rightarrow t = -1 \\ 1 &= -t && \Rightarrow t = -1 \\ 0 &= 2 + 2t && \Rightarrow 2t = -2 \Rightarrow t = -1 \end{aligned}\right\} \text{Same value for } t\text{, so your answer was correct.}$$

[Now try TJ3, p 12, Ex 3.]

The Intersection of Two Straight Lines (in 3-D space)

Two lines on a 2-D plane are either parallel or they intersect (and if they do meet, we use simultaneous equations to find the point of intersection). However, for two lines in 3-D space, there are three possibilities: parallel, intersecting or skew. (Lines are skew when they are not parallel, but fail to intersect when extended.)

Q15 For the following pairs of straight lines, establish whether the lines are parallel, intersecting or skew and, if intersecting, find the coordinates of the point of intersection.

(a) $\dfrac{x+9}{4} = \dfrac{y+5}{1} = \dfrac{z+1}{-2}$ and $\dfrac{x-8}{-5} = \dfrac{y-2}{-4} = \dfrac{z+15}{8}$

(b) $\dfrac{x+9}{4} = \dfrac{y+5}{1} = \dfrac{z+1}{-2}$ and $\dfrac{x-8}{-5} = \dfrac{y-2}{-4} = \dfrac{z-5}{8}$

Ans 15 (a) $\dfrac{x+9}{4} = \dfrac{y+5}{1} = \dfrac{z+1}{-2}$ has parametric form

point on line.

$$\begin{aligned} x &= -9 + 4t \quad &\text{------(1)} \\ y &= -5 + t \quad &\text{------(2)} \\ z &= -1 - 2t \quad &\text{------(3)} \end{aligned}$$

direction

since the symmetrical form shows that $(-9,-5,-1)$ lies on the line and the direction vector is $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

Similarly, $\dfrac{x-8}{-5} = \dfrac{y-2}{-4} = \dfrac{z+15}{8}$ passes through $(8,2,-15)$ and

is parallel to $\begin{pmatrix}-5\\-4\\8\end{pmatrix}$ so its parametric form is

$$x = 8 - 5\lambda \text{ ...(4)}$$
$$y = 2 - 4\lambda \text{ ...(5)}$$
$$z = -15 + 8\lambda \text{ ...(6)}$$

Couldn't use "t" again in the same question.

The two lines have direction vectors which are not multiples of each other

i.e. $\begin{pmatrix}-5\\-4\\8\end{pmatrix}$ is not a multiple of $\begin{pmatrix}4\\1\\-2\end{pmatrix}$

so the lines are not parallel. (Therefore they either intersect or are skew.)

If they intersect,

$-9 + 4t = 8 - 5\lambda$ (from (1) and (4)) and $-5 + t = 2 - 4\lambda$ (from (2) & (5))

$4t + 5\lambda = 17$ ....(7) $\qquad t + 4\lambda = 7$ ...(8)

$$4t + 5\lambda = 17 \text{ ...(7)}$$
$$4t + 16\lambda = 28 \text{ ...(8)}\times 4 \text{ ...(9)}$$
$$(9)-(7) \quad 11\lambda = 11$$
$$\lambda = 1$$

Sub. $\lambda = 1$ into (8):

$$t + 4(1) = 7$$
$$t + 4 = 7$$
$$t = 3$$

However, we MUST now check that these values of the parameters satisfy the two remaining equations — in this case, (3) and (6).

(3): $z = -1 - 2t$, where $t = 3$

$z = -1 - 2(3)$

$z = -7$

(6): $z = -15 + 8\lambda$, where $\lambda = 1$

$z = -15 + 8(1)$

$z = -7$

$\lambda = 1$ and $t = 3$ satisfy equations (6) and (3), so the lines DO intersect!

Either substitute $t = 3$ into (1) and (2) — we already know (3) — or sub. $\lambda = 1$ into (4) and (5) — we already know about (6).

(1): $x = -9 + 4(3) = 3$

(2): $y = -5 + 3 = -2$

(3): $z = -7$

OR

(4): $x = 8 - 5(1) = 3$

(5): $y = 2 - 4(1) = -2$

(6): $z = -7$

so lines intersect at $(3,-2,-7)$

(b) From (a), first line has parametric form $x = -9 + 4t$ ---(1)
$y = -5 + t$ ---(2)
$z = -1 - 2t$ ---(3)

Similarly, the other line has parametric form:
$x = 8 - 5\mu$ ---(10)
$y = 2 - 4\mu$ ---(11)
$z = 5 + 8\mu$ ---(12)

Same direction vectors as part (a), so <u>not</u> parallel since <u>not</u> multiples of each other.

If they intersect,

$-9 + 4t = 8 - 5\mu$ (using (1) & (10)) and $-5 + t = 2 - 4\mu$

$4t + 5\mu = 17$ ---(13) $\qquad t + 4\mu = 7$ ---(14)

Simultaneous equations using (13) and (14) leads to

$t = 3$ and $\mu = 1$

Again, we MUST check that these values of the parameters satisfy the two remaining equations – in this case (3) and (12).

(3): $z = -1 - 2t$, where $t = 3$
$z = -1 - 2(3)$
$z = -7$

(12): $z = 5 + 8\mu$, where $\mu = 1$
$z = 5 + 8(1)$
$z = 13$

We have different values of z at a possible point of intersection, so this means that the second pair of lines do <u>not</u> intersect (despite the fact that they are not parallel).

Therefore, the lines $\dfrac{x+9}{4} = \dfrac{y+5}{1} = \dfrac{z+1}{-2}$ and $\dfrac{x-8}{-5} = \dfrac{y-2}{-4} = \dfrac{z-5}{8}$

are skew!

Notice that parts (a) and (b) were almost identical – the only difference between those lines intersecting or being skew (having easily ruled out parallel) was whether the final check worked or not!

Q15 could have been given in terms of vector equations rather than as symmetrical / Cartesian equations, but it is still recommended to convert them into parametric equations of lines before proceeding as before.
Part (a) could have been phrased as:

"Lines L_1 and L_2 have vector equations

$$\underline{r} = \begin{pmatrix} -9 \\ -5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \underline{r} = \begin{pmatrix} 8 \\ 2 \\ -15 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ -4 \\ 8 \end{pmatrix}.$$

Establish whether the lines are parallel, intersecting or skew and, if intersecting, find the position vector of the point of intersection."

The solution would be the same as before since equations (1), (2) and (3) should be obvious from the first vector equation, and equations (4), (5) and (6) should be clear from the second vector equation.
However, notice that the final sentence requires a position vector, in which case the solution would be $\begin{pmatrix} 3 \\ -2 \\ -7 \end{pmatrix}$ or $3\underline{i} - 2\underline{j} - 7\underline{k}$ rather than $(3, -2, -7)$.

On the other hand, Q15(a) could have been given in terms of the following types of vector equations:

"Lines L_1 and L_2 have vector equations:

$\underline{r} = -9\underline{i} - 5\underline{j} - \underline{k} + \lambda(4\underline{i} + \underline{j} - 2\underline{k})$ and

$\underline{r} = 8\underline{i} + 2\underline{j} - 15\underline{k} + \mu(-5\underline{i} - 4\underline{j} + 8\underline{k})$, etc."

The UPM book uses (in its question 15) the phrase "non-parallel coplanar lines" rather than saying simply "intersecting lines".

[Now try TJ3, p22, Ex5, Q4

and

UPM, p.420 & 421, Ex17C, Q14, 13 and then 12
(in that order)]