Vectors : Equation of a Plane

A plane can be written in Cartesian form, parametric form and in vector form, but there are two types of vector form depending on where you are reading about it — there is the "parametric vector form" (not to be confused with the "parametric form") and there is the "scalar product vector form".

Cartesian Equation of a Plane (Most widely used)

The Cartesian equation of a plane is $ax + by + cz = k$ where $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underline{n}$ is the vector representing a normal to the plane.

A normal is a vector which is perpendicular to the plane, and is therefore perpendicular to every vector that lies on the plane.

Q16 Find the Cartesian equation of the plane π with normal vector $\underline{n} = 2\underline{i} - \underline{j} + \underline{k}$ and containing the point $A(2,3,7)$.

Ans16 [Annoyingly, planes are sometimes labelled as π — it's not $3 \cdot 14\ldots$ or $180°$ — it is just a weird choice of (Greek) letter which has been chosen.]

$\underline{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ so plane is of the form $2x - y + z = k$. But $A(2,3,7)$ lies on plane:

$$2(2) - 3 + 7 = k$$
$$k = 8$$

Cartesian equation of the plane is $2x - y + z = 8$.

Q17 $A(2,3,-1)$, $B(3,1,1)$ and $C(5,-2,3)$ lie on the same plane. Find the Cartesian equation of the plane.

Ans17 Unlike Q16, we have not been given the direction of the normal. Find two vectors which have a common point, and use these to create the normal (and then proceed as in Q16). Let's choose A as the common point.

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$$

The normal to the plane is perpendicular to the plane, so the normal is perpendicular to <u>both</u> $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Previously, we found out that the Vector (or Cross) Product of two vectors produces a vector which is perpendicular to both vectors. Therefore, the normal is (a multiple of) the vector product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\underline{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 2 \\ 3 & -5 & 4 \end{vmatrix}$$

$$\underline{n} = \underline{i}\left(-8-(-10)\right) - \underline{j}\left(4-6\right) + \underline{k}\left(-5-(-6)\right)$$

$$\underline{n} = 2\underline{i} + 2\underline{j} + \underline{k}$$

[OR

$$\begin{matrix} 1 & & -2 & & 2 & & 1 \\ & \times & & \times & & \times & \\ 3 & & -5 & & 4 & & 3 \end{matrix} \qquad \begin{pmatrix} -8-(-10) \\ 6-4 \\ -5-(-6) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \underline{n}$$
]

$\therefore$ Plane is of the form $2x + 2y + z = k$.

But $A(2,3,-1)$ lies on this plane (<u>or</u> use B <u>or</u> use C – all give same k)

$$2(2) + 2(3) + (-1) = k$$
$$k = 9$$

$\therefore$ Plane has equation $\underline{\underline{2x + 2y + z = 9}}$

<u>Q18</u> Find the Cartesian equation of the plane through $(-1,2,3)$ which contains the vectors $8\underline{i} + 5\underline{j} + \underline{k}$ and $-4\underline{i} + 5\underline{j} + 7\underline{k}$

<u>Ans 18</u> This is like Q17 but involves slightly less working since we do not have to create two vectors. The <u>normal is perpendicular to every vector lying on the plane</u>, so it must be perpendicular to these two.

Let $\underline{u} = 8\underline{i} + 5\underline{j} + \underline{k}$ and $\underline{v} = -4\underline{i} + 5\underline{j} + 7\underline{k}$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 8 & 5 & 1 \\ -4 & 5 & 7 \end{vmatrix} = \underline{i}\left(35-5\right) - \underline{j}\left(56-(-4)\right) + \underline{k}\left(40-(-20)\right)$$

$$= 30\underline{i} - 60\underline{j} + 60\underline{k}$$

(or use shortcut)

$$= \begin{pmatrix} 30 \\ -60 \\ 60 \end{pmatrix} = 30\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$ We could use this as the normal, but it is easier to use the simplest form rather than a multiple of it.

$$\therefore \underline{n} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$\therefore$ Plane is of the form $x - 2y + 2z = k$.

But $(-1, 2, 3)$ lies on the plane:

$$-1 - 2(2) + 2(3) = k$$
$$-1 - 4 + 6 = k$$
$$k = 1$$

$\therefore$ Plane has equation $\underline{\underline{x - 2y + 2z = 1}}$

Q19 Find the Cartesian equation of the plane containing the point $P(3, -2, -7)$ and the line $\dfrac{x-5}{3} = \dfrac{y}{1} = \dfrac{z+6}{4}$.

Ans 19 The line has direction vector $\underline{d} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. The point $(5, 0, -6)$ lies on the line (and since the line lies on the plane, that point lies on the plane too). Let's call $(5, 0, -6)$ point A.

$$\vec{PA} = \underline{a} - \underline{p} = \begin{pmatrix} 5 \\ 0 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

It is now like Q18 – we know two vectors lying on the plane, and although we know 2 points, we need only one.

$$\underline{n} = \vec{PA} \times \underline{d} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = \underline{i}(8-1) - \underline{j}(8-3) + \underline{k}(2-6)$$
$$= 7\underline{i} - 5\underline{j} - 4\underline{k}$$

[OR
$$\begin{matrix} 2 & 2 & 1 & 2 \\ 3 & 1 & 4 & 3 \end{matrix} \qquad \begin{pmatrix} 8-1 \\ 3-8 \\ 2-6 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ -4 \end{pmatrix} = \underline{n}$$
(3) (1) (2)]

$\therefore$ Plane is of the form of $7x - 5y - 4z = k$

But $P(3, -2, -7)$ lies on plane, so $7(3) - 5(-2) - 4(-7) = k$

[or use $A(5, 0, -6)$]

$$k = 59$$

$\therefore$ Plane has equation $\underline{7x - 5y - 4z = 59}$

Other things that you might come across (but won't need to worry about in terms of SQA exams) include the following.

- If a normal is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then its components are called "direction ratios".
- If the normal happens to be a <u>unit</u> vector, its components are called "direction <u>cosines</u>".
- If you are given three points lying on a plane and are trying to find the Cartesian equation of the plane but end up getting the normal to be $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, it means that the three points must have been collinear and an infinite number of planes pass through the three points.

We will now look at three more forms of equations of planes, starting with the two types of "vector forms": the scalar product vector form and then the parametric vector form.
It is highly likely that the scalar product vector form will not come up in SQA exams, and that in the unlikely event that the parametric vector form is required, it will probably be referred to as simply the vector form (to avoid confusion with the parametric form).

Scalar Product Vector Form of a Plane (rare)

If point A on the plane has position vector $\underset{\sim}{a}$ and if $R(x, y, z)$ with position vector $\underset{\sim}{r}$ also lies on the plane, and if the normal of the plane is $\underset{\sim}{n}$, then

$$\boxed{\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}}$$

and if $\underset{\sim}{a} \cdot \underset{\sim}{n} = k$, then

$$\boxed{\underset{\sim}{r} \cdot \underset{\sim}{n} = k}$$

Q20 Find the scalar product vector equation of the plane Π with normal vector $\underline{n} = 2\underline{i} - \underline{j} + \underline{k}$ which contains $A(2,3,7)$.

Ans 20 $\underline{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\underline{a} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ so $\underline{a}.\underline{n} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\Rightarrow \underline{a}.\underline{n} = (2)(2) + (3)(-1) + (7)(1) = 8$

$$\therefore \quad \underline{r}.(2\underline{i} - \underline{j} + \underline{k}) = 8$$

[or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 8$ which leads to the Cartesian form: $2x - y + z = 8$ as before]

III. Parametric Vector Form of a Plane

To write the equation of a plane in parametric vector form, we need to know a point on the plane and the directions of two (non-zero) vectors which lie on the plane.
Alternatively, if we know 3 points that all lie on the plane, we can create **two** vectors that lie on the plane and proceed as before.
If points A, B and C lie on the plane and have position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively, then the (parametric) vector form of the plane can be written as:

$$\underline{r} = \underline{a} + \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$$

or $$\underline{r} = \underline{b} + \lambda \overrightarrow{BA} + \mu \overrightarrow{BC}$$

or $$\underline{r} = \underline{c} + \lambda \overrightarrow{CA} + \mu \overrightarrow{CB}$$

If a question says "find the vector form of a plane" assume that the question wants this form and <u>not</u> the scalar product vector form.

Q21 Let A be the point $(2,3,-1)$, B be $(3,1,1)$ and $C(5,-2,3)$. If all three points lie on a plane, find the (parametric) <u>vector equation</u> of the plane.

Ans 21 $\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix}3\\1\\1\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$

$\vec{AC} = \underline{c} - \underline{a} = \begin{pmatrix}5\\-2\\3\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}3\\-5\\4\end{pmatrix}$

$\Rightarrow$ (parametric) vector equation of the plane is

$$\underline{r} = 2\underline{i} + 3\underline{j} - \underline{k} + \lambda(\underline{i} - 2\underline{j} + 2\underline{k}) + \mu(3\underline{i} - 5\underline{j} + 4\underline{k})$$

$$\left[\text{or } \underline{r} = \begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\2\end{pmatrix} + \mu\begin{pmatrix}3\\-5\\4\end{pmatrix}\right]$$

although other forms of answers are possible if using the alternative versions at the foot of the previous page.

Q22 Find the (parametric) <u>vector equation</u> of the plane π with normal vector $\underline{n} = 2\underline{i} - \underline{j} + \underline{k}$ which contains $A(2,3,7)$. [Same data as Q16]

Ans 22 The parametric vector equation does not rely on the normal $\underline{n}$ but we appear to know only one point, A, lying on the plane. However, we could repeat Q 16 to find the Cartesian equation of the plane: $2x - y + z = 8$ and then find two more points that satisfy this equation; for example $B(0,3,11)$ and $C(2,-1,3)$ — simply choose an x-value and use commonsense to find a y-value and z-value which will lead to the value on the right hand side (RHS).
Now proceed as in Q21 above.

(P.T.O.)

$A(2,3,7)$, $B(0,3,11)$, $C(2,-1,3)$

$$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix}0\\3\\11\end{pmatrix} - \begin{pmatrix}2\\3\\7\end{pmatrix} = \begin{pmatrix}-2\\0\\4\end{pmatrix}$$

$$\vec{AC} = \underline{c} - \underline{a} = \begin{pmatrix}2\\-1\\3\end{pmatrix} - \begin{pmatrix}2\\3\\7\end{pmatrix} = \begin{pmatrix}0\\-4\\-4\end{pmatrix}$$

$$\Rightarrow \quad \underline{r} = \begin{pmatrix}2\\3\\7\end{pmatrix} + \lambda\begin{pmatrix}-2\\0\\4\end{pmatrix} + \mu\begin{pmatrix}0\\-4\\4\end{pmatrix}$$

i.e. $\underline{r} = 2\underline{i} + 3\underline{j} + 7\underline{k} + \lambda(-2\underline{i} + 4\underline{k}) + \mu(-4\underline{j} + 4\underline{k})$

(although this answer is not unique).

IV. Parametric Form of a Plane

The parametric equation of a plane is similar to parametric forms of straight lines but have one extra parameter.

If a plane contains a point $A(a,b,c)$ and has two (non-zero) vectors (non-parallel) lying on the plane with directions $\begin{pmatrix}e\\f\\g\end{pmatrix}$ and $\begin{pmatrix}h\\m\\n\end{pmatrix}$ then the parametric equation of the plane is:

$$\begin{aligned} x &= a + \lambda e + \mu h \\ y &= b + \lambda f + \mu m \\ z &= c + \lambda g + \mu n \end{aligned}$$

Q23 Find the parametric equation of the plane passing through the points $A(2,3,-1)$, $B(3,1,1)$ and $C(5,-2,3)$. [see Q21]

Ans 23 $\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix}3\\1\\1\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$ and $\vec{AC} = \underline{c} - \underline{a} = \begin{pmatrix}5\\-2\\3\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}3\\-5\\4\end{pmatrix}$

$$\Rightarrow \quad \left.\begin{aligned} x &= 2 + \lambda + 3\mu \\ y &= 3 - 2\lambda - 5\mu \\ z &= -1 + 2\lambda + 4\mu \end{aligned}\right\}$$ is the parametric equation of the plane.

(using A – we could use B with $\vec{BA}$ & $\vec{BC}$ or use C with $\vec{CA}$ & $\vec{CB}$)

Q24 Find the parametric equation of a plane through $(2,3,-1)$ and parallel to vectors $-2\underline{i} - 3\underline{j} + \underline{k}$ and $-\underline{i} + 3\underline{j} + 4\underline{k}$.

Ans 24 It makes no difference if the question says that the plane "is parallel to the vectors" or if it says the plane "contains the vectors" (because the vectors could be moved parallel to their original positions onto the plane itself).

The vector form would have been $\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v}$

$$\underline{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$$

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$$

We could skip this and go straight to the required form.

so the parametric equation is:

$$x = 2 - 2\lambda - \mu$$
$$y = 3 - 3\lambda + 3\mu$$
$$z = -1 + \lambda + 4\mu$$

Point lying on plane

One of the vectors on or parallel to the plane

The other vector on or parallel to the plane.

[Now try TJ 3, pages 15 & 16, Ex4 which concentrates solely on the Cartesian form of a plane.]

(UPM, pages 429 and 430, Ex17D, Q1→12 features the other forms of planes.)