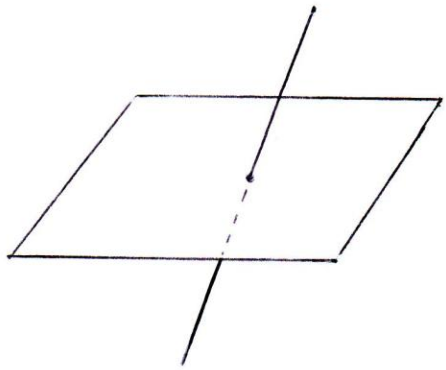
Vectors: The Intersection of Two Lines — see Q15 from "Vectors: Equation of a Straight Line"

Vectors: The Intersection of a Line and a Plane

There are 3 possibilities:

(i) The line intersects the plane at a single point.



(ii) If we end up with an infinite number of solutions, then the line lies ON the plane.
(Show that the line is parallel to the plane by showing that the plane's normal and the line are perpendicular, and then show that the point on the line satisfies the equation of the plane.)

(iii) If there is no solution, then the line and the plane are parallel.

Q29 Find the point of intersection (if any) between the line

$$\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{2} \quad \text{and the plane } x-y-2z=-15.$$

Ans 29 To find where a line intersects a plane, we use the PARAMETRIC equation of a line (as we did for the intersection of two lines — see Q15).

The parametric equation of the line passing through (1, 2, 3) with direction vector $\begin{pmatrix}1\\1\\2\end{pmatrix}$ is:

$$x = 1 + t \quad \text{---} (1)$$
$$y = 2 + t \quad \text{---} (2)$$
$$z = 3 + 2t \quad \text{---} (3)$$

The plane has equation $x - y - 2z = -15$ ......(4)

Substitute (1), (2) & (3) into (4):

$$(1+t) - (2+t) - 2(3+2t) = -15$$

$$1 + t - 2 - t - 6 - 4t = -15$$

$$-7 - 4t = -15$$

$$8 = 4t$$

$$t = 2$$

Substitute $t = 2$ back into (1), (2) & (3):

$$x = 1 + 2 = 3$$

$$y = 2 + 2 = 4$$

$$z = 3 + 2(2) = 7$$

The point of intersection is $\underline{\underline{(3,4,7)}}$.

Q30 Show that the line $\frac{x-3}{1} = \frac{y-3}{1} = \frac{z+2}{-1}$ lies <u>on</u> the plane with equation $3x - 2y + z = 1$.

Ans 30 The line has direction vector $\underline{d} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and the plane's normal is $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \underline{n}$.

$$\underline{d} \cdot \underline{n} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 3 + -2 + -1 = 0 \Rightarrow \underline{d} \text{ and } \underline{n} \text{ are } \underline{\text{perpendicular}}$$

$\Rightarrow$ the line is $\underline{\text{parallel}}$ to the plane.

The point $(3, 3, -2)$ lies on the line. Substitute it into the equation of the plane:

$$3(3) - 2(3) + (-2) = 1$$

$$9 - 6 - 2 = 1$$

$$3 - 2 = 1$$

$$1 = 1$$

- The point satisfies the equation of the plane, so the point lies <u>on</u> the plane.
- The line is parallel to the plane and yet it has a point lying on the plane $\Rightarrow$ line must lie <u>on</u> the plane.

Vectors: The Intersection of Two Planes

If two planes have normals $\underline{n}_1$ and $\underline{n}_2$, and if one normal is a multiple of the other (or if $\underline{n}_1 = \underline{n}_2$) then the two planes are parallel.

However, if two planes are non-parallel, they intersect in a straight line.

There are two different methods – use whichever one you prefer!

Unfortunately, the answers are <u>not</u> unique (regardless of which method you choose) so it can be awkward to tell if your answer is right or wrong when checking the answer from a book (but in an exam, the marker will recognise what is right and what is not). Initially, we will use two methods and show that they give identical answers, but then we will acknowledge that other possibilities exist.

Q31 Find the Cartesian equation of the line of intersection of the planes $3x-5y+z=8$ & $2x-3y+z=3$.

Ans31 (Method 1) [If stuck at a question involving planes, finding the normal of a plane is usually a good idea!]

Knowing that a <u>line needs a direction vector and a point</u> lying on it, we firstly use the logic that the <u>line of intersection</u> of 2 planes will be <u>perpendicular</u> to <u>both</u> <u>normals</u>.

$3x-5y+z=8$ has normal $\underline{n}_1 = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$ and $2x-3y+z=3$ has normal $\underline{n}_2 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

Direction vector of line is perpendicular to <u>both</u> $\underline{n}_1$ and $\underline{n}_2$:

$$\underline{n}_1 \times \underline{n}_2 \quad \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -5 & 1 \\ 2 & -3 & 1 \end{vmatrix} = \underline{i}(-5+3) - \underline{j}(3-2) + \underline{k}(-9+10)$$

$$= -2\underline{i} - \underline{j} + \underline{k} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \text{ or } -1\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

or $\begin{matrix} 3 & -5 & 1 & 3 \\ 2 & -3 & 1 & 2 \end{matrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

fewer negatives.

Take $\underline{d} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, although could use $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

Now we need any point on the line of intersection and there are an infinite number of possibilities (which makes checking answers awkward). It is common to choose the value zero for x, y or z — usually z — but to show that methods 1 & 2 can give identical answers, let's choose $y=0$ for now (and return to $z=0$ later) and find the corresponding values of x and z.

One plane is $3x - 5y + z = 8$ ------(1)
and other is $2x - 3y + z = 3$ -------(2)

Let $y=0$: (1) $\Rightarrow$ $3x + z = 8$ -----(3)
(2) $\Rightarrow$ $2x + z = 3$ ------(4)

(3) − (4): $x = 5$. Sub. $x=5$ into (4): $2(5) + z = 3$
$10 + z = 3 \Rightarrow z = -7$.

$\therefore (5, 0, -7)$ lies on the line with direction vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

so its Cartesian equation is $\dfrac{x-5}{2} = \dfrac{y}{1} = \dfrac{z+7}{-1}$ --------($*$)

(or $\dfrac{x-5}{-2} = \dfrac{y}{-1} = \dfrac{z+7}{1}$)

[Had we picked $z=0$ instead, we would do the following:

(1) $\Rightarrow$ $3x - 5y = 8$ -----(5) $\rightarrow$ $9x - 15y = 24$ ---(5)×3 ---(7)
(2) $\Rightarrow$ $2x - 3y = 3$ ---(6) $\quad$ $-10x + 15y = -15$ ---(6)×−5 ---(8)

(7) + (8): $-x = 9 \Rightarrow x = -9$.

Sub. $x = -9$ into (5): $3(-9) - 5y = 8$
$-27 - 8 = 5y$
$-35 = 5y$
$y = -7$.

$(-9, -7, 0)$ is another point that lies on the line of intersection which has direction vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ so the Cartesian equation could have been

given as $\dfrac{x+9}{2} = \dfrac{y+7}{1} = \dfrac{z}{-1}$ (which looks different to ($*$) but it's the <u>same</u> line)

or even $\dfrac{x+9}{-2} = \dfrac{y+7}{-1} = \dfrac{z}{1}$.]

[In an exam, it will not matter which one you chose, but to save irritation when working from a book, have a quick peek at the answer before you start to see which letter was chosen to be zero, and then start the question, choosing that same value!]

Ans31 (Method 2)

$$3x - 5y + z = 8 \quad ---- (1)$$
$$2x - 3y + z = 3 \quad --- (2)$$

Eliminate a letter (whichever looks easiest – in this question, z looks easy to eliminate).

(1)−(2): $x - 2y = 5$ ....(9) (Notice for y: $-5-(-3) = -5+3 = -2$)

There are an infinite number of values of x and y which satisfy equation (9), so choose a value (parameter) for y and solve for x (or vice versa).

Let $y = t$ Sub. $y = t$ into (9):

$x - 2t = 5 \Rightarrow x = 2t + 5$.

Sub. $x = 2t+5$ and $y = t$ into (1) or (2).

(2): $2(2t+5) - 3t + z = 3$

$4t + 10 - 3t + z = 3$

$t + 10 + z = 3$

$z = -7 - t$

Therefore, the line of intersection has parametric equation

$$x = 2t+5 \qquad\qquad x = 5+2t$$
$$y = t \qquad \text{i.e.} \qquad y = t$$
$$z = -7-t \qquad\qquad z = -7-t$$

so the Cartesian equation of the line is $\dfrac{x-5}{2} = \dfrac{y}{1} = \dfrac{z+7}{-1}$

(which is equation (✱) from Method 1)

[Notice that choosing $y = 0$ in method 1 gave the same answer as choosing $y = t$ in method 2. Had we eliminated x or y instead in method 2 and later let $z = t$, it would have given the answer that we wrote in red in method 1 when $z = 0$.]

Vectors: The Intersection of 3 Planes

When there are 3 planes, there are 3 possibilities:

(i) The 3 planes can intersect at a single point — a unique solution!

Q32 Three planes have equations

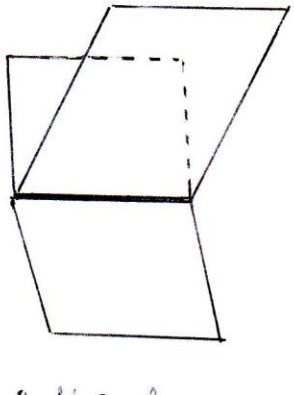
$$x + 2y + z = 8$$
$$3x + y - 2z = -1$$
$$x + 5y - z = 8.$$

Find the point of intersection of where the 3 planes meet.

Ans 32 Use Gaussian Elimination! This is a reworded version of Q2 in our earlier notes "Introduction to Matrices / Solving Systems of Linear Equations (Gaussian Elimination)", so please look back to see the answer/method.

Point of intersection $= (1, 2, 3)$

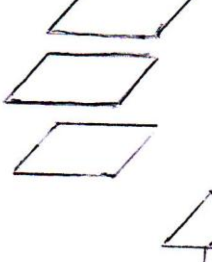
(ii) All 3 planes can intersect at a line (i.e. there is an infinite set of points common to all three planes).



(See Q33)

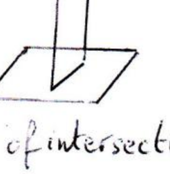
(iii) There is no intersection of all 3 planes.

- 3 planes are parallel



(See Q34)

- 2 planes are parallel



- One plane is parallel to the line of intersection of the other two.

Q33 Show that the planes

$$A: \ 2x - y + 5z = -4$$
$$B: \ 3x - y + 2z = -1$$
$$C: \ 4x - y - z = 2$$

intersect on a line and find its equation.

Ans33 [If we had used Gaussian Elimination here, we would end up with a row of zeros, indicating that we have a redundant equation and consequently an infinite number of solutions, so the answer is a line.]

$$\underline{n}_A = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \quad \underline{n}_B = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \text{ and } \underline{n}_C = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$$

The line where A and B intersect is parallel to $\underline{n}_A \times \underline{n}_B = \underline{d}$

(see method 1 of Q).

$$\underline{d} = \underline{n}_A \times \underline{n}_B = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 5 \\ 3 & -1 & 2 \end{vmatrix} = \underline{i}(-2+5) - \underline{j}(4-15) + \underline{k}(-2+3)$$
$$= 3\underline{i} + 11\underline{j} + \underline{k}$$

OR

$$\begin{matrix} 2 & & -1 & & 5 & & 2 \\ 3 & & -1 & & 2 & & 3 \\ & ③ & & ① & & ② & \end{matrix} \qquad \begin{pmatrix} -2+5 \\ 15-4 \\ -2+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 1 \end{pmatrix}$$

We have still to show:

- that the normal ($\underline{n}_c$) of the third plane (C) is perpendicular to $\underline{d}$, and
- that there is a coordinate that lies on the line of intersection of planes A and B that also lies on plane C.

$$\underline{d} \cdot \underline{n}_c = \begin{pmatrix} 3 \\ 11 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} = 12 + -11 + -1 = 0$$

$\Rightarrow$ $\underline{d}$ and $\underline{n}_c$ are perpendicular (as required)

Make a (random) decision and set x, y or z equal to zero (as in method 1 of Q31).

Let $z = 0$: A becomes $2x - y = -4$ ....(1)
B becomes $3x - y = -1$ .....(2)

(2) − (1): $x = 3$ Sub. $x = 3$ into (1):

$$2(3) - y = -4 \Rightarrow 6 + 4 = y \Rightarrow y = 10$$

$(3, 10, 0)$ lies on the line of intersection of planes A and B.

However, we have still to establish that $(3,10,0)$ also lies on plane C (to avoid it possibly being like the situation described in the last statement two pages back).

$$C: \ 4x - y - z = 2$$

$$\begin{aligned} 4(3) - 10 - 0 &= 2 \\ 12 - 10 - 0 &= 2 \\ 2 - 0 &= 2 \\ 2 &= 2 \end{aligned}$$

which confirms that $(3,10,0)$ lies on plane C in addition to lying on the line of intersection of planes A and B

$\Rightarrow$ Planes A, B and C do intersect on a line

This question simply asked for "its equation", so possibilities for the equation of the line of intersection are:

$$\frac{x-3}{3} = \frac{y-10}{11} = \frac{z}{1} \quad \text{OR} \quad \begin{aligned} x &= 3 + 3t \\ y &= 10 + 11t \\ z &= t \end{aligned} \quad \text{OR} \quad \underline{r} = \begin{pmatrix} 3 \\ 10 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 11 \\ 1 \end{pmatrix}$$

$$\text{OR} \quad \underline{r} = 3\underline{i} + 10\underline{j} + t(3\underline{i} + 11\underline{j} + \underline{k})$$

Remember that had we chosen $x=0$ or $y=0$ instead, the answer would appear to be very different (so have a peek at the answer before starting to see if the book used $x=0$, $y=0$ or $z=0$).

Q 34 Show that the planes
$$\begin{aligned} x + 2y + 2z &= 11 \\ 2x - y + z &= 8 \\ 3x + y + 3z &= 18 \end{aligned}$$
do not intersect (at a point or on a line).

Ans 34 This is a reworded version of Q4 from our previous notes on "Introduction to Matrices / Solving Systems of Linear Equations (Gaussian Elimination)"

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 2 & -1 & 1 & 8 \\ 3 & 1 & 3 & 18 \end{array}\right) \text{ became } \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & 5 & 3 & 14 \\ 0 & 0 & 0 & -1 \end{array}\right)$$ after various elementary row operations (E.R.O.s) were performed.

The third row shows inconsistency and there are no solutions, so the planes do not intersect.

(Two pages back, we saw three possible cases for non-intersection of 3 planes, but we do not need to comment on which of these three scenarios applies here.)

[Now try TJ 3, pages 22 and 23, Ex 5, Q5 → 12
and all of the Vector past paper questions from pbworks.]