Vectors: Calculating the Angle Between Two Lines in 3-D

The angle between 2 lines is the angle between their direction vectors

and can be found using the <u>scalar</u> product (even if the lines are skew).
We take the angle to be <u>acute</u>, so if it turns out to be obtuse, subtract it from 180° (i.e. find the supplement of the angle). We don't worry about whether the vectors are tail-to-tail any more.

<u>Q25</u> Find the acute angle between the lines $\frac{x+4}{2} = \frac{y-5}{4} = \frac{z-3}{1}$

and $\underline{r} = 3\underline{j} + 2\underline{k} + \lambda(-\underline{i} - \underline{j} + \underline{k})$.

<u>Ans 25</u> $\underline{d}_1 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ and $\underline{d}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ $\qquad \cos\theta = \dfrac{\underline{d}_1 \cdot \underline{d}_2}{|\underline{d}_1||\underline{d}_2|}$

$$\underline{d}_1 \cdot \underline{d}_2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -2 - 4 + 1 = -5$$

$$|\underline{d}_1| = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{4+16+1} = \sqrt{21}$$

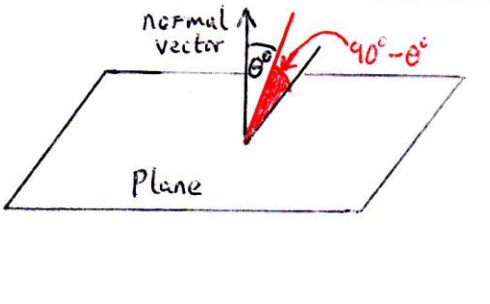
$$|\underline{d}_2| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$\cos\theta = \frac{-5}{\sqrt{21}\sqrt{3}} = \frac{-5}{\sqrt{63}} \Rightarrow \theta = 129.0°$$

<u>Acute</u> angle between the lines is $180° - 129.0° = \underline{\underline{51.0°}}$

Calculating the Angle Between a Line and a Plane

If a line and a plane intersect, the angle between them is the <u>complement</u>* of the angle between the line and the normal to the plane. In other words, find the <u>acute</u> angle between the <u>direction vector of the line</u> and the <u>normal</u> to the plane, and then subtract your answer from 90°.



* Complementary angles add up to 90°

Q26 Find the angle between the line

$$x = 1 - 4t$$
$$y = 2 - 3t$$
$$z = -3 - 2t$$

and the plane $10x - 2y + z = -1$.

Ans 26 The direction vector $\underline{d}$ of the line is $\begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}$ and the normal vector for the plane is $\begin{pmatrix} 10 \\ -2 \\ 1 \end{pmatrix} = \underline{n}$.

$$\cos\theta = \frac{\underline{d} \cdot \underline{n}}{|\underline{d}||\underline{n}|} \qquad \underline{d} \cdot \underline{n} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -2 \\ 1 \end{pmatrix} = -40 + 6 + -2 = -36$$

$$|\underline{d}| = \sqrt{(-4)^2 + (-3)^2 + (-2)^2} = \sqrt{16 + 9 + 4} = \sqrt{29}$$

$$|\underline{n}| = \sqrt{10^2 + (-2)^2 + 1^2} = \sqrt{100 + 4 + 1} = \sqrt{105}$$

$$\cos\theta = \frac{-36}{\sqrt{29}\sqrt{105}} = \frac{-36}{\sqrt{3045}} \Rightarrow \theta = 130 \cdot 7^\circ$$

Acute angle between $\underline{d}$ and $\underline{n}$ is $180^\circ - 130 \cdot 7^\circ = 49 \cdot 3^\circ$.

Angle between the line and the plane is $90^\circ - 49 \cdot 3^\circ = \underline{\underline{40 \cdot 7^\circ}}$

Calculating the Angle Between Two Planes

If 2 planes intersect, the angle between them is equal to the angle between their normal vectors.

Q27 Find the acute angle between the planes with equations $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$.

Ans 27 $\underline{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $\quad \underline{n}_2 = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$ $\qquad \cos\theta = \dfrac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}$

$$\underline{n}_1 \cdot \underline{n}_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} = 6 + -6 + 4 = 4$$

$$|\underline{n}_1| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$|\underline{n}_2| = \sqrt{3^2 + (-6)^2 + (-2)^2} = \sqrt{9+36+4} = \sqrt{49} = 7$$

$$\cos\theta = \frac{4}{3 \times 7} = \frac{4}{21} \Rightarrow \underline{\underline{\theta = 79.0^\circ}}$$

<u>Q28</u> Find the acute angle between the planes
$\underline{r} \cdot (6\underline{i} + 2\underline{j} + 5\underline{k}) = 4$ and $\underline{r} \cdot (\underline{i} - 4\underline{j} + 3\underline{k}) = 9$

<u>Ans 28</u> These planes are written in vector form $\underline{r} \cdot \underline{n} = k$ where $\underline{n}$ is the normal vector,

$$\underline{n}_1 = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \underline{n}_2 = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$$

$$\cos\theta = \frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}$$

$$\underline{n}_1 \cdot \underline{n}_2 = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = 6 + -8 + 15 = 13$$

$$|\underline{n}_1| = \sqrt{6^2 + 2^2 + 5^2} = \sqrt{36+4+25} = \sqrt{65}$$

$$|\underline{n}_2| = \sqrt{1^2 + (-4)^2 + 3^2} = \sqrt{1+16+9} = \sqrt{26}$$

$$\cos\theta = \frac{13}{\sqrt{65}\sqrt{26}} = \frac{13}{\sqrt{1690}}$$

$$\underline{\underline{\theta = 71.6^\circ}}$$

[Now try TJ3, Ex5, Q1, 2 & 3]