Separable Differential Equations

1.   The greatest chef in the world is making his masterpiece, a completely spherical oyster and kiwi fruit soufflé, which he serves with three complementary sauces. No one knows how he does it, but visiting gourmets and scientists have noticed that when the soufflé is in the oven it retains its spherical shape and its volume increases at a rate which is proportional to its radius.

It can be shown that the radius r cm of the soufflé at time t minutes after it has gone into the oven satisfies the differential equation

$$\frac{dr}{dt} = \frac{k}{r}$$

where k is constant.

Given that the radius of the soufflé is 8 cm when it goes in the oven and an unbelievable 12 cm when it is done, 30 minutes later, find its radius after 15 minutes in the oven.

2.   When a bullet is fired from a gun at a speed of 210 ms^{-1}, its retardation $\frac{dV}{dt}$ satisfies the differential equation

$$\frac{dV}{dt} = -kV$$

where V ms^{-1} is its speed at time t seconds. When t = 0·5 seconds, V = 105 ms^{-1}.
(a)   Show that k = ln 4.
(b)   After how many seconds will the speed be reduced to 10 ms^{-1}?

3.   A cylindrical water tank, which is initially full of water, has a small hole in its horizontal base. The water leaks out at a rate which is proportional to the square of the depth of water at any time. Consequently, the situation can be modelled by the differential equation:

$$\frac{dh}{dt} = -kh^2$$

If it is half full after 1½ hours, find how long it takes for it to become a quarter full.

4.   The rate of increase of the number of cells of a yeast is proportional to the number of cells present.

(a)   If the number of cells at time t is n, show that $n = n_0 e^{kt}$, where n_0 is the initial number of cells and k is a constant.

(b)   If the yeast takes 30 minutes to double in volume, show that $k = \frac{1}{30} \ln 2$.

(c)   How long will it take to treble in volume?

5.   A sample of radium loses mass at a rate which is proportional to the amount present.

(a)   If its mass at time t is m, show that $m = m_0 e^{-kt}$, where m_0 is its initial mass and k is a constant.
(b)   Sketch a graph of m against t.
(c)   If its half-life (i.e. the time taken for its mass to be halved) is 1600 years, show that $k \approx 0·000433$ years^{-1}.

6. Newton's Law of Cooling states that the temperature of a body in a room (at constant temperature) reduces at a rate proportional to the difference between the body's temperature and the temperature of the room. A body with temperature 90°C is placed in a room with temperature 18°C. The body then cools by 3°C in the next minute.
   (a) Find how long it takes for the body's temperature to drop to 60°C.
   (b) What will be the temperature 24 minutes from the time the body was put in the room?

7. A bath is full of water and is at a temperature of 80°C. It is in a room where the temperature remains constant at 20°C.
   (a) Using Newton's Law of Cooling (see question 6), show that after time t (minutes) its temperature θ (°C) is given by
   $$\theta = 20 + 60e^{-kt}$$
   where k is a constant.
   (b) If it takes 5 minutes to reach 70°C, show that $k \approx 0.036464$ min^{-1}.
   (c) Find the temperature after 10 minutes.
   (d) Find the time taken for the temperature to reach 60°C.
   (e) Sketch a graph of θ against t.

8. A lock on a canal is filled by opening the sluices. Water then pours in to the lock at a rate which is inversely proportional to the depth of water in the lock. Initially, the water is at its minimum depth of 2 m. Given that the lock can be half-filled (depth 5 m) in 30 seconds, find the time taken to fill it to its maximum depth of 8 m.

9. In a certain chemical reaction, a compound is being made from two other compounds. At a time t hours, there is an amount x of the required substance present.

   The speed of reaction $\dfrac{dx}{dt}$ is defined by $\dfrac{dx}{dt} = 2(x - 3)(x - 6)$. Initially, x = 0.

   Find approximately how many *seconds* elapse before x = 2.

10. The change in gene frequency of the mosquito may be described by the differential equation
    $$\frac{dq}{dn} = \frac{q(1 - q)}{3 + q}$$
    where q is the gene frequency after n generations. If the initial gene frequency is q_0, show that
    $$n = 3\ln \frac{q}{q_0} + 4\ln \left\{ \frac{1 - q_0}{1 - q} \right\}.$$

11. A direct e.m.f. E is applied to a coil of inductance L and resistance R. The current i in the circuit satisfies the equation
    $$E - L \frac{di}{dt} = iR \qquad \text{(where E, L and R are constants).}$$
    (a) If i = 0 when t = 0, show that $i = \dfrac{E}{R} (1 - e^{-Rt/L})$.

    (b) Show that the current settles to a steady value $i_0 = E/R$.

    (c) With the current at its steady value, the circuit is then broken, and the current now satisfies the equation
    $$-L \frac{di}{dt} = iR$$
    Show that $i = i_0 e^{-Rt/L}$. (L/R is called the *time constant* of the circuit.)

    (d) Sketch a rough graph of i against t.