Solutions to "Separable Differential Equations"

[These are all variable separable first order differential equations.]

(1)

$$\frac{dr}{dt} = \frac{k}{r}$$

$$\int r\,dr = k\int dt$$

$$\frac{1}{2}r^2 = kt + c \qquad \text{[But when } t=0,\ r=8\text{]}$$

$$\frac{1}{2}8^2 = 0 + c \quad \Rightarrow c = 32.$$

$$\therefore \quad \frac{1}{2}r^2 = kt + 32 \quad \Rightarrow r^2 = 2kt + 64$$

[But $r = 12$ when $t = 30$]

$$144 = 60k + 64$$

$$80 = 60k$$

$$\frac{80}{60} = k$$

$$k = \frac{4}{3}$$

$$r^2 = 2\left(\frac{4}{3}\right)t + 64 \quad \Rightarrow r^2 = \frac{8}{3}t + 64$$

$$\Rightarrow r^2 = \frac{8}{3} \times 15 + 64 = 40 + 64 = 104$$

$$\Rightarrow r = \sqrt{104}$$

$$\Rightarrow r = 10.2\text{cm}.$$

(2) $\frac{dV}{dt} = -kV$

$\int \frac{dV}{V} = -k\int dt$

$\ln V = -kt + c$ [But when $t=0$, $V=210$]

$\ln 210 = 0 + c = c$

$\therefore \ln V = -kt + \ln 210$

$\ln V = \ln e^{-kt} + \ln 210 = \ln 210e^{-kt}$

$V = 210e^{-kt}$ [But when $t=0.5$, $V=105$]

$105 = 210e^{-kt}$

$\frac{1}{2} = e^{-0.5k}$

$\ln\frac{1}{2} = -0.5k$

$-2\ln\left(\frac{1}{2}\right) = k$

$k = \ln(2^{-1})^{-2} = \ln 2^2 = \ln 4.$

ie. $V = 210e^{-t(\ln 4)} = 210e^{\ln 4^{-t}} = (210)4^{-t} = \frac{210}{4^t}$

When $V=10$, $10 = \frac{210}{4^t} \Rightarrow 10(4^t) = 210 \Rightarrow 4^t = 21$

$\Rightarrow \ln 4^t = \ln 21$

$\Rightarrow t\ln 4 = \ln 21$

$\Rightarrow t = \frac{\ln 21}{\ln 4} = 2.196$ secs.

distance travelled between $t=1$ and $t=2$ $= \int_1^2 \frac{210}{4^t}.dt = 210\int_1^2 4^{-t}.dt$

Let $u = -t$ Also, $t=1 \Rightarrow u=-1$

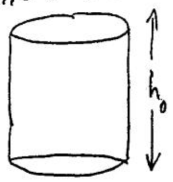
$du = -dt$. $t=2, \Rightarrow u=-2$

$\therefore$ dist $= -210\int_{-1}^{-2} 4^u.du = -\frac{210}{\ln 4}\left[4^u\right]_{-1}^{-2} = -\frac{210}{\ln 4}\left[\frac{1}{16} - \frac{1}{4}\right] = -\frac{210}{\ln 4}\left(-\frac{3}{16}\right)$

Uses: $\int a^x.dx = \frac{1}{\ln a}a^x + c$

$= \frac{3\times 210}{16\ln 4} = 28.4$m.

(3) At $t=0$:



$$\frac{dh}{dt} \propto h^2$$

$$\frac{dh}{dt} = -kh^2$$

$$\int \frac{dh}{h^2} = -k\int dt$$

$$\frac{h^{-1}}{-1} = -kt + c$$

$$\Rightarrow -\frac{1}{h} = c - kt$$

$$\Rightarrow h = \frac{1}{kt - c}$$

But at $t=0$, $h = h_0$

$$\Rightarrow h_0 = \frac{1}{-c} \quad \Rightarrow \quad c = -\frac{1}{h_0}$$

$$\therefore h = \frac{1}{kt + \frac{1}{h_0}} = \frac{h_0}{h_0 kt + 1}$$

But $t = \frac{3}{2} \Rightarrow h = \frac{1}{2}h_0$

$$\frac{1}{2}h_0 = \frac{h_0}{h_0 k \frac{3}{2} + 1} = \frac{2h_0}{3kh_0 + 2}$$

$$h_0(3kh_0 + 2) = 4h_0$$

$$3kh_0^2 + 2h_0 = 4h_0$$

$$3kh_0^2 = 2h_0$$

$$k = \frac{2h_0}{3h_0^2} = \frac{2}{3h_0}$$

ie. $$h = \frac{h_0}{h_0 \frac{2}{3h_0} t + 1} = \frac{3h_0}{2t + 3}$$

When $h = \frac{1}{4}h_0$, $$\frac{1}{4}\cancel{h_0} = \frac{3\cancel{h_0}}{2t+3}$$

$$2t + 3 = 12$$

$$2t = 9$$

$$t = 4{\cdot}5 \text{ hours.}$$

4 (a) $\dfrac{dn}{dt} = kn$

$$\int \frac{dn}{n} = k\int dt$$

$$\ln|n| = kt + c$$

$$n = e^{kt+c} = e^{kt}e^{c} = Ae^{kt} \quad (\text{where } A = e^{c})$$

At $t=0$, $n = n_0$

$$\Rightarrow \quad n_0 = Ae^{0} = A$$

ie. $n = n_0 e^{kt}$

(b) $$2n_0 = n_0 e^{30k}$$

$$\frac{2n_0}{n_0} = e^{30k}$$

ie $e^{30k} = 2$

$$30k = \ln 2$$

$$k = \frac{1}{30(\text{min})}\ln 2 = 0.023\,\text{min}^{-1}$$

(c) $$n = n_0 e^{0.023t}$$

$$3\cancel{n_0} = \cancel{n_0} e^{0.023t}$$

$$\ln 3 = 0.023t$$

$$t = \frac{\ln 3}{0.023} = 47.7\,\text{mins.}$$

OR

$$n = n_0 e^{\frac{1}{30}(\ln 2)t}$$

$$3\cancel{n_0} = \cancel{n_0} e^{\frac{1}{30}(\ln 2)t}$$

$$\ln 3 = \frac{1}{30}(\ln 2)t$$

$$30\ln 3 = (\ln 2)t$$

$$t = \frac{30\ln 3}{\ln 2} = 47.55\,\text{mins.}$$

(5)(a) $\frac{dm}{dt} = -km$

$$\int \frac{dm}{m} = -k \int dt$$

$$\ln|m| = -kt + c$$

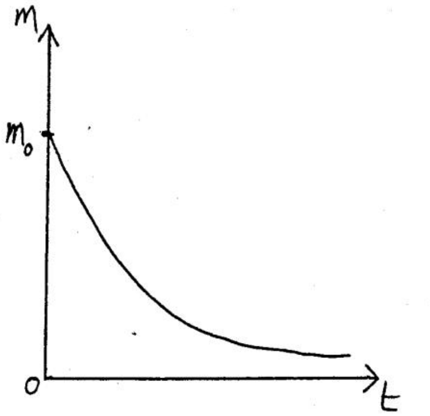
$$m = e^{-kt+c} = e^{c} \cdot e^{-kt} = Ae^{-kt}$$

When $t=0$, $m=m_0$

$$m_0 = Ae^{0} = A$$

$$m = m_0 e^{-kt}$$

(b)



(c) $\frac{1}{2} m_0 = m_0 e^{-1600k}$

$$\ln 0.5 = -1600k$$

$$-\frac{\ln 0.5}{1600_{\text{years.}}} = k$$

$$k = 0.000433 = 4.33 \times 10^{-4} \text{ years}^{-1}$$

(6) Let θ be the temperature (°C) and t be the time body has been cooling (mins)

$$\frac{d\theta}{dt} = -k(\theta - 18)$$

$$\int \frac{d\theta}{\theta - 18} = -k\int dt$$

$$\ln(\theta - 18) = -kt + c$$

$$\theta - 18 = e^{-kt+c} = e^{c}e^{-kt} = Ae^{-kt} \quad (\text{where } A = e^{c})$$

$$\theta = 18 + Ae^{-kt}$$

But when $t=0$, $\theta = 90$

$$90 = 18 + Ae^{0} = 18 + A$$

$$\Rightarrow A = 72$$

ie. $\theta = 18 + 72e^{-kt}$

When $t = 1$, $\theta = 87$

$$87 = 18 + 72e^{-k}$$

$$69 = 72e^{-k}$$

$$e^{-k} = \frac{69}{72}$$

$$-k = \ln\left(\frac{69}{72}\right)$$

$$k = -\ln\left(\frac{69}{72}\right) = 0.0426 \quad (\text{to 3 sig. fig}).$$

$$\therefore \boxed{\theta = 18 + 72e^{-0.0426t}}$$

(a)

$$60 = 18 + 72e^{-0.0426t}$$

$$42 = 72e^{-0.0426t}$$

$$\frac{42}{72} = e^{-0.0426t}$$

$$-0.0426t = \ln\left(\frac{42}{72}\right)$$

$$t = \frac{\ln\left(\frac{42}{72}\right)}{-0.0426} = 12.7 \text{mins}$$

(b) $\theta = 18 + 72e^{(-0.0426 \times 24)}$

$\theta = 43.9$°C

7 (a) Let the body's temperature be θ.

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

$$\int \frac{d\theta}{\theta - 20} = -k\int dt$$

$$\ln(\theta - 20) = -kt + c$$

$$\theta - 20 = e^{-kt+c} = e^{c}.e^{-kt} = Ae^{-kt}$$

But when $t=0$, $\theta = 80$

$$80 - 20 = Ae^{0} = A$$

$$A = 60$$

$$\theta - 20 = 60e^{-kt}$$

$$\theta = 20 + 60e^{-kt}$$

(b)

$$70 = 20 + 60e^{-5k}$$

$$50 = 60e^{-5k}$$

$$\frac{50}{60} = e^{-5k}$$

$$\ln\left(\frac{5}{6}\right) = -5k$$

$$-\frac{1}{5}\ln\left(\frac{5}{6}\right) = k \Rightarrow k = 0.036464 \qquad \text{ie. } \theta = 20 + 60e^{-0.036464t}$$

(c)

$$\theta = 20 + 60e^{-0.36464} = 61.67°C$$

(d)

$$60 = 20 + 60e^{-0.036464t}$$

$$40 = 60e^{-0.036464t}$$

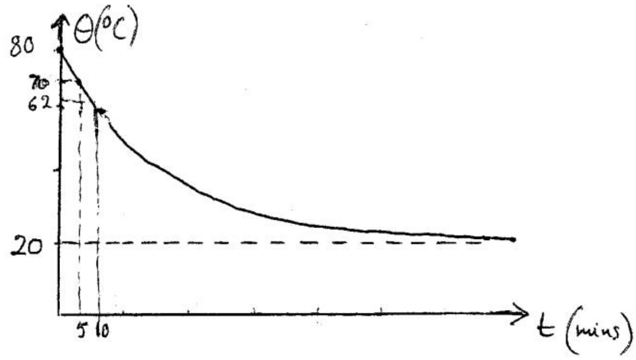
$$\frac{2}{3} = e^{-0.036464t}$$

$$\ln\frac{2}{3} = -0.036464t$$

$$t = \ln\left(\tfrac{2}{3}\right)/-0.036464$$

$t = 11.12$ mins

7(e)



(8)

$$\frac{dh}{dt} = \frac{k}{h}$$

$$\int h\,dh = k\int dt$$

$$\frac{h^2}{2} = kt + c.$$

$$kt = \frac{h^2}{2} - c$$

But $t=0 \Rightarrow h=2$.

$$0 = \frac{2^2}{2} - c$$

$$\therefore c = 2$$

$$kt = \frac{h^2}{2} - 2$$

But $t=30$, $h=5$

$$30k = \frac{25}{2} - 2 = \frac{25}{2} - \frac{4}{2} = \frac{21}{2}.$$

$$k = \frac{21}{60} = \frac{7}{20}$$

$$\frac{7}{20}t = \frac{h^2}{2} - 2$$

But $h=8$

$$\frac{7}{20}t = \frac{8^2}{2} - 2 = \frac{64}{2} - 2 = 32 - 2$$

$$\frac{7}{20}t = 30$$

$$7t = 600$$

$$t = \frac{600}{7} = \underline{\underline{85.7}} \text{ secs.}$$

(9) $\dfrac{dx}{dt} = 2(x-3)(x-6) \quad ; \quad t=0 \Rightarrow x=0.$

$$\int \frac{dx}{(x-3)(x-6)} = 2\int dt$$

$$\left[\text{But } \frac{1}{(x-3)(x-6)} = \frac{A}{x-3} + \frac{B}{x-6} = \frac{A(x-6)+B(x-3)}{(x-3)(x-6)} \right.$$

$x=6 : \; 3B = 1 \Rightarrow B = \frac{1}{3}.$

$x=3 : \; -3A = 1 \Rightarrow A = -\frac{1}{3}. \;\Big]$

$$2t = -\frac{1}{3}\int \frac{1}{x-3}\,dx + \frac{1}{3}\int \frac{1}{x-6}\,dx$$

$$2t = -\frac{1}{3}\ln|x-3| + \frac{1}{3}\ln|x-6| + \ln C$$

$$2t = -\ln\left|(x-3)^{1/3}\right| + \ln\left|(x-6)^{1/3}\right| + \ln C = \ln\left|\frac{C(x-6)^{1/3}}{(x-3)^{1/3}}\right|$$

But $x=0$ when $t=0$:

$$0 = \ln\left|C\frac{(-6)^{1/3}}{(-3)^{1/3}}\right| = \ln\left|C\left(\frac{-6}{-3}\right)^{1/3}\right| = \ln\left[C\left(2^{1/3}\right)\right]$$

$$\Rightarrow C\left(2^{1/3}\right) = 1 \Rightarrow C = \frac{1}{2^{1/3}} = 2^{-1/3}.$$

$$\therefore \; 2t = \ln\left|\left(\frac{x-6}{2(x-3)}\right)^{1/3}\right|$$

When $x=2$,

$$2t = \ln\left|\left(\frac{-4}{2\times -1}\right)^{1/3}\right| = \ln 2^{1/3} = \frac{1}{3}\ln 2$$

$$\Rightarrow t = \left[\frac{1}{6}\ln 2\right]\text{hours} = \left[\frac{1}{6}\ln 2 \times 60 \times 60\right]\text{seconds} = 600\ln 2 = 415.9\text{secs}$$

$$\approx \underline{\underline{416 \text{ seconds.}}}$$

As $x \to 3$, $t \to \frac{1}{2}\ln\left|\left(\frac{-3}{2\times 0}\right)^{1/3}\right|$ ie. $\underline{\underline{\text{as } x\to 3,\ t\to\infty}}$

⑩ $$\frac{dq}{dn} = \frac{q(1-q)}{3+q}$$

$$\int \frac{(3+q)}{q(1-q)} \cdot dq = \int dn$$

[Let $\frac{3+q}{q(1-q)} = \frac{A}{q} + \frac{B}{1-q} = \frac{A(1-q)+Bq}{q(1-q)}$

$\Rightarrow A(1-q)+Bq = 3+q$

$q=1: \quad B = 3+1=4.$

$q=0: \quad A = 3.$]

$$n = 3\int \frac{1}{q} \cdot dq + 4\int \frac{dq}{1-q} = 3\ln|q| - 4\ln|1-q| + \ln|C|$$

$$= \ln|q^3| - \ln|(1-q)^4| + \ln|C|$$

$$n = \ln\left|\frac{Cq^3}{(1-q)^4}\right|$$

But when $n=0$, $q=q_0$ $\quad \therefore \quad 0 = \ln\left|\frac{Cq_0^3}{(1-q_0)^4}\right|$

$$\Rightarrow \frac{Cq_0^3}{(1-q_0)^4} = 1 \quad \Rightarrow Cq_0^3 = (1-q_0)^4 \quad \Rightarrow C = \frac{(1-q_0)^4}{q_0^3}$$

$$\therefore \quad n = \ln\left|\frac{\frac{(1-q_0)^4}{q_0^3} \cdot q^3}{(1-q)^4}\right| = \ln\left|\frac{(1-q_0)^4 q^3}{q_0^3(1-q)^4}\right|$$

$$n = \ln|(1-q_0)^4 q^3| - \ln|q_0^3(1-q)^4|$$

$$n = \ln|(1-q_0)^4| + \ln q^3 - \left[\ln|q_0^3| + \ln|(1-q)^4|\right]$$

$$n = 4\ln|1-q_0| + 3\ln q - 3\ln q_0 - 4\ln(1-q)$$

$$n = 3\ln\left|\frac{q}{q_0}\right| + 4\ln\left|\frac{1-q_0}{1-q}\right|$$

(11) (a) $E - L\dfrac{di}{dt} = iR$

$$E - iR = L\frac{di}{dt}$$

$$\frac{1}{L}\int dt = \int \frac{di}{E-iR}$$

$$\frac{1}{L}t = \frac{1}{-R}\ln|E-iR| + \ln|A|$$

$$\frac{t}{L} = \ln\left|(E-iR)^{-\frac{1}{R}}\right| + \ln|A| = \ln\left|A(E-iR)^{-\frac{1}{R}}\right|$$

$$e^{t/L} = \frac{A}{(E-iR)^{1/R}}$$

But $i=0$ when $t=0$

$$\Rightarrow 1 = \frac{A}{E^{1/R}}$$

$$\Rightarrow A = E^{1/R}$$

ie. $e^{t/L} = \dfrac{E^{1/R}}{(E-iR)^{1/R}}$

$$\Rightarrow e^{Rt/L} = \frac{E}{E-iR}$$

$$\Rightarrow (E-iR)e^{Rt/L} = E$$

$$\Rightarrow Ee^{Rt/L} - iRe^{Rt/L} = E$$

$$\Rightarrow Ee^{Rt/L} - E = iRe^{Rt/L}$$

$$\Rightarrow \frac{1}{R}Ee^{Rt/L} - \frac{E}{R} = ie^{Rt/L}$$

$$\Rightarrow e^{-Rt/L}\frac{E}{R}e^{Rt/L} - \frac{E}{R}e^{-Rt/L} = i$$

$$\Rightarrow \frac{E}{R} - \frac{E}{R}e^{-Rt/L} = i$$

$$\Rightarrow i = \frac{E}{R}\left(1 - e^{-Rt/L}\right).$$

(b) As $t \to \infty$, $e^{-\frac{Rt}{L}} \to 0$

$$\Rightarrow i \to \frac{E}{R}(1-0)$$

ie. $i \to \frac{E}{R}$

ie. current settles to a steady value $i_0 = \frac{E}{R}$.

(c)

$$-L\frac{di}{dt} = iR$$

$$\int \frac{di}{i} = -\frac{R}{L}\int dt$$

$$\ln i = -\frac{R}{L}t + c$$

$$i = e^{-Rt/L + c} = e^{c} e^{-Rt/L} = Ae^{-Rt/L}$$

At the instant that the current was broken, its value was $i_0 \left(= \frac{E}{R}\right)$

$$\therefore\ i_0 = Ae^{0} = A$$

$$\therefore\ i = i_0 e^{-Rt/L} \quad \left(= \frac{E}{R} e^{-Rt/L} \right)$$

(d)