The Binomial Theorem

Prerequisites: Cancelling fractions; summation notation; rules of indices.

Maths Applications: Proving trig. identities using complex numbers; probability.

Real-World Applications: Counting problems; Hardy-Weinberg Formula (biology).

Factorials and Binomial Coefficients

In a race with 3 people, in how many ways can the runners finish ? There are 3 possibilities for the first place; for *each of these 3 possibilities*, there are 2 possibilities for the remaining 2 places; for *each of these 2 possibilities*, there is only 1 possibility for the final place. So, there are $3 \times 2 \times 1 = 6$ ways the runners can finish.

Definition:

For $n \in \mathbb{N}$, the **factorial of *n*** (aka ***n* factorial** or **factorial *n***) is,

$$n! \overset{def}{=} n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$$

Note that $1! = 1$ and the convention $0! \overset{def}{=} 1$ is made.

In how many ways can a child pick 3 crayons from a selection of 5 coloured crayons (all different colours) ? There are $5 \times 4 \times 3 = 60$ ways of choosing 3 crayons, if the order in which they are taken matters. However, the child isn't interested in which order they're picked, so this answer of 60 is too big. Whichever 3 colours are picked, there are $3\,! = 6$ ways of doing so. Therefore, the value of 60 is 6 times too large; hence, the actual number of ways of choosing 3 crayons from 5 without worrying about the order is $60 \div 6 = 10$.

The order matters in a *permutation*, as opposed to a combination.

Definition:

The **number of ways of choosing r objects from n without taking into account the order** (aka **n choose r** or the **number of combinations of r objects from n**) is given by the **binomial coefficient** ^nC_r defined by,

$$^nC_r \equiv \binom{n}{r} \stackrel{def}{=} \frac{n!}{r!\,(n-r)!}$$

Evaluating a Binomial Coefficient Without a Calculator

Example 1

$$\begin{aligned} ^7C_4 &= \frac{7!}{4!\,(7-4)!} \\ &= \frac{7\times6\times5\times4\times3\times2\times1}{(4\times3\times2\times1)\times(3\times2\times1)} \\ &= \frac{7\times6\times5}{3\times2\times1} \\ &= 35 \end{aligned}$$

Properties of Binomial Coefficients

- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ **(Khayyam-Pascal Identity)**

Example 2

Solve the equation $\binom{n}{n-2} = 15$.

$$\frac{n!}{2!\,(n-2)!} = 15$$

$$\frac{n\,(n-1)\,(n-2)!}{2\,(n-2)!} = 15$$

$$\frac{n\,(n-1)}{2} = 15$$

$$n^2 - n = 30$$

$$n^2 - n - 30 = 0$$

$$(n-6)\,(n+5) = 0$$

Hence, $n = 6$ or $n = -5$. However, as n cannot be negative, $n = 6$.

Example 3

Show that $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$.

A very simple proof can be obtained by replacing r with 2 in the Khayyam-Pascal Identity. However a more ' get your fingers dirty ' method will be given, which illustrates some general techniques when manipulating binomial coefficients.

Starting with the LHS,

$$\binom{n+1}{3} - \binom{n}{3} = \frac{(n+1)!}{3!\,(n-2)!} - \frac{n!}{3!\,(n-3)!}$$

$$= \frac{(n+1)\,n!}{3!\,(n-2)\,(n-3)!} - \frac{n!}{3!\,(n-3)!}$$

$$= \frac{n!}{3!\,(n-3)!}\left(\frac{n+1}{n-2} - 1\right)$$

$$= \frac{n!}{3!\,(n-3)!}\left(\frac{n+1-n+2}{n-2}\right)$$

$$= \frac{n!}{3!\,(n-3)!}\left(\frac{3}{n-2}\right)$$

$$= \frac{n!\,3}{3\,.\,2!\,(n-3)!\,(n-2)}$$

$$= \frac{n!}{2!\,(n-2)!}$$

$$= \binom{n}{2}$$

which equals the RHS.

Pascal's Triangle

Pascal's triangle - also known as *Pingala's triangle*, *Khayyam's triangle*, *Yang Hui's triangle* and *Tartaglia's triangle*, after mathematicians who discovered or studied the triangle before Pascal – is the following infinite arrangement of evaluated binomial coefficients:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \vdots & & & & & \end{array}$$

The top row consisting of the single entry 1 is the 0^{th} row. Each number not on an edge of the triangle is obtained by adding the 2 numbers in the previous row and just to the right and left of that entry (this is the Khayyam-Pascal identity).

The Binomial Theorem

Taking powers of a binomial can be achieved via the following theorem.

Theorem (**Binomial Theorem**):

For whole numbers r and n,

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} \, y^r$$

Written out fully, the RHS is called the **binomial expansion** of $(x + y)^n$.

Using the first property of the binomial coefficients and a little relabelling, the Binomial Theorem can be written slightly differently.

Corollary:

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r \, x^r \, y^{n-r}$$

Note that there are $n + 1$ terms in any binomial expansion.

Expanding a Binomial

Example 4

Expand $(x + y)^5$.

$$(x + y)^5 = \sum_{r=0}^{5} \binom{5}{r} x^{5-r} \, y^r$$

$$= \binom{5}{0} x^5 \, y^0 + \binom{5}{1} x^4 \, y^1 + \binom{5}{2} x^3 \, y^2 + \binom{5}{3} x^2 \, y^3 + \binom{5}{4} x^1 \, y^4 + \binom{5}{5} x^0 \, y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Note that the powers of x go up by 1 as the powers of y go down by 1, and that the sum of the powers of x and y equal 5. Also, the number of terms in the expansion is one more than the value of n. The binomial coefficients are evaluated using Pascal's triangle.

Example 5

Expand $\left(x^2 - \frac{2}{x}\right)^4$.

$$\left(x^2 - \frac{2}{x}\right)^4 = \sum_{r=0}^{4} \binom{4}{r} \left(x^2\right)^{4-r} \left(-\frac{2}{x}\right)^r$$

$$= \binom{4}{0}\left(x^2\right)^4\left(-\frac{2}{x}\right)^0 + \binom{4}{1}\left(x^2\right)^3\left(-\frac{2}{x}\right)^1 + \binom{4}{2}\left(x^2\right)^2\left(-\frac{2}{x}\right)^2$$

$$+ \binom{4}{3}\left(x^2\right)^1\left(-\frac{2}{x}\right)^3 + \binom{4}{4}\left(x^2\right)^0\left(-\frac{2}{x}\right)^4$$

$$= x^8(1) + 4x^6\left(-\frac{2}{x}\right) + 6x^4\left(\frac{4}{x^2}\right) + 4x^2\left(-\frac{8}{x^3}\right) + \left(\frac{16}{x^4}\right)$$

$$= x^8 - 8x^5 + 24x^2 - \frac{32}{x} + \frac{16}{x^4}$$

Finding the General Term

Definition:

The **general term** in a binomial expansion is,

$${}^nC_r \; x^{n-r} \; y^r$$

Essentially, the general term is everything in the Binomial Theorem apart from the summation sign.

Example 6

Find and simplify the general term in the expansion of $\left(x^2 + \frac{3}{x}\right)^{13}$.

$$\left(x^2 + \frac{3}{x}\right)^{13} = \sum_{r=0}^{13} \binom{13}{r} \left(x^2\right)^{13-r} \left(\frac{3}{x}\right)^r$$

The general term is the expression after the summation sign on the RHS of the above equation. So,

$$\text{General term} = \binom{13}{r} \left(x^2\right)^{13-r} \left(\frac{3}{x}\right)^r = \binom{13}{r} 3^r x^{26-2r} x^{-r} = \binom{13}{r} 3^r x^{26-3r}$$

Finding a Specific Term or Coefficient

Example 7

Find the term independent of x in the expansion of $\left(x - \frac{5}{x^2}\right)^9$.

$$\text{General term} = \binom{9}{r} x^{9-r} \left(-\frac{5}{x^2}\right)^r = \binom{9}{r} (-5)^r x^{9-3r}$$

The term independent of x occurs when the index $9 - 3r = 0$, i.e. when $r = 3$. Thus, the required term is,

$$\binom{9}{3} (-5)^3 = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{(3\times2\times1)\times(6\times5\times4\times3\times2\times1)} \times (-125) = -10\,500$$

Example 8

Find the term containing x^{28} in the expansion of $\left(x^2 + \frac{x}{2}\right)^{15}$.

$$\text{General term} = \binom{15}{r} \left(x^2\right)^{15-r} \left(\frac{x}{2}\right)^r = \binom{15}{r} 2^{-r} x^{30-r}$$

The term containing x^{28} occurs when the index $30 - r = 28$, i.e. when $r = 2$. Thus, the required term is,

$$\binom{15}{2} 2^{-2} x^{28} = \frac{15!}{2!\ 13!} \times \frac{1}{4} x^{28} = \frac{105}{4} x^{28}$$

Example 9

Find the coefficient of x^4 in the expansion of $\left(2 - x\right)^{11}$.

$$\text{General term} = \binom{11}{r} 2^{11-r} (-x)^r = \binom{11}{r} (-1)^r\ 2^{11-r}\ x^r$$

The coefficient of x^4 occurs when $r = 4$. Thus, the required coefficient is,

$$\binom{11}{4} (-1)^4\ 2^7 = \frac{11!}{4!\ 7!} \times 128 = 42\ 240$$

Evaluating a Natural Power of a Decimal

Example 10

Calculate the value of $(0 \cdot 8)^4$ using the Binomial Theorem.

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

Letting $x = -0 \cdot 2$,

$$(0 \cdot 8)^4 = 1 + 4\,(-0 \cdot 2) + 6\,(-0 \cdot 2)^2 + 4\,(-0 \cdot 2)^3 + (-0 \cdot 2)^4$$

$$= 1 - 0 \cdot 8 + 6\,(0 \cdot 04) - 4\,(0 \cdot 008) + 0 \cdot 001\ 6$$

$$= 1 - 0 \cdot 8 + 0 \cdot 24 - 0 \cdot 032 + 0 \cdot 001\ 6$$

$$= 0 \cdot 409\ 6$$

Obviously, a calculator should be used for questions similar in spirit to Example 10.

Applications of the Binomial Theorem

The Binomial Theorem is often used to solve probabilistic problems.

<u>Example 11</u>

A fair coin is flipped 5 times. Calculate the probability of obtaining exactly 3 heads.

This problem can be solved by listing all possible combinations of heads and tails (e.g. HHHHH, HTHTH, etc.), picking those that give exactly 2 heads and dividing by the total number of possibilities. No doubt, this is a very lengthy process. However, the Binomial Theorem can be used to sort this out very quickly.

If p stands for the probability of obtaining a head and q for the probability of obtaining a tail, then obviously $p + q = 1$ (total probability equals 100 %). Hence, and this is where the link to the Binomial Theorem comes in, $(p + q)^n = 1$. In our case, $n = 5$. Then,

$$(p + q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$$

The first term in this expansion gives the probability of obtaining 5 heads, the second term 4 heads and a tail etc. (check this using a tree diagram – for example, the coefficient 5 in the second term gives the number of ways 4 heads and 1 tail can be obtained). As the coin is fair, $p = q = \frac{1}{2}$. Thus, the required probability is,

$$\begin{aligned} \text{P (exactly 3 heads)} &= 5p^4q \\ &= 5.\left(\frac{1}{2}\right)^4.\frac{1}{2} \\ &= \frac{5}{32} \end{aligned}$$

Notice that the sum of the coefficients in the above expansion equals 32 (the total number of possibilities for 5 flips).

<u>Example 12</u>

If the probabilities of dominant (T) and recessive (t) alleles, denoted by p and q respectively, for an organism with n copies of the same chromosome are linked via the equation,

$$(p + q)^n = 1$$

find the probability that an organism with 4 copies of the same chromosome has the genotype TTtt if $q = 0 \cdot 2$.

The biological details of the problem are not relevant, only the general structure of the problem. We have $n = 4$ and the genotype TTtt corresponds to the term p^2q^2 (2 T's and 2 t's) in the binomial expansion of $(p + q)^4$. Expanding this gives,

$$(p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$$

Since $p + q = 1$, $p = 0 \cdot 8$. The required probability is thus,

$$\begin{aligned} \text{P (TTtt)} &= 6p^2q^2 \\ &= 6\,(0 \cdot 2)^2\,(0 \cdot 8)^2 \\ &= 0 \cdot 153\,6 \end{aligned}$$