Complex Numbers

Prerequisites: Expanding brackets; solving quadratics; finding angles using basic trigonometry; exact values.

Maths Applications: Deriving trig. identities; solving polynomials.

Real-World Applications: Electrical circuits; quantum mechanics; relativity.

Number Systems and Complex Numbers

History of Complex Numbers

There are lots of different types of numbers. The ones we know about include whole numbers ($\mathbb{W}$), natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$) and real numbers ($\mathbb{R}$). Historically, the above types of numbers arose out of the need to solve real world problems, eventually extending to the need for solving equations.

Complex numbers arose out of a similar need to solve cubic equations. There is a very complicated formula, called the *Cubic Formula*, for solving any cubic equation (just like for the quadratic equation there is the Quadratic Formula). The equation $x^3 - x = 0$ obviously has the 3 real roots $x = 0$, 1 and -1. However, the Cubic Formula gives (ask your teacher how),

$$x = \frac{1}{\sqrt{3}}\left(\left(\sqrt{-1}\right)^{1/3} + \frac{1}{\left(\sqrt{-1}\right)^{1/3}} \right)$$

Clearly, there has to be some way of getting the 3 real roots from this. The square root of -1 has a crucial role here. Clearly, we can't ' take the square root of -1 '. Or can we ? Until fairly recent times, people did not believe in negative numbers. During the 18th century, negative solutions to equations were ignored. So, what we do to reconcile the above discrepancy is to introduce a new symbol, denoted by i, for the square root of -1 (just like -4 is a symbol for the ' solution ' to $x + 4 = 0$; in the 18th century, a ' solution ' was normally a positive number). Then we just get on with it. We will solve the above problem in a later section. Numbers involving i were called *imaginary numbers*, but they now go by a different name.

Cartesian Form of a Complex Number

<u>Definition:</u>

A **complex number** is a number of the form $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. The real number x is called the **real part of z** (**Re(z)**) and the real number y is called the **imaginary part of z** (**Im(z)**).

Writing a complex number as $z = x + iy$ is known as the **Cartesian form of z**.

<u>Theorem:</u>

Complex numbers are equal if they have the same real parts and the same imaginary parts (and vice versa).

<u>Example 1</u>

If the complex numbers $z = 4 + 5\,i$ and $w = (2p - q)x + (p + q)$ i are equal, find p and q.

As z and w are equal, their real and imaginary parts can be equated to give,

$$2p - q = 4$$

$$p + q = 5$$

The solution of these simultaneous equations is $p = 3$ and $q = 2$.

<u>Definition:</u>

The **set of all complex numbers** is the set defined by,

$$\mathbb{C} \stackrel{def}{=} \{ x + iy : x, y \in \mathbb{R}, i^2 = -1 \}$$

The Algebra of Complex Numbers

Addition, Subtraction and Multiplication

Theorem:

Complex numbers are added (subtracted) by adding (subtracting) the real parts together and by adding (subtracting) the imaginary parts together,

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$$

Example 2

Add the complex numbers $z = 4 + 5i$ and $w = -7 + i$.

$$\begin{aligned} z + w &= (4 + 5i) + (-7 + i) \\ &= (4 - 7) + (5 + 1)i \\ &= -3 + 6i \end{aligned}$$

Example 3

Find $z - w$ when $z = 5 + 2i$ and $w = 4 - 9i$.

$$\begin{aligned} z - w &= (5 + 2i) - (4 - 9i) \\ &= (5 - 4) + (2 + 9)i \\ &= 1 + 11i \end{aligned}$$

Multiplication has a slightly more complicated rule.

Theorem:

Complex numbers are multiplied according to the rule,

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

In practice, this rule isn't memorised; just expand brackets, remember to use $i^2 = -1$ and simplify.

<u>Example 4</u>

Find the product of $z = -6 + 4i$ and $w = 2 - 3i$.

$$\begin{aligned} zw &= (-6 + 4i)(2 - 3i) \\ &= -12 + 8i + 18i - 12i^2 \\ &= -12 + 26i + 12 \\ &= 26i \end{aligned}$$

The Complex Conjugate and Division

<u>Definition:</u>

The **complex conjugate** of a complex number $z = x + iy$ is the complex number defined by,

$$\bar{z} \overset{def}{=} x - iy$$

The complex conjugate is obtained by changing the sign of the imaginary part while keeping the real part unchanged.

<u>Theorem:</u>

The complex conjugate of $z = x + iy$ satisfies,

$$z\bar{z} = x^2 + y^2$$

Notice that this product is always a real number.

<u>Example 5</u>

Evaluate $z\bar{z} + 2\bar{z}$ when $z = 8 - 7i$.

The conjugate is $\bar{z} = 8 + 7i$. Hence,

$$z\bar{z} + 2\bar{z} = 8^2 + 7^2 + 2(8 + 7i)$$

$$= 64 + 49 + 16 + 14i$$

$$= 129 + 14i$$

The conjugate is used to divide complex numbers. The trick is to multiply the denominator of the fraction by the complex conjugate of the denominator.

Example 6

Divide $-3 + 4i$ by $2 + 3i$, expressing the answer in Cartesian form.

$$\frac{-3 + 4i}{2 + 3i} = \frac{-3 + 4i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i}$$

$$= \frac{(-3 + 4i)(2 - 3i)}{(2 + 3i)(2 - 3i)}$$

$$= \frac{-6 + 8i + 9i - 12i^2}{4 + 9}$$

$$= \frac{6 + 17i}{13}$$

$$= \frac{6}{13} + \frac{17}{13}i$$

Solving any Quadratic Equation

Complex numbers allow any quadratic equation to be solved (more lies from your Higher teacher ?).

Example 7

Solve $z^2 - 2z + 5 = 0$.

The Quadratic Formula gives,

$$z = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

$$z = \frac{2 \pm \sqrt{-16}}{2}$$

$$z = \frac{2 \pm 4\,\mathrm{i}}{2}$$

$$z = 1 \pm 2\,\mathrm{i}$$

Example 8

Find the square roots of $15 - 8\,\mathrm{i}$.

Let $a + b\,\mathrm{i}$ be a square root of $15 - 8\,\mathrm{i}$, i.e.,

$$(a + b\,\mathrm{i})^2 = 15 - 8\,\mathrm{i}$$

$$(a^2 - b^2) + (2ab)\,\mathrm{i} = 15 - 8\,\mathrm{i}$$

Equating real and imaginary parts gives,

$$a^2 - b^2 = 15$$

$$2ab = -8$$

The second equation gives, since $a \neq 0$ (why ?),

$$b = -\frac{4}{a}$$

Substituting this into the first equation then gives,

$$a^2 - \frac{16}{a^2} = 15$$

$$a^4 - 16 = 15a^2$$

$$a^4 - 15a^2 - 16 = 0$$

This is a quadratic equation for a^2 which factorises nicely as,

$$(a^2 - 16)(a^2 + 1) = 0$$

Hence, either $a^2 = 16$ or $a^2 = -1$. This second possibility cannot arise, as $a \in \mathbb{R}$. Hence, $a = \pm 4$. Thus, $b = \mp 1$. So, the square roots of $15 - 8\,\mathrm{i}$ are $4 - \mathrm{i}$ and $-4 + \mathrm{i}$.

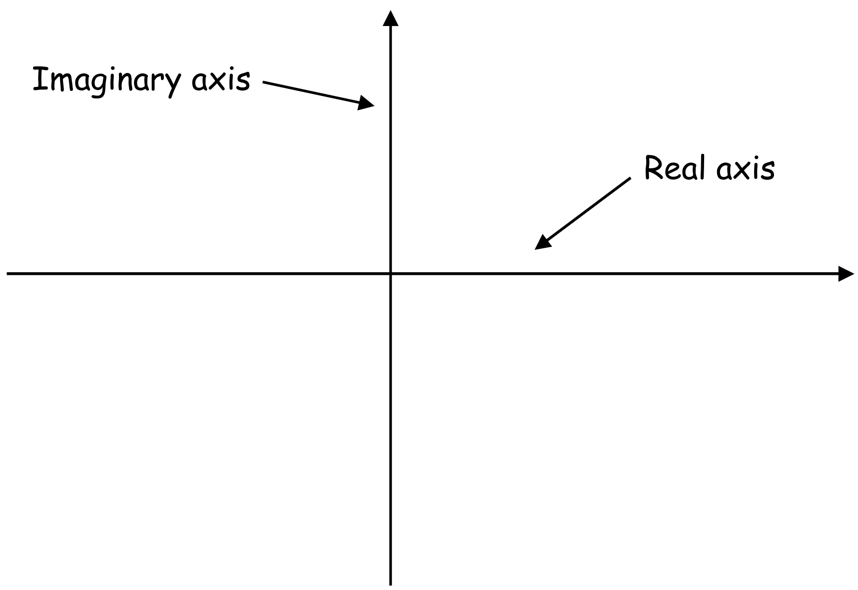
The Geometry of Complex Numbers

Modulus, Argument and Argand Diagrams

Complex numbers can be thought of as 2D vectors, the real and imaginary parts corresponding to x and y components, respectively. Vectors are also thought of as quantities that have magnitude (size) and direction. These quantities correspond to quantities known as the modulus and argument, respectively.

Definition:

The **Complex Plane** (aka **Argand Plane**) is the 2D plane showing $\mathbb{C}$. The horizontal axis is called the **real axis** (consisting of all complex numbers of the form $a + 0\,\mathrm{i}$), whereas the vertical axis is called the **imaginary axis** (consisting of all complex numbers of the form $0 + b\,\mathrm{i}$).



Definition:

An **Argand diagram** (aka **Wessel diagram**) is a plot of one or more complex numbers in the Complex Plane.

The above 2 concepts are sometimes used interchangeably, but technically there is a difference.

Definition:

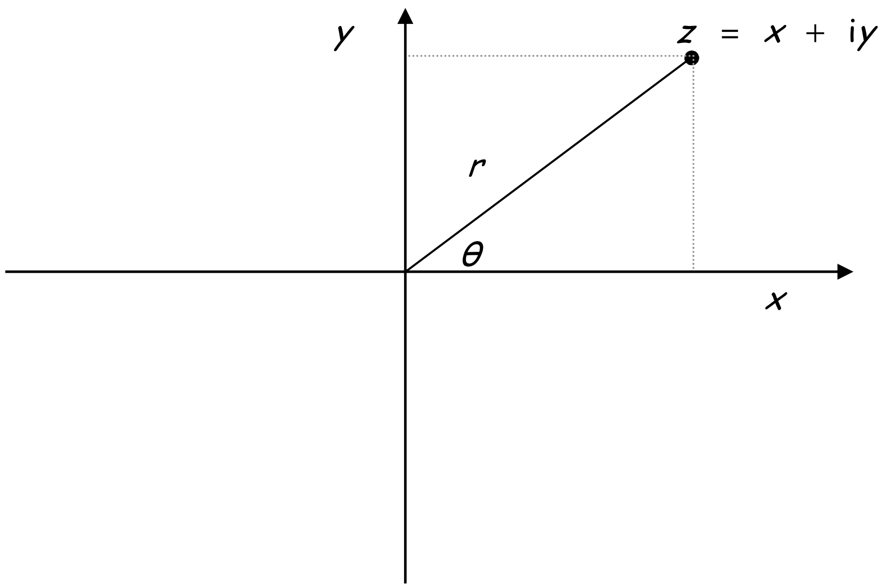
The **modulus** of a complex number $z = x + iy$ is the distance of the complex number from the origin of the Complex Plane and defined as,

$$r \equiv |z| \overset{def}{=} \sqrt{x^2 + y^2}$$

Definition:

The **principal argument** of a complex number z is the angle in the interval $(-\pi, \pi]$ from the positive x – axis to the ray joining the origin to z and defined as,

$$\theta \equiv \arg z \overset{def}{=} \tan^{-1}\left(\frac{y}{x}\right)$$



The convention for the angle range is chosen randomly, another common one being $(0, 2\pi]$. We will use the one in the definition.

Definition:

An **argument** of a complex number $z = x + iy$, denoted by Arg z, is defined as,

$$\text{Arg } z \overset{def}{=} \{\arg z + 2\pi n : n \in \mathbb{Z}\}$$

A complex number has infinitely many arguments, but obviously only 1 principal argument. When asked for the ' argument ' of a complex number, it almost always means the principal argument.

Example 9

Find the modulus and argument (in degrees) of $z = 3 + 4\text{ i}$.

The modulus is $|z| = \sqrt{3^2 + 4^2}$, i.e. $|z| = 5$. The Argand diagram for $3 + 4\text{ i}$ shows that it lies in the first quadrant. The argument is found by solving $\tan\theta = 4/3$. The related angle is $\tan^{-1}(4/3) = 53 \cdot 1\,°$ to 1 d.p.. So the principal argument is $\theta = 53 \cdot 1\,°$.

Example 10

Find the modulus and argument (in degrees, to 1 d.p., and radians, to 3 s.f.) of $z = -3 - 4\text{ i}$.

The modulus is obviously $r = 5$. A plot of $-3 - 4\text{ i}$ shows that it lies in the third quadrant. The related angle is $53 \cdot 1\,°$. Hence, the principal argument is $180\,° - 53 \cdot 1\,°$, i.e. $\theta = 126 \cdot 9\,°$ or $2 \cdot 21$ rads.

Complex Loci

Definition:

A **complex locus** (plural: **loci**) is a subset of the complex plane.

In English, a complex locus is the set of all complex numbers satisfying a given condition. The following examples should clarify this.

Example 11

Describe the locus of points $z = x + iy$ in the Complex Plane satisfying $|z| = 8$.

Putting $z = x + iy$ into the condition $|z| = 8$ gives,

$$\sqrt{x^2 + y^2} = 8$$

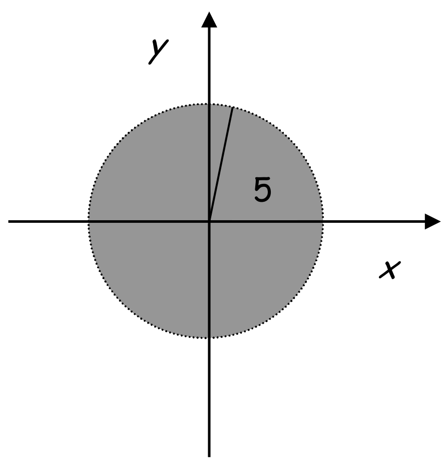
$$x^2 + y^2 = 8^2$$

This is the equation of a circle with centre the origin and radius 8. Hence, the locus is the set of all points on the circle with centre (0, 0) and radius 8.

Example 12

Describe the locus of points $z = x + iy$ in the Complex Plane satisfying $|z| < 5$.

Based on the analysis of Example 11, the locus is the set of all points inside the circle with centre (0, 0) and radius 5.



Example 13

Describe the locus of points $z = x + iy$ in the Complex Plane satisfying
$|z| > 7$.

The locus is the set of all points outside the circle with centre (0, 0) and radius 7.

Example 14

Describe the locus of points $z = x + iy$ in the Complex Plane satisfying $|z - 6 + i| \geq 3$.

We have,

$$|(x + iy) - 6 + i| \geq 3$$

$$|(x - 6) + (y + 1)\,i| \geq 3$$

$$\sqrt{(x - 6)^2 + (y + 1)^2} \geq 3$$

$$(x - 6)^2 + (y + 1)^2 \geq 3^2$$

This locus is the set of all points on or outside the circle with centre $(6, -1)$ and radius 3.

Example 15

Describe the locus of points $z = x + iy$ in the Complex Plane satisfying $|z - 3| = |z + 4i|$.

Putting $z = x + iy$ into the condition gives,

$$|x + iy - 3| = |x + iy + 4i|$$

$$|(x - 3) + iy| = |x + i\,(y + 4)|$$

$$\sqrt{(x - 3)^2 + y^2} = \sqrt{x^2 + (y + 4)^2}$$

$$(x - 3)^2 + y^2 = x^2 + (y + 4)^2$$

$$x^2 - 6x + 9 + y^2 = x^2 + y^2 + 8y + 16$$

$$-6x + 9 = 8y + 16$$

$$6x + 8y + 7 = 0$$

Hence, the locus is the set of all points on the straight line $6x + 8y + 7 = 0$.

Example 16

Describe the locus of points $z = x + iy$ in the Complex Plane satisfying $\arg z = -\frac{3\pi}{4}$.

Complex numbers satisfying the stated condition have an argument of $-\frac{3\pi}{4}$. Hence, they lie in the third quadrant and satisfy,

$$\tan^{-1}\left(\frac{y}{x}\right) = -\frac{3\pi}{4}$$

$$\frac{y}{x} = \tan\left(-\frac{3\pi}{4}\right)$$

$$\frac{y}{x} = 1$$

$$y = x$$

Be careful. The locus is not the straight line $y = x$. The locus is the set of points on the straight line $y = x$ with $x < 0$ (make a sketch).

Polar Form of a Complex Number

The modulus and argument of a complex number can be used to write it in another form. From the diagram on page 8, $x = r\cos\theta$ and $y = r\sin\theta$.

Definition:
A complex number $z = x + iy$ is in **polar form** when it is written as,
$z = r(\cos\theta + i\sin\theta) \equiv r\,\text{cis}\,\theta$

Here, r is the modulus of z and θ is the principal argument. Note that, with z as above, the complex conjugate of z is written in polar form as $\bar{z} = r(\cos\theta - i\sin\theta)$.

Any complex number can be changed from Cartesian form into polar from (and vice versa).

Example 17

Change $z = 3 + 3i$ into polar form.

We need the modulus and principal argument. $|z| = \sqrt{18} = 3\sqrt{2}$. z lies in the first quadrant. $\theta = \tan^{-1} 1 = \frac{\pi}{4}$ rads. or 45 °. Hence,

$$z = 3\sqrt{2}\ \text{cis}\,\frac{\pi}{4} = 3\sqrt{2}\,(\cos 45° + i\sin 45°)$$

Example 18

Change $z = \sqrt{3}\,(\cos \pi/6 + i\sin \pi/6)$ into Cartesian form.

We use $x = r\cos\theta$ and $y = r\sin\theta$. Here, $r = \sqrt{3}$ and $\theta = \pi/6$. Hence, $x = \sqrt{3}\,\cos(\pi/6) = 3/2$ and $y = \sqrt{3}\,\sin(\pi/6) = \sqrt{3}/2$. Hence,

$$z = \frac{3}{2} + \frac{\sqrt{3}}{2}\,i$$

Writing complex numbers in polar form makes multiplying and dividing them a lot easier.

> Theorem:
>
> Given 2 complex numbers z and w in polar form, the following hold,
>
> $$|zw| = |z|\,|w| \quad , \quad \text{Arg}\, zw = \text{Arg}\, z + \text{Arg}\, w$$

' Multiply the moduli and add the arguments ' is a nice wee way of remembering these results.

Example 19

Simplify $z\,w$ where $z = \sqrt{7}\ (\cos 45° + i \sin 45°)$ and $w = (\cos 135° + i \sin 135°)$.

$$
\begin{aligned}
z\,w &= \sqrt{7} \times 1 \,.\, \text{cis}\,(45 + 135)° \\
&= \sqrt{7}\ \text{cis}\ 180° \\
&= \sqrt{7}\ (-1 + 0\,i) \\
&= -\sqrt{7}
\end{aligned}
$$

As is evident, the shorthand polar form notation is very handy.

> Theorem:
>
> Given 2 complex numbers z and w in polar form, the following hold,
>
> $$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \quad , \quad \text{Arg}\, \frac{z}{w} = \text{Arg}\, z - \text{Arg}\, w$$

' Divide the moduli and subtract the arguments ' is a way of remembering these results.

Example 20

Simplify $z \div w$ where $z = \sqrt{2}\ \text{cis}\ \frac{\pi}{4}$ and $w = 2\ \text{cis}\ 3\pi$.

$$z \div w = \frac{\sqrt{2}}{2}\ \text{cis}\left(\frac{\pi}{4} - 3\pi\right)$$

$$= \frac{1}{\sqrt{2}}\ \text{cis}\left(-\frac{11\pi}{4}\right)$$

As $-\frac{11\pi}{4}$ is not in the required range for a principal argument (it is too negative), we must add multiples of 2π to it to get it in $(-\pi, \pi]$. As $-\frac{11\pi}{4}$ is slightly bigger than -3π, add 2π, which is the same as $\frac{8\pi}{4}$, to get,

$$z \div w = \frac{1}{\sqrt{2}}\ \text{cis}\left(-\frac{3\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\right)$$

$$= -\frac{1}{2} - \frac{1}{2}\,i$$

Powers and de Moivre's Theorem

Theorem (**de Moivre's Theorem**):

Given a complex number $z = r(\cos\theta + i\sin\theta)$, then for $k \in \mathbb{R}$,

$$z^k = r^k(\cos k\theta + i\sin k\theta)$$

De Moivre's Theorem makes taking powers of complex numbers much easier than the traditional approach (expanding brackets). It is a very powerful (pun intended) theorem.

Example 21

Express $(1 + i)^{19}$ in Cartesian form.

In polar form, $1 + i$ becomes $\sqrt{2}\ \text{cis}\ \frac{\pi}{4}$. Hence,

$$(1 + i)^{19} = \left(\sqrt{2}\ \text{cis}\ \frac{\pi}{4}\right)^{19}$$

Using de Moivre's Theorem this becomes,

$$\begin{aligned}(1 + i)^{19} &= \left(\sqrt{2}\right)^{19}\ \text{cis}\ \frac{19\pi}{4} \\ &= 2^9\sqrt{2}\ \text{cis}\ \frac{19\pi}{4} \\ &= 512\sqrt{2}\ \text{cis}\ \frac{19\pi}{4} \\ &= 512\sqrt{2}\ \text{cis}\ \frac{3\pi}{4} \\ &= 512\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\right) \\ &= -512 + 512\,i\end{aligned}$$

Trigonometric Identities

A funky application of de Moivre's Theorem occurs in deriving trigonometric identities by using the Binomial Theorem.

It is to be noted that the Binomial Theorem applies to complex numbers.

Example 22

Derive the double angle formulae for sine and cosine.

The idea is to write $(\cos\theta + i\sin\theta)^2$ in 2 ways. On the one hand, de Moivre's Theorem states that,

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

On the other hand, the Binomial Theorem says that,

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$$

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + (2\sin\theta\cos\theta)\, i$$

Equating these 2 expressions for $(\cos\theta + i\sin\theta)^2$ gives,

$$\cos 2\theta + i\sin 2\theta = \cos^2\theta - \sin^2\theta + (2\sin\theta\cos\theta)\, i$$

Equating real and imaginary parts gives,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

<u>Example 23</u>

Express $\cos 3\theta$ in terms of powers of $\cos\theta$.

By de Moivre's Theorem,

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

By the Binomial Theorem, this also equals (after simplification),

$$\cos^3\theta - 3\cos\theta\sin^2\theta + 3i\cos^2\theta\sin\theta - i\sin^3\theta$$

Equating real parts gives,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

The Pythagorean identity then gives,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

The n^{th} Roots of any Complex Number

As de Moivre's Theorem holds for fractional powers as well, this makes it a useful tool for taking roots.

Solving $z^n = w$

Theorem (**n^{th} Roots Theorem**):

Given $w = r(\cos\theta + i\sin\theta)$, then the n solutions of the equation $z^n = w$ are given by,

$$z_k = r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + \sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

$$(k = 0, 1, 2, \ldots, n - 1)$$

The following examples will illustrate how to use this horrible formula.

Example 24

Find the fourth roots of $4 + 4i$.

We translate the problem so that the theorem can be used. Letting $w = 2 + 2i$, the problem is now to solve $z^4 = w$. Writing w in polar form gives (check !),

$$z^4 = 4\sqrt{2}\ \text{cis}\ \frac{\pi}{4}$$

By the theorem,

$$z_k = 4^{1/4}\, 2^{1/8}\ \text{cis}\left(\frac{\pi/4 + 2\pi k}{4}\right)$$

$$z_k = 2^{5/8} \text{ cis } \left(\frac{\pi\,(8k + 1)}{16} \right) \quad (k = 0, 1, 2 \text{ and } 3)$$

So,

$$z_0 = 2^{5/8} \text{ cis } \frac{\pi}{16}$$

$$z_1 = 2^{5/8} \text{ cis } \frac{9\pi}{16}$$

$$z_2 = 2^{5/8} \text{ cis } \frac{17\pi}{16}$$

$$z_3 = 2^{5/8} \text{ cis } \frac{25\pi}{16}$$

As z_2 and z_3 are not in $(-\pi, \pi]$, we must take away multiples of 2π from the arguments given above to get,

$$z_0 = 2^{5/8} \text{ cis } \frac{\pi}{16}$$

$$z_1 = 2^{5/8} \text{ cis } \frac{9\pi}{16}$$

$$z_2 = 2^{5/8} \text{ cis } \left(-\frac{15\pi}{16} \right)$$

$$z_3 = 2^{5/8} \text{ cis } \left(-\frac{7\pi}{16} \right)$$

These are the roots of $4 + 4\,\text{i}$. Don't expect a horrible formula to give unhorrible answers.

There is a geometric interpretation to the problem of finding n^{th} roots of a complex number. The next example will demonstrate this.

Example 25

Solve $z^3 = 8\,\mathrm{i}$.

In polar form, $8\,\mathrm{i} = 8 \text{ cis } \frac{\pi}{2}$. Hence,

$$z_k = 2 \text{ cis } \left(\frac{\pi/2 + 2\pi k}{3} \right)$$

$$z_k = 2 \text{ cis } \left(\frac{\pi\,(4k + 1)}{6} \right) \qquad (k = 0, 1 \text{ and } 2)$$

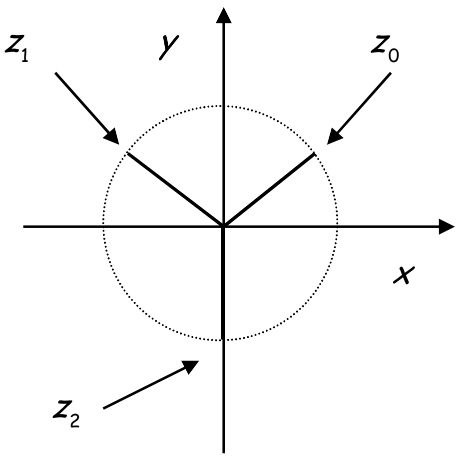
Check that the solutions are,

$$z_0 = \sqrt{3} + \mathrm{i}$$

$$z_1 = -\sqrt{3} + \mathrm{i}$$

$$z_2 = -2\,\mathrm{i}$$

Now plot these solutions on an Argand diagram.



As all the solutions have the same modulus, they lie on a circle with centre (0, 0) and radius 2 . The angle between any 2 of the lines (loosely referred to as the ' angle between the solutions ') is $2\pi/3$ radians. This situation is a specific case of a more general phenomenon.

<u>Theorem:</u>

The n solutions of $z^n = w$ lie on a circle with radius $|w|^{1/n}$ and are equally spaced, the angle between any 2 successive roots being $2\pi/n$ radians.

The problem given near the start of this topic can now be solved. It is easy to see that the cube roots of i are $\frac{\sqrt{3}}{2} + \frac{1}{2}\,\mathrm{i}$, $-\frac{\sqrt{3}}{2} + \frac{1}{2}\,\mathrm{i}$ and $-\,\mathrm{i}$. Taking each of these roots and substituting them into the formula on page 1 gives the 3 real solutions of $x^3 - x = 0$. We have come full circle (no pun intended).

Notice that the sum of all the solutions of $z^n = w$ add up to 0. This is no coincidence, especially by remembering the vector interpretation of complex numbers.

<u>Theorem:</u>

The n^{th} roots of $z^n = w$ sum to 0.

Roots of Unity

The solution to the problem $z^n = 1$ is called finding the $\boldsymbol{n}^{\textbf{th}}$ **roots of unity**. The special case of the last theorem for roots of unity is often stated separately.

<u>Theorem:</u>

The n^{th} roots of unity ($n > 1$) satisfy,

$$\sum_{k=0}^{n-1} z_k = 0$$

<u>Example 26</u>

Find the 6^{th} roots of unity.

To solve $z^6 = 1$, we use the n^{th} Roots Theorem, noting that $1 = \operatorname{cis} 0$,

$$z_k = \text{cis}\left(\frac{\pi k}{3}\right) \qquad (k = 0, 1, 2, 3, 4 \text{ and } 5)$$

Thus,

$$z_0 = 1$$

$$z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}\,\text{i}$$

$$z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\text{i}$$

$$z_3 = -1$$

$$z_4 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,\text{i}$$

$$z_5 = \frac{1}{2} - \frac{\sqrt{3}}{2}\,\text{i}$$

Notice that the roots occur in conjugate pairs (z_0 and z_3, z_1 and z_5, z_2 and z_4).

Solving Polynomials

The true power of complex numbers stems from the fact that they can be used to solve any polynomial equation.

> Definition:
>
> A repeated root (occurring m times) of a polynomial p is called a root of **multiplicity m**.

The next theorem is one of the most important results in mathematics.

> Theorem (**Fundamental Theorem of Algebra**):
>
> Every non-constant polynomial with complex coefficients has at least one complex root.

Compare this theorem with the case of polynomials with only real coefficients; many quadratics can't be solved using real numbers. The Fundamental Theorem of Algebra implies the following result.

Corollary:

Every polynomial of degree n (≥ 1) with complex coefficients has exactly n complex roots (including multiplicities).

Theorem:

A polynomial p of degree at least 1 with complex coefficients can be factorised into a product of n linear factors,

$$p(z) = \prod_{r=1}^{n} (z - z_r)$$

Theorem:

If a polynomial p of degree n with all coefficients real has a non-real root, then the conjugate of this root is also a root of p.

Theorem:

A polynomial of degree n with all coefficients real can be factorised into a product of real linear factors and real irreducible quadratic factors,

$$p(z) = \prod_{r=1}^{t} (z - d_r) \times \prod_{s=1}^{(n-t)/2} (a_s z^2 + b_s z + c_s)$$

where the second product only exists if $n > t$.

Cubic with 2 Complex Roots and 1 Real Root

Example 27

Given that $z = 3$ is a root $p(z) = z^3 - 3z^2 + 4z - 12$, find the other roots of p.

As $z = 3$ is a root, $(z - 3)$ is a factor of p. Dividing p by $(z - 3)$ gives (check !),

$$p(z) = (z - 3)(z^2 + 4)$$

In the language of the last theorem, $n = 3$, $t = 1$ (and hence $r = 1$ only), $d_1 = 3$, $s = 1$ only, $a_1 = 1$, $b_1 = 1$ and $c_1 = 4$. The roots of p are obtained by solving $(z - 3)(z^2 + 4) = 0$; hence, $z - 3 = 0$ and $z^2 + 4 = 0$, which implies $z = 3$ (which we already know about) and $z = \pm 2\,i$. Thus, p has roots $z = 3$ and $z = \pm 2\,i$.

Cubic with 3 Real Roots

Example 28

Factorise $z^3 - 8z^2 + 11z + 20$ in to a product of 3 linear factors.

It can be checked that $z = -1$ is a solution of $z^3 - 8z^2 + 11z + 20 = 0$. Hence, $(z + 1)$ is a factor of the cubic. Long division yields,

$$z^3 - 8z^2 + 11z + 20 = (z + 1)(z^2 - 9z + 20)$$

The quadratic can be easily factorised, or, if that is too much for your brain to handle, use the Quadratic Formula to get the roots (and hence the factors), into $z^2 - 9z + 20 = (z - 4)(z - 5)$. Hence,

$$z^3 - 8z^2 + 11z + 20 = (z + 1)(z - 4)(z - 5)$$

In terms of the last theorem, $n = 3$, $t = 3$ (and r ranges from 1 to 3), $d_1 = -1$, $d_2 = 4$ and $d_3 = 5$ (there are no irreducible quadratic factors, as can be deduced from the fact that $n = t$).

Quartic with 4 Complex Roots

Example 29

Given that $z = 2 + 3\,i$ is a root of the quartic $p(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$, find the other 3 roots of p and express p as (i) a product of complex linear factors (ii) a product of real irreducible quadratic factors.

As p has all coefficients real, the conjugate $\overline{z} = 2 - 3\,\mathrm{i}$ is also a root. Hence, $z - (2 + 3\,\mathrm{i})$ and $z - (2 - 3\,\mathrm{i})$ are factors of p; thus, $(z - (2 + 3\,\mathrm{i}))(z - (2 - 3\,\mathrm{i})) = z^2 - 4z + 13$ is also a factor of p. To find the other roots, divide this quadratic factor into p (the long division process also works for quadratic factors),

$$
\begin{array}{r|l}
 & z^2 - 2z + 2 \\
\hline
z^2 - 4z + 13 & z^4 - 6z^3 + 23z^2 - 34z + 26 \\
 & z^4 - 4z^3 + 13z^2 \\
\hline
 & \quad -2z^3 + 10z^2 - 34z + 26 \\
 & \quad -2z^3 + 8z^2 - 26z \\
\hline
 & \qquad 2z^2 - 8z + 26 \\
 & \qquad 2z^2 - 8z + 26 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

As the remainder is 0, we have that $z^2 - 2z + 2$ is a factor of p. Hence, the roots of p are given by,

$$(z^2 - 4z + 13)(z^2 - 2z + 2) = 0$$

Putting the first factor equal to zero gives us the roots we know about. Putting the second factor equal to zero and using the quadratic formula gives $z = 1 \pm \mathrm{i}$. Hence, the roots of p are $z = 2 \pm 3\,\mathrm{i}$ and $z = 1 \pm \mathrm{i}$.

For part (i),

$$p(z) = (z - (2 + 3\,\mathrm{i}))(z - (2 - 3\,\mathrm{i}))(z - (1 + \mathrm{i}))(z - (1 - \mathrm{i}))$$

For part (ii),

$$p(z) = (z^2 - 4z + 13)(z^2 - 2z + 2)$$

Quartic with 2 Complex Roots and 2 Real Roots

Example 30

Verify that $z = 1 + 3\,i$ is a root of $p(z) = z^4 - 2z^3 + 9z^2 + 2z - 10$ and find all the other roots.

Substituting $z = 1 + 3\,i$ into $p(z)$ gives,

$$p(1 + 3\,i) = (1 + 3\,i)^4 - 2(1 + 3\,i)^3 + 9(1 + 3\,i)^2 + 2(1 + 3\,i) - 10$$

We need to work out $(1 + 3\,i)^2$, $(1 + 3\,i)^3$ and $(1 + 3\,i)^4$. We have (check !),

$$(1 + 3\,i)^2 = -8 + 6\,i$$

$$(1 + 3\,i)^3 = -26 - 18\,i$$

$$(1 + 3\,i)^4 = 28 - 96\,i$$

Thus,

$$p(1 + 3\,i) = 28 - 96\,i - 2(-26 - 18\,i) + 9(-8 + 6\,i) + (2 + 6\,i) - 10$$

$$= 28 - 96\,i + 52 + 36\,i - 72 + 54\,i + 2 + 6\,i - 10$$

$$= (28 + 52 - 72 - 8) + (-96 + 36 + 54 + 6)\,i$$

$$= 0$$

Hence, as $p(1 + 3\,i) = 0$, $z = 1 + 3\,i$ is a root of p. As p has all coefficients real, $\bar{z} = 1 - 3\,i$ is also a root. Hence, $z - (1 + 3\,i)$ and $z - (1 - 3\,i)$ are factors of p; thus, $(z - (1 + 3\,i))(z - (1 - 3\,i)) = z^2 - 2z + 10$ is also a factor of p. Long dividing p by $z^2 - 2z + 10$ gives,

$$p(z) = (z^2 - 2z + 10)(z^2 - 1)$$

Solving $p(z) = 0$ gives $z = 1 \pm 3i$ and $z = \pm 1$.

Quartic with 4 Real Roots

Example 31

Show that $z^4 - 8z^3 + 17z^2 + 2z - 24$ can be written in the form $(z + 1)(z - 2)(z - 3)(z - 4)$.

Performing synthetic division on the polynomial gives,

$z^4 - 8z^3 + 17z^2 + 2z - 24 = (z + 1)(z^3 - 8z^2 + 11z + 20)$

Performing synthetic division on the cubic gives,

$$z^3 - 8z^2 + 11z + 20 = (z - 2)(z^2 - 7z + 12)$$

Factorising the quadratic gives,

$$z^2 - 7z + 12 = (z - 3)(z - 4)$$

Putting these 3 pieces of information together gives,

$$z^4 - 8z^3 + 17z^2 + 2z - 24 = (z + 1)(z - 2)(z - 3)(z - 4)$$

Examples 27, 29 and 30 are the most likely cases that you will encounter in this course (as they are ones that involve complex numbers !); the other 2 examples are given for the sake of completeness.