Differential Calculus and Applications

Prerequisites: Differentiating x^n, sin x and cos x; sum/difference and chain rules; finding max./min.; finding tangents to curves; finding stationary points and their nature; optimising a function.

Maths Applications: Higher derivatives; integration.

Real-World Applications: Particle motion; chemical reaction rates; generic rates of change.

Differentiation from First Principles

Recall from Higher that the derivative of a function f is defined by,

$$f'(x) \stackrel{def}{=} \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

This frightening formula gives a recipe for calculating the derivative of any function. Using this recipe to calculate the derivative of a function is called *differentiation from first principles*. Although not required, it is useful to see just how painful it is to differentiate a function from first principles, as we can then be grateful for simpler differentiation techniques (which come from the definition).

Example 1

Differentiate $f(x) = x^{1/2}$ from first principles.

$$f'(x) = \lim_{h \to 0}\left(\frac{(x+h)^{1/2} - x^{1/2}}{h}\right)$$

Multiplying the numerator and denominator by $(x+h)^{1/2} + x^{1/2}$ gives,

$$f'(x) = \lim_{h \to 0}\left(\frac{((x+h)^{1/2} - x^{1/2})((x+h)^{1/2} + x^{1/2})}{h((x+h)^{1/2} + x^{1/2})}\right)$$

Multiplying out the numerator and simplifying gives,

$$f'(x) = \lim_{h \to 0} \left(\frac{x + h - x}{h((x + h)^{\frac{1}{2}} + x^{\frac{1}{2}})} \right)$$

Further simplification and rewriting the denominator gives,

$$f'(x) = \lim_{h \to 0} \left(\frac{1}{\sqrt{x + h} + \sqrt{x}} \right)$$

Taking the limit as $h \to 0$ finally gives,

$$f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Notation for Derivatives

The derivative of a function can be written in different ways. Below are listed some common types.

Notation:

The derivative of a function $y = f(x)$ is written variously as,

y' or $f'(x)$ (**Dash notation** aka **prime notation** or **Lagrange notation**)

$\frac{df}{dx}$ or $\frac{d}{dx} f(x)$ or $\frac{dy}{dx}$ (**d by dx notation** aka **Leibniz notation**)

Dy or Df (**Big D notation** aka **Euler notation**)

The derivative of a function of time t has an alternative notation.

Notation:

The derivative of a function of time t is often written as,

$\dot{f}(t) \overset{def}{=} \frac{df}{dt}$ (**Dot notation** aka **Newton notation**)

Higher-Order Derivatives

Sometimes, the derivative of a function can be differentiated.

Definition:

Given a function $y = f(x)$, the **higher-order derivative of order n** (aka the **n^{th} derivative**) is defined by,

$$\frac{d^n f}{dx^n} \overset{def}{=} \underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\cdots\left(\frac{d}{dx}\left(\frac{d}{dx}\right.\right.\right.\right.}_{n \text{ times}} f \left.\left.\left.\left.\right)\right)\cdots\right)\right)$$

An important special case of this is the **second derivative**,

$$\frac{d^2 f}{dx^2} \overset{def}{=} \frac{d}{dx}\left(\frac{df}{dx}\right)$$

In the Euler and Lagrange notations, higher-order derivatives are written respectively as,

$$D^n f \text{ and } f^{(n)}(x)$$

this Lagrange notation mainly being used for derivatives usually greater than or equal to 4 (as then the dashes become too numerous). To avoid ambiguity, sometimes the case when $n = 1$ is called the **first derivative**.

The derivatives we consider are technically called *ordinary derivatives*, in comparison to *partial derivatives*, which are derivatives of functions that are of more than 1 variable. Most interesting equations in science involve partial derivatives (as the description of most phenomena depends on more than 1 physical quantity).

Example 2

Find the first and second derivatives of $f(x) = x^4 + \sin 3x$.

$$f'(x) = 4x^3 + 3\cos 3x$$

$$f''(x) = 12x^2 - 9\sin 3x$$

Example 3

Find the smallest value of n for which $f^{(n)}(x) = 0$ when $f(x) = x^3$.

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$f'''(x) = 6$$

$$f^{(4)}(x) = 0$$

Clearly, all other higher derivatives will be 0 too, so $n = 4$.

Product and Quotient Rules

The sum, difference and chain rules for differentiating functions are assumed in this course. In order to differentiate more types of functions, 2 new rules will be introduced, the first to enable differentiation of a product of functions, the second to differentiate a quotient.

Theorem (**Product Rule**):

Given 2 functions f and g, the derivative of their product is given by,

$$(fg)' = f'g + fg'$$

Theorem (**Quotient Rule**):

Given 2 functions f and g, the derivative of their quotient (f/g) is given by,

$$(f/g)' = \frac{1}{g^2}(f'g - fg')$$

The Product Rule is easy to remember because of the + sign. The Quotient Rule takes more care because of the $-$ sign and the division by g^2.

Try writing both these rules in Leibniz notation and Euler notation to see which form is easier (or preferable) to remember.

Example 4

Differentiate $h(x) = x^3 \cos 2x$.

Let $f(x) = x^3$ and $g(x) = \cos 2x$. The following way of setting out the working is useful,

$$\begin{array}{lcl} f(x) = x^3 & , & g(x) = \cos 2x \\ f'(x) = 3x^2 & , & g'(x) = -2\sin 2x \end{array}$$

Now substitute everything into the Product Rule,

$$h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$h'(x) = (3x^2)\cos 2x + x^3\,(-2\sin 2x)$$

$$h'(x) = 3x^2 \cos 2x - 2x^3 \sin 2x$$

$$h'(x) = x^2(3\cos 2x - 2x\sin 2x)$$

Example 5

Differentiate $h(x) = \dfrac{x^2}{(x-1)^3}$ $(x \neq 1)$, simplifying as much as possible.

Let $f(x) = x^2$ and $g(x) = (x-1)^3$. Then,

$$\begin{array}{lcl} f(x) = x^2 & , & g(x) = (x-1)^3 \\ f'(x) = 2x & , & g'(x) = 3(x-1)^2 \end{array}$$

and $g(x)^2 = (x-1)^6$ (remember rules of indices – power of a power). Then apply the Quotient Rule to get,

$$h'(x) = \frac{(2x)(x-1)^3 - x^2(3(x-1)^2)}{(x-1)^6}$$

$$h'(x) = \frac{x\,(x - 1)^2\,(2\,(x - 1) - 3x)}{(x - 1)^6}$$

$$h'(x) = \frac{x\,(x - 1)^2\,(-x - 2)}{(x - 1)^6}$$

$$h'(x) = \frac{-x(x + 2)}{(x - 1)^4}$$

Reciprocal Trigonometric Functions

There are 3 new trigonometric functions that will be studied in this course. They are defined using the sine, cosine and tangent functions.

Definition:

The **secant function** is defined by,

$$\sec x \stackrel{def}{=} \frac{1}{\cos x}$$

The domain and range of the secant function are,

$$\text{dom}(\sec x) = \mathbb{R} \setminus \left\{ x \in \mathbb{R} : x = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z} \right\}$$

$$\text{ran}(\sec x) = \mathbb{R} \setminus (-1, 1)$$

Definition:

The **cosecant function** is defined by,

$$\text{cosec } x \stackrel{def}{=} \frac{1}{\sin x}$$

The domain and range of the cosecant function are,

$$\text{dom}(\text{cosec } x) = \mathbb{R} \setminus \{ x \in \mathbb{R} : x = n\pi,\ n \in \mathbb{Z} \}$$

$$\text{ran}(\text{cosec } x) = \mathbb{R} \setminus (-1, 1)$$

Definition:

The **cotangent function** is defined by,

$$\cot x \overset{def}{=} \frac{\cos x}{\sin x}$$

The domain and range of the cotangent function are,

$$\text{dom}(\cot x) = \mathbb{R} \setminus \{x \in \mathbb{R} : x = n\pi,\ n \in \mathbb{Z}\}$$

$$\text{ran}(\cot x) = \mathbb{R}$$

Derivatives of Trigonometric Functions

The derivatives of tan x, sec x, cosec x and tan x can be obtained by using the quotient rule and the derivatives of sin x and cos x.

- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \sec x = \sec x \tan x$
- $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

Derivatives of exp x and ln x

The derivatives of exp x ($\equiv e^x$) and ln x ($\equiv \log_e x$) require more effort to obtain. The results will be stated here.

- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \ln x = \frac{1}{x}$

Example 6

Differentiate $p(x) = e^{2x} \sin x$.

$$p'(x) = (2e^{2x}) \sin x + e^{2x} (\cos x)$$

$$p'(x) = e^{2x} (2 \sin x + \cos x)$$

Example 7

Differentiate $m(x) = \dfrac{x^2}{\ln 4x}$.

$$m'(x) = \frac{(2x) \ln 4x - \left(\dfrac{4}{4x}\right) x^2}{(\ln 4x)^2}$$

$$m'(x) = \frac{2x \ln 4x - x}{(\ln 4x)^2}$$

$$m'(x) = \frac{x (2 \ln 4x - 1)}{(\ln 4x)^2}$$

Rectilinear Motion

The derivative of a function gives a measure of the rate of change of a function with respect to a certain variable. Historically, Calculus was invented to satisfy the needs of describing physical phenomena. In particular, the rate of change of functions with respect to time played a very important role in understanding dynamics through mathematics.

Definition:
Motion in a straight line is called **rectilinear motion**.

Unless otherwise stated, time will be measured in seconds and distance in metres.

Definition:

The **displacement** of a particle from the origin O is the location of the particle from O after time t. Displacement is denoted by $s(t)$.

Definition:

The **velocity** of a particle is defined by,

$$v(t) \overset{def}{=} \frac{ds}{dt} = \dot{s}(t)$$

Definition:

The **acceleration** of a particle is defined by,

$$a(t) \overset{def}{=} \frac{dv}{dt} = \frac{d^2s}{dt^2} = \ddot{s}(t)$$

Example 8

The displacement of a particle from the origin after time t is described by the function $s(t) = 4t^2 - 3t^3$. Find the velocity and acceleration of the particle when $t = 3$.

$$\dot{s}(t) = 8t - 9t^2 \Rightarrow \dot{s}(3) = 8(3) - 9(3)^2 \Rightarrow \dot{s}(3) = -57 \text{ m/s}$$

$$\ddot{s}(t) = 8 - 18t \Rightarrow \ddot{s}(3) = 8 - 18(3) \Rightarrow \ddot{s}(3) = -46 \text{ m/s}^2$$

Differentiability of a Function

Recall that a function is differentiable at a point if a unique tangent line can be drawn through that point. This can be understood in terms of tangent lines approaching from the left and right to the point in question. As tangent lines approach from the left and right, the gradients of these tangent lines should approach the same value. The function should be 'smooth' and 'curvy' at the point in question.

Definition:

The **left-hand derivative of f at $x = a$** is defined by,

$$f_{-}'(a) \stackrel{def}{=} \lim_{h \to 0^-} \left(\frac{f(a + h) - f(a)}{h} \right)$$

The **right-hand derivative of f at $x = a$** is defined by,

$$f_{+}'(a) \stackrel{def}{=} \lim_{h \to 0^+} \left(\frac{f(a + h) - f(a)}{h} \right)$$

Theorem:

A function is differentiable at a point if and only if both the left-hand and right-hand derivatives exist at that point and are equal to each other.

Thus, to determine if a function is not differentiable at a point, it suffices to show one of the following (i) the left-hand derivative does not exist (ii) the right-hand derivative does not exist (iii) if they both exist, they are not equal. In terms of the graph of a function, this usually means identifying a sharp corner (aka ' kink ') or an endpoint (at an endpoint, obviously one of the derivatives won't exist).

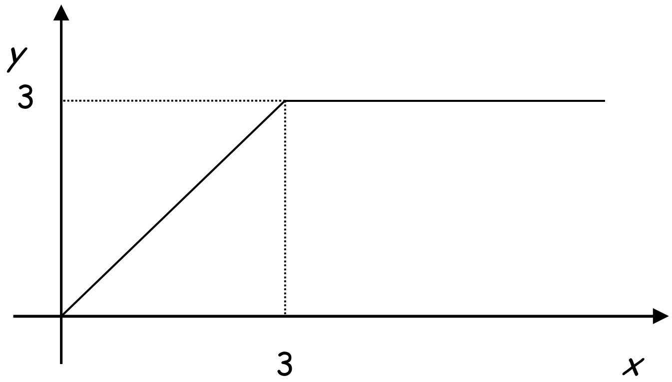
Example 9

Determine where the function f defined by,

$$f(x) = \begin{cases} x & (0 \le x \le 3) \\ 3 & (x > 3) \end{cases}$$

is differentiable and where it is not differentiable.

First sketch the graph of f.



For $0 < x < 3$, $f\,' = 1$, so f is differentiable for $0 < x < 3$.
For $x > 3$, $f\,' = 0$, so f is differentiable for $x > 3$.

The only unclear points are where $x = 0$ and $x = 3$. For $x = 0$, the left-hand derivative does not exist, so f is not differentiable at $x = 0$. For $x = 3$, $f_-\,'(3) = 1$ and $f_+\,'(3) = 0$, so f is not differentiable at $x = 3$.

Critical Points

A function may have points where the derivative is 0 or where the derivative does not exist. Such points have a special name.

Definition:

A function f has a **critical point at $x = a$** if either $f\,'(a) = 0$ or $f\,'(a)$ does not exist. a is called the **critical number** and $f(a)$ is called the **critical value**.

Note that all stationary points are critical points, but not all critical points are stationary points. Referring to the previous example, f has critical points at (0, 0), (3, 3) and (x, 3) ($x > 3$). At (0, 0) and (3, 3), the derivative doesn't exist, whereas for (x, 3) ($x > 3$) the derivative is 0 (i.e. all the points (x, 3) ($x > 3$) are stationary points).

Extrema of a Function

It is important to know the maximum and minimum values of a function.

Definition:

Maxima and minima of a function are known as **extrema**.

There are 3 types of extrema.

- Local.
- Endpoint.
- Global.

Definition:

A function f has a **local (relative) maximum at $x = a$** if $\exists$ an open interval I about a for which $f(a) \geq f(x)$ $\forall x \in \mathrm{I}$.

A function f has a **local (relative) minimum at $x = a$** if $\exists$ an open interval I about a for which $f(a) \leq f(x)$ $\forall x \in \mathrm{I}$.

Theorem:

All local extrema occur at critical points. Stated differently, if f has a local extremum, then that local extremum point is also a critical point.

Note that not all critical points are local extrema. Referring back to Example 9, the point (0, 0) is not a local minimum, but it is a critical point.

Definition:

If a is an endpoint in dom f, f has an **endpoint maximum at $x = a$** if $\exists$ $p \in$ dom f for which $f(a) \geq f(x)$ ($\forall x \in [a, p)$ or $(p, a]$).

If a is an endpoint in dom f, f has an **endpoint minimum at $x = a$** if $\exists$ $p \in$ dom f for which $f(a) \leq f(x)$ ($\forall x \in [a, p)$ or $(p, a]$).

Definition:

A function f function has a **global (absolute) maximum at $x = a$** if $f(a) \geq f(x)$ ($\forall x \in$ dom f).

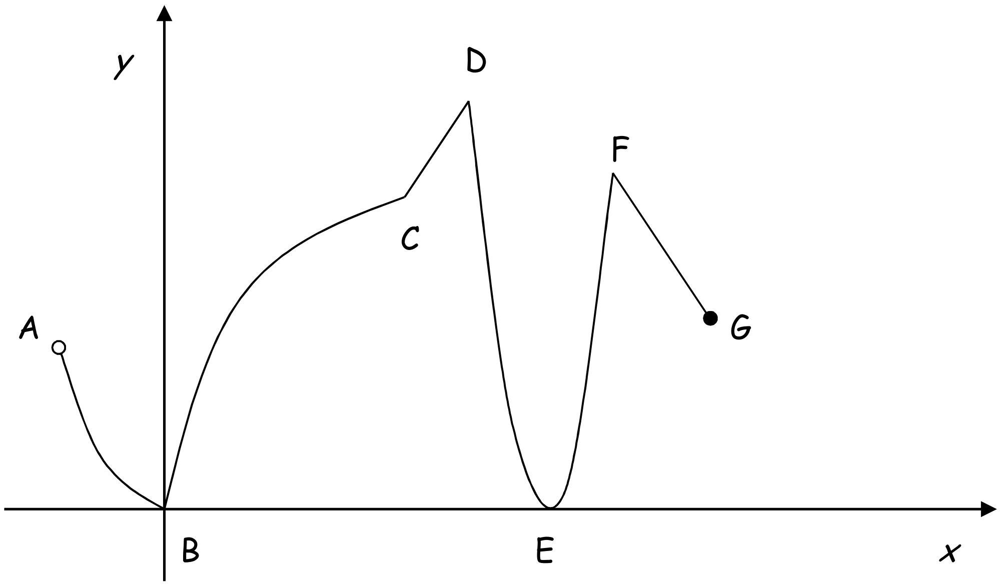
A function f function has a **global (absolute) minimum at $x = a$** if $f(a) \leq f(x)$ ($\forall x \in$ dom f).

Theorem:

Every global extremum is either a local extremum or endpoint extremum.

Example 10

Classify the extrema for the function f with the graph shown.



The small circles indicate the extent of where f is defined. The white circle means that the point is not in dom f, whereas the black circle means that the point is in dom f.

Let's analyse the labelled points in turn. Remember, all extrema occur at critical points.

A is not an endpoint max., as A $\notin$ dom f (if A were in dom f, then A would be an endpoint max.). A is not a critical point.

B is a critical point, as $f_-'(\text{B}) < 0$ and $f_+'(\text{B}) > 0$, so $\nexists\, f'(\text{B})$. In particular, a local min. occurs at B, as all y – values in the immediate vicinity of $f(\text{B})$ are bigger than $f(\text{B})$; the local min. is also a global min., as is clear from the graph. So, B is a global minimum.

C is a critical point (even though $f_-'(C) > 0$ and $f_+'(C) > 0$, the actual values are different and there is clearly no smooth transition between these values; the function is kinky at C, so $\nexists\, f'(C)$). C is not a local extremum, as there y – values close to $f(C)$ which are both bigger and smaller than $f(C)$. C is a critical point.

D is a critical point, as $f_{-}'(\text{D}) > 0$ and $f_{+}'(\text{D}) < 0$, so $\nexists\, f'(\text{D})$. D is obviously a local max. and from the graph it clearly has the largest y – value. So, D is a global maximum.

E is a critical point, as $f'(\text{E}) = 0$. Thus, E is a stationary point. It is a local as well as a global min. So, E is a global minimum.

F is clearly a critical point (for a similar reason cited for D). It is also a local max., but not a global max. So, F is a local maximum.

G is a critical point, as $\nexists\, f_{+}'(G)$. There are x – values to the left of G for which $f(x) > f(G)$ and G is an endpoint. So, G is an endpoint minimum.

Second Derivatives, Concavity and Points of Inflexion

The second derivative provides a useful measure of how curvy a function is and how the curviness changes.

Definition:

A function f is **concave up** on an interval if $f''(x) \geq 0$ on that interval.

A function f is **concave down** on an interval if $f''(x) \leq 0$ on that interval.

Definition:

A function f has a **point of inflexion (inflection) at $x = a$** if there is a change of concavity as the graph passes through $x = a$.

The above definitions lead to an alternative way of spotting inflexion points.

Theorem:

If a function f has a point of inflexion at $x = a$, then either $\nexists\, f''(x)$ or $f''(x) = 0$.

Note that if a function has $f''(a) = 0$, this does not necessarily mean that $x = a$ is a P of I. For example, $f(x) = x^4$ has $f''(x) = 12x^2$ and thus $f''(0) = 0$. However, (0, 0) is clearly a minimum (sketch the graph).

Inflexion points may be further classified depending on the value of f'.

<u>Definition</u>:

A function f has a **horizontal (stationary) point of inflexion at $x = a$** if $f'(a) = 0$.

A function has a **non-horizontal (non-stationary) point of inflexion at $x = a$** if $f'(a) \neq 0$.

For stationary points, there is a useful test for finding local maxima and minima.

<u>Theorem (**Second Derivative Test**)</u>:

If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a.

If $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at a.

If $f'(a) = 0$ and $f''(a) = 0$, then f may have a local maximum, a local minimum or a (horizontal) point of inflexion.

In the third case, the P of I must be horizontal, as $f' = 0$.

A nature table is normally used to find maxima, minima and inflexion points. The Second Derivative Test gives a quick way of deciding whether a function has maxima or minima. In the third case, a nature table must be drawn to determine the stationary point.

<u>Example 11</u>

Find all maxima, minima and inflexion points of $f(x) = \sin x$ in the interval $(0, 2\pi)$.

$f'(x) = \cos x$. The stationary points occur at $(\pi/2, 1)$ and $(3\pi/2, -1)$.

$f''(x) = -\sin x$. $f''(\pi/2) = -1 < 0$ and $f''(3\pi/2) = 1 > 0$. So, $(\pi/2, 1)$ is a global maximum and $(3\pi/2, -1)$ is a global minimum. As neither the max. nor min. have $f'' = 0$, any inflexion points will be found by solving this equation. So, solving $-\sin x = 0$ gives $x = \pi$. When $x = \pi$, $f(\pi) = 0$ and $f'(\pi) = -1 \neq 0$. Thus, $(\pi, 0)$ is a non-horizontal P of I.

If a function f has a P of I (and f'' exists at the point in question), then $f'' = 0$. Hence, if $f'' \neq 0$, there do not exist any inflexion points.

<u>Example 12</u>

Show that $f(x) = \dfrac{x - 5}{x + 3}$ has no stationary points and no inflexion points.

Rewriting f using long division gives,

$$f(x) = 1 - \frac{8}{x + 3}$$

$$f'(x) = \frac{8}{(x + 3)^2}$$

For stationary points, $f'(x) = 0$. However, the equation,

$$\frac{8}{(x + 3)^2} = 0$$

obviously has no solutions. Hence, there are no maxima or minima. For any inflexion points, we need to analyse the second derivative,

$$f''(x) = -\frac{16}{(x + 3)^3}$$

Similarly, the equation

$$-\frac{16}{(x + 3)^3} = 0$$

has no solutions. Thus, there are no inflexion points.